Stephen W. Wi

A COURSE OF MATHEMATICS.

IN TWO VOLUMES.

FOR THE USE OF ACADEMIES,

AS WELL AS

PRIVATE TUITION.

BY

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A

COURSE

OF

MATHEMATICS, &c.

PLANE TRIGONOMETRY CONSIDERED ANALYTICALLY.

ART. 1.

THERE are two methods which are adopted by mathematicians in investigating the theory of Trigonometry: the one *Geometrical*, the other *Algebraical*. In the former, the various relations of the sines, cosines, tangents, &c, of single or multiple arcs or angles, and those of the sides and angles of triangles, are deduced immediately from the figures to which the several enquiries are referred; each individual case requiring its own particular method, and resting on evidence peculiar to itself. In the latter, the nature and properties of the linear-angular quantities (sines tangents &c,) being first defined, some general relation of these quantities, or of them in connection with a triangle, is expressed by one or more algebraical equations; and then every other theorem or precept, of use in this branch of science, is developed by the simple reduction and transformation of the primitive equation. Thus, the rules for the three fundamental cases in Plane Trigonometry, which are deduced by three independent geometrical investigations, in the first volume of this Course of Mathematics, are obtained algebraically, by forming, between the three data

and the three unknown quantities, three equations, and obtaining in expressions of known terms, the value of each of the unknown quantities, the others being exterminated by the usual processes. Each of these general methods has its peculiar advantages. The geometrical method carries conviction at every step; and by keeping the objects of enquiry constantly before the eye of the student, serves admirably to guard him against the admission of error: the algebraical method, on the contrary, requiring little aid from first principles, but merely at the commencement of its career, is more properly mechanical than mental, and requires frequent checks to prevent any deviation from truth. The geometrical method is direct, and rapid in producing the requisite conclusions at the outset of trigonometrical science; but slow and circuitous in arriving at those results which the modern state of the science requires: while the algebraical method, though sometimes circuitous in the developement of the mere elementary theorems, is very rapid and fertile in producing those curious and interesting formulæ, which are wanted in the higher branches of pure analysis, and in mixed mathematics, especially in Physical Astronomy. This mode of developing the theory of Trigonometry is, consequently, well suited for the use of the more advanced student; and is therefore introduced here with as much brevity as is consistent with its nature and utility.

2. To save the trouble of turning very frequently to the 1st volume, a few of the principal definitions, there given, are here repeated, as follows:

The SINE of an arc is the perpendicular let fall from one of its extremities upon the diameter of the circle which passes through the other extremity.

The COSINE of an arc, is the sine of the complement of that arc, and is equal to the part of the radius comprised between the centre of the circle and the foot of the sine.

The TANGENT of an arc, is a line which touches the circle in one extremity of that arc, and is continued from thence till it meets a line drawn from or through the centre and through the other extremity of the arc.

The SECANT of an arc, is the radius drawn through one of the extremities of that arc and prolonged till it meets the tangent drawn from the other extremity.

The VERSED SINE of an arc, is that part of the diameter of the circle which lies between the beginning of the arc and the foot of the sine.

The COTANGENT, COSECANT, and COVERSED SINE of an arc, are the tangent, secant, and versed sine, of the complement of such arc.

3. Since

3. Since arcs are proper and adequate measures of plane angles, (the ratio of any two plane angles being constantly equal to the ratio of the two arcs of any circle whose centre is the angular point, and which are intercepted by the lines whose inclinations form the angle), it is usual, and it is perfectly safe, to apply the above names without circumlocution as though they referred to the angles themselves; thus, when we speak of the sine, tangent, or secant, of an angle, we mean the sine, tangent, or secant, of the arc which measures that angle; the radius of the circle employed being known.

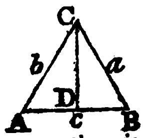
4. It has been shown in the 1st vol. (pa. 382), that the tangent is a fourth proportional to the cosine, sine, and radius; the secant, a third proportional to the cosine and radius; the cotangent, a fourth proportional to the sine, cosine, and radius; and the cosecant a third proportional to the sine and radius. Hence, making use of the obvious abbreviations, and converting the analogies into equations, we have

$$\text{tan.} = \frac{\text{rad.} \times \text{sine.}}{\text{cos.}},\ \text{cot.} = \frac{\text{rad} \times \text{cos.}}{\text{sine}},\ \text{sec.} = \frac{\text{rad}^2.}{\text{cos.}},\text{cos.} = \frac{\text{rad}^2.}{\text{sine}}.$$

Or, assuming unity for the rad. of the circle, these will become

$$\text{tan.} = \frac{\text{sin.}}{\text{cos.}} \ldots \text{cot.} = \frac{\text{cos.}}{\text{sin.}} \ldots \text{sec.} = \frac{1}{\text{cos}} \ldots \text{cos.} = \frac{1}{\text{sin.}}.$$

These preliminaries being borne in mind, the student may pursue his investigations.



5. Let ABC be any plane triangle, of which the side BC opposite the angle A is denoted by the small letter a, the side AC opposite the angle B by the small letter b, and the side AB opposite the angle C by the small letter c, and CD perpendicular to AB: then is, $c = a \,.\, \cos \text{B} + b \,.\, \cos \text{A}$.

For, since AC $= b$, AD is the cosine of A to that radius; consequently, supposing radius to be unity, we have AD $= b$. cos. A. In like manner it is BD $= a$. cos. B. Therefore, AD + BD = AB $= c = a$. cos. B + b . cos. A. By pursuing similar reasoning with respect to the other two sides of the triangle, exactly analogous results will be obtained. Placed together, they will be as below:

$$\left.\begin{aligned} a &= b\,.\,\text{cos. C} + c\,.\,\text{cos. B} \\ b &= a\,.\,\text{cos. C} + c\,.\,\text{cos. A} \\ c &= a\,.\,\text{cos. B} + b\,.\,\text{cos. A} \end{aligned}\right\} \quad \text{(I.)}$$

6. Now, if from these equations it were required to find expressions for the angles of a plane triangle, when the sides are

are given; we have only to multiply the first of these equations by a, the second by b, the third by c, and to subtract each of the equations thus obtained from the sum of the other two. For thus we shall have

$$\left.\begin{array}{ll} b^2+c^2-a^2=2bc\,.\cos.\text{A}, \text{ whence } \cos.\text{A} = & \dfrac{b^2+c^2-a^2}{2bc} \\ a^2+c^2-b^2=2ac\,.\cos.\text{B}, \quad \ldots \cos.\text{B}, = & \dfrac{a^2+c^2-b^2}{2ac} \\ a^2+b^2-c^2=2ab\,.\cos.\text{C}, \quad \ldots \cos.\text{C}, = & \dfrac{a^2+b^2-c^2}{2ab} \end{array}\right\}\text{(II.)}$$

7. More convenient expressions than these will be deduced hereafter: but even these will often be found very convenient, when the sides of triangles are expressed in integers, and tables of sines and tangents, as well as a table of squares, (like that in our first vol.) are at hand.)

Suppose, for example, the sides of the triangle are $a=320$, $b=562$, $c=800$, being the numbers given in prop. 4, pa. 161, of the Introduction to the Mathematical Tables: then we have

$b^2+c^2-a^2 = 853444$ . . . . . . log. = 5·9311751
$2bc$ . . = 899200 . . . . . . log. = 5·9538080
The remainder being log. cos. A, or of 18°20′ = 9·9773671
Again, $a^2+c^2-b^2 = 426556$ . . . log. = 5·6299760
$2ac$ . . . = 512000 . . . log. = 5·7092700
The remainder being log. cos. B, or of 33°35′ = 9·9207060

Then 180° − (18°20′ + 33°35′) = 128°5′ = C: where all the three triangles are determined in 7 lines.

8. If it were wished to get expressions for the sines, instead of the cosines, of the angles; it would merely be necessary to introduce into the preceding equations (marked II), instead of cos. A, cos. B, &c, their equivalents cos. A $=\sqrt{(1-\sin^2.\text{A})}$, cos B $=\sqrt{(1-\sin^2.\text{B})}$, &c. For then, after a little reduction, there would result,

$$\left.\begin{array}{l} \sin.\text{A} = \frac{1}{2bc}\sqrt{2a^2b^2+2a^2c^2+2b^2c^2-(a^4+b^4+c^4)} \\ \sin.\text{B} = \frac{1}{2ac}\sqrt{2a^2b^2+2a^2c^2+2b^2c^2-(a^4+b^4+c^4)} \\ \sin.\text{C} = \frac{1}{2ab}\sqrt{2a^2b^2+2a^2c^2+2b^2c^2-(a^4+b^4+c^4)} \end{array}\right\}$$

Or, resolving the expression under the radical into its four constituent factors, substituting s for $a+b+c$, and reducing, the equations will become

Sin.

$$\left.\begin{aligned}\sin.\,A &= \frac{2}{bc}\sqrt{\tfrac{1}{2}s(\tfrac{1}{2}s-a)(\tfrac{1}{2}s-b)(\tfrac{1}{2}s-c)}\\ \sin.\,B &= \frac{2}{ac}\sqrt{\tfrac{1}{2}s(\tfrac{1}{2}s-a)(\tfrac{1}{2}s-b)(\tfrac{1}{2}s-c)}\\ \sin.\,C &= \frac{2}{ab}\sqrt{\tfrac{1}{2}s(\tfrac{1}{2}s-a)(\tfrac{1}{2}s-b)(\tfrac{1}{2}s-c)}\end{aligned}\right\}\ (\text{III.})$$

These equations are moderately well suited for computation in their latter form; they are also perfectly symmetrical: and as indeed the quantities under the radical are identical, and are constituted of known terms, they may be represented by the same character; suppose K: then shall we have

$$\sin.\,A = \frac{2K}{bc}\ \ldots\ \sin.\,B = \frac{2K}{ac}\ \ldots\ \sin.\,C = \frac{2K}{ab}\ \ldots\ (iii.)$$

Hence we may immediately deduce a very important theorem: for, the first of these equations, divided by the second, gives $\frac{\sin.\,A}{\sin.\,B} = \frac{a}{b}$, and the first divided by the third gives $\frac{\sin.\,A}{\sin.\,C} = \frac{a}{c}$: whence we have

$$\sin.\,A : \sin.\,B : \sin.\,C \propto a : b : c\ \ldots\ (\text{IV.})$$

Or, in words, *the sides of plane triangles are proportional to the sines of their opposite angles.* (See th. 1. Trig. vol. i).

9. Before the remainder of the theorems, necessary in the solution of plane triangles, are investigated, the fundamental proposition in the theory of sines, &c, must be deduced, and the method explained by which Tables of these quantities, confined within the limits of the quadrant, are made to extend to the whole circle, or to any number of quadrants whatever. In order to this, expressions must be first obtained for the sines, cosines, &c, of the sums and differences of any two arcs or angles. Now, it has been found (I) that $a = b\,.\cos.\,C + c\,.\cos.\,B$. And the equations (IV) give $b = a\,.\frac{\sin.\,B}{\sin.\,A}\ \ldots\ c = a\,.\frac{\sin.\,C}{\sin.\,A}$. Substituting these values of b and c for them in the preceding equation, and multiplying the whole by $\frac{\sin.\,A}{a}$, it will become

$$\sin.\,A = \sin.\,B\,.\cos.\,C + \sin.\,C\,.\cos.\,B.$$

But, in every plane triangle, the sum of the three angles is two right angles; therefore B and C are equal to the supplement of A: and, consequently, since an angle and its supplement have the same sine (cor. 1, pa. 378, vol. i), we have $\sin.\,(B + C) = \sin.\,B\,.\cos.\,C + \sin\,C\,.\cos.\,B.$

10. If,

10 If, in the last equation, c become subtractive, then would sin. c manifestly become subtractive also, while the cosine of c would not change its sign, since it would still continue to be estimated on the same radius in the same direction. Hence the preceding equation would become

$$\sin.\ (B - C) = \sin\ B\ .\ \cos.\ C - \sin.\ C\ .\ \cos.\ B.$$

11. Let c′ be the complement of c, and $\frac{1}{4}O$ be the quarter of the circumference: then will $c' = \frac{1}{4}O - c$, $\sin. c' = \cos. c$, and $\cos. c' = \sin. c$. But (art. 10), $\sin. (B - c') = \sin. B$, $\cos. c' - \sin. c' \cos B$. Therefore, substituting for sin. c′, cos c′, their values, there will result $\sin (B - c') = \sin. B\ .\ \sin. c - \cos. B\ .\ \cos. c$. But because $c' = \frac{1}{4}O - c$, we have $\sin. (B - c) = \sin. (B + c - \frac{1}{4}O) = \sin. [(B + c) - \frac{1}{4}O] = -\sin [\frac{1}{4}O - (B + c)] = -\cos. (B + c)$. Substituting this value of sin. (B − c′) in the equation above, it becomes $\cos. (B + c) = \cos. B\ .\ \cos. c - \sin. B\ .\ \sin c$.

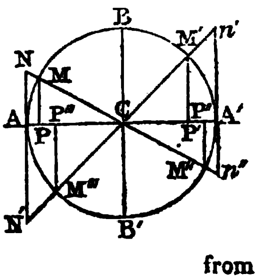
12. In this latter equation, if c be made subtractive, sin. c will become − sin c, while cos. c will not change: consequently the equation will be transformed to the following, viz. $\cos (B - c) = \cos. B\ .\ \cos. c + \sin. B\ .\ \sin. c$.

If, instead of the angles B and C, the angles had been A and B; or, if A and B represented the *arcs* which measure those angles, the results would evidently be similar: they may therefore be expressed generally by the two following equations, for the sines and cosines of the sums or differences of any two arcs or angles:

$$\left.\begin{array}{l}\sin.\ (A \pm B) = \sin.\ A\ .\ \cos.\ B \pm \sin.\ B\ .\ \cos.\ A.\\ \cos.\ (A \pm B) = \cos.\ A\ .\ \cos.\ B \mp \sin\ A\ .\ \sin.\ B.\end{array}\right\}\ (V.)$$

13. We are now in a state to trace completely the mutations of the sines, cosines, &c, as they relate to arcs in the various parts of a circle; and thence to perceive that tables which apparently are included within a quadrant, are, in fact, applicable to the whole circle.

Imagine that the radius MC of the circle, in the marginal figure, coinciding at first with AC, turns about the point C (in the same manner as a rod would turn on a pivot), and thus forming successively with AC all possible angles: the point M at its extremity passing over all the points of the circumference ABA′B′A, or describing the whole circle. Tracing this motion attentively, it will appear, that at the point A, where the arc is nothing, the sine is nothing also, while the cosine does not differ



from

from the radius. As the radius MC recedes from AC, the sine PM keeps increasing, and the cosine CP decreasing, till the describing point M has passed over a quadrant, and arrived at B: in that case PM becomes equal to CB the radius, and the cosine CP vanishes. The point M continuing its motion beyond B, the sine P′M′ will diminish, while the cosine CP′, which now falls on the *contrary* side of the centre C will increase. In the figure, P′M′ and CP′ are respectively the sine and cosine of the arc A′M′, or the sine and cosine of ABM′, which is the supplement of A′M′ to ½○, half the circumference: whence it follows that an obtuse angle (measured by an arc greater than a quadrant) has *the same sine and cosine as its supplement;* the cosine however, being reckoned subtractive or negative, because it is situated contrariwise with regard to the centre C.

When the describing point M has passed over ½○, or half the circumference, and has arrived at A′, the sine P′M′ vanishes, or becomes *nothing*, as at the point A, and the cosine is again equal to the radius of the circle. Here the angle ACM has attained its maximum limit; but the radius CM may still be supposed to continue its motion, and pass *below* the diameter AA′. The sine, which will then be P″M″, will consequently fall below the diameter, and will augment as M moves along the third quadrant, while on the contrary CP″, the cosine, will diminish. In this quadrant too, both sine and cosine must be considered as negative; the former being on a contrary side of the diameter, the latter a contrary side of the centre, to what each was respectively in the first quadrant. At the point B′, where the arc is three-fourths of the circumference, or ¾○, the sine P″M″ becomes equal to the radius CB, and the cosine CP″ vanishes. Finally, in the fourth quadrant, from B′ to A, the sine P‴M‴, always *below* AA′, diminishes in its progress, while the cosine CP‴, which is then found on the same side of the centre as it was in the first quadrant, augments till it becomes equal to the radius CA. Hence, the sine in this quadrant is to be considered as negative or subtractive, the cosine as positive. If the motion of M were continued through the circumference again, the circumstances would be exactly the same in the fifth quadrant as in the first, in the sixth as in the second, in the seventh as in the third, in the eighth as in the fourth: and the like would be the case in any subsequent revolutions.

14. If the mutations of the *tangent* be traced in like manner, it will be seen that its magnitude passes from nothing to infinity in the first quadrant; becomes negative, and decreases from infinity to nothing in the second; becomes positive again, and increases from nothing to infinity in the third

third quadrant; and lastly, becomes negative again, and decreases from infinity to nothing, in the fourth quadrant.

15. These conclusions admit of a ready confirmation; and others may be deduced, by means of the analytical expressions in arts. 4 and 12. Thus, if A be supposed equal to $\frac{1}{4}\bigcirc$, in equa. v, it will become

$$\cos.\ (\tfrac{1}{4}\bigcirc \pm \text{B}) = \cos.\ \tfrac{1}{4}\bigcirc \,.\, \cos.\ \text{B} \mp \sin.\ \tfrac{1}{4}\bigcirc \,.\, \sin.\ \text{B},$$
$$\sin.\ (\tfrac{1}{4}\bigcirc \pm \text{B}) = \sin\ \tfrac{1}{4}\bigcirc \,.\, \cos.\ \text{B} \pm \sin.\ \text{B} \,.\, \cos.\ \tfrac{1}{4}\bigcirc.$$

But $\sin.\ \frac{1}{4}\bigcirc = \text{rad.} = 1$; and $\cos.\ \frac{1}{4}\bigcirc = 0$:
so that the above equations will become

$$\cos.\ (\tfrac{1}{4}\bigcirc \pm \text{B}) = \mp \sin.\ \text{B}.$$
$$\sin.\ (\tfrac{1}{4}\bigcirc \pm \text{B}) = \cos.\ \text{B}.$$

From which it is obvious, that if the sine and cosine of an arc, less than a quadrant, be regarded as positive, the cosine of an arc greater than $\frac{1}{4}\bigcirc$ and less than $\frac{1}{2}\bigcirc$ will be negative, but its sine positive. If B also be made $= \frac{1}{4}\bigcirc$; then shall we have $\cos,\ \frac{1}{2}\bigcirc = -1$; $\sin.\ \frac{1}{2}\bigcirc = 0$.

Suppose next, that in the equa. v, $\text{A} = \frac{1}{2}\bigcirc$; then shall we obtain

$$\cos.\ (\tfrac{1}{2}\bigcirc \pm \text{B}) = -\cos.\ \text{B}.$$
$$\sin.\ (\tfrac{1}{2}\bigcirc \pm \text{B}) = \mp \sin.\ \text{B};$$

which indicates, that every arc comprised between $\frac{1}{2}\bigcirc$ and $\frac{3}{4}\bigcirc$, or that terminates in the third quadrant, will have its sine and its cosine both negative. In this case too, when $\text{B} = \frac{1}{4}\bigcirc$, or the arc terminates at the *end* of the third quadrant, we shall have $\cos.\ \frac{3}{4}\bigcirc = 0$, $\sin.\ \frac{3}{4}\bigcirc = -1$.

Lastly, the case remains to be considered in which $\text{A} = \frac{3}{4}\bigcirc$, or in which the arc terminates in the fourth quadrant. Here the primitive equations (V) give

$$\cos.\ (\tfrac{3}{4}\bigcirc \pm \text{B}) = \pm \sin.\ \text{B}$$
$$\sin.\ (\tfrac{3}{4}\bigcirc \pm \text{B}) = -\cos.\ \text{B};$$

so that in all arcs between $\frac{3}{4}\bigcirc$ and $\bigcirc$, the cosines are positive and the sines negative.

16. The changes of the tangents, with regard to positive and negative, may be traced by the application of the preceding results to the algebraic expression for the tangent; viz, $\tan. = \frac{\sin.}{\cos}$. For it is hence manifest, that when the sine and cosine are either both positive or both negative, the tangent will be positive; which will be the case in the first and third quadrants. But when the sine and cosine have different signs, the tangents will be negative, as in the second and fourth quadrants. The algebraic expression for the cotangent, viz, $\cot. = \frac{\cos.}{\sin.}$, will produce exactly the same results.

The

The expressions for the secants and cosecants, viz, sec. $= \frac{1}{\text{cos.}}$, cosec. $= \frac{1}{\text{sin.}}$ show, that the signs of the secants are the same as those of the cosines; and those of the cosecants the same as those of the sines.

The *magnitude* of the tangent at the end of the first and third quadrants will be infinite; because in those places the sine is equal to radius, the cosine equal to zero, and therefore $\frac{\text{sin.}}{\text{cos.}} = \infty$ (infinity). Of these, however, the former will be reckoned positive, the latter negative.

17. The magnitudes of the cotangents, secants, and cosecants, may be traced in like manner; and the results of the 13th, 14th, and 15th articles, recapitulated and tabulated as below.

	0°	90°	180°	270°	360°	
Sin.	0	R	0	−R	0	(VI.)
Tan.	0	∞	0	−∞	0	
Sec.	R	∞	−R	−∞	R	
Cos.	R	0	−R	0	R	
Cot.	∞	0	−∞	0	∞	
Cosec.	∞	R	−∞	−R	∞	

The changes of signs are these:

quadrants				sin.	cos.	tan.	cot.	sec.	cosec.	
1st.	5th.	9th.	13th.	+	+	+	+	+	+	(VII.)
2d.	6th.	10th.	14th.	+	−	−	−	−	+	
3d.	7th.	11th.	15th.	−	−	+	+	−	−	
4th.	8th.	12th.	16th.	−	+	−	−	+	−	

We have been thus particular in tracing the mutations, both with regard to value and algebraic signs, of the principal trigonometrical quantities, because a knowledge of them is absolutely necessary in the application of trigonometry to the solution of equations, and to various astronomical and physical problems.

18. We may now proceed to the investigation of other expressions relating to the sums, differences, multiples, &c, of arcs; and in order that these expressions may have the more generality, give to the radius any value R instead of confining it to unity. This indeed may always be done in an expression, however complex, by merely rendering all the terms homogeneous; that is, *by multiplying each term by such a power of* R *as shall make it of the same dimension, as the term in the equation which has the highest dimension.* Thus, the expression for a triple arc

 sin.

$\sin. 3A = 3 \sin. A - 4 \sin^3 . A$ (radius $= 1$)
becomes when radius is assumed $= R$,
$R^2 \sin. 3A = R^2\, 3 \sin. A - 4 \sin^3 . A$

or $\sin. 3A = \dfrac{3R^2 \sin A - 4 \sin^3 . A}{R^2}$.

Hence then, if consistently with this precept, R be placed for a denominator of the second member of each equation v (art. 12), and if A be supposed equal to B, we shall have

$$\sin. (A + A) = \frac{\sin. A . \cos. A + \sin. A \cos. A}{R}.$$

That is, $\sin. 2A = \dfrac{2 \sin. A . \cos. A}{R}$.

And, in like manner, by supposing B to become successively equal to 2A, 3A, 4A, &c, there will arise

$$\left.\begin{aligned} \sin. 3A &= \frac{\sin. A . \cos. 2A + \cos. A . \sin. 2A}{R} \\ \sin. 4A &= \frac{\sin. A . \cos. 3A + \cos. A . \sin. 3A}{R} \\ \sin. 5A &= \frac{\sin. A . \cos. 4A + \cos. A . \sin. 4A}{R} \end{aligned}\right\} \text{(VIII.)}$$

And, by similar processes, the second of the equations just referred to, namely, that for $\cos. (A + B)$, will give successively,

$$\left.\begin{aligned} \cos. 2A &= \frac{\cos^2 . A - \sin^2 . A}{R} \\ \cos. 3A &= \frac{\cos. A . \cos 2A - \sin. A . \sin. 2A}{R} \\ \cos. 4A &= \frac{\cos. A . \cos. 3A - \sin. A . \sin. 3A}{R} \\ \cos. 5A &= \frac{\cos. A . \cos. 4A - \sin. A . \sin. 4A}{R} \end{aligned}\right\} \text{(IX.)}$$

19. If, in the expressions for the successive multiples of the sines, the values of the several cosines in terms of the sines were substituted for them; and a like process were adopted with regard to the multiples of the cosines, other expressions would be obtained, in which the multiple sines would be expressed in terms of the radius and sine, and the multiple cosines in terms of the radius and cosines.

$$\text{As}\quad\left.\begin{aligned} \sin. A &= s \\ \sin. 2A &= 2s\sqrt{R^2 - s^2} \\ \sin. 3A &= 3s - 4s^3 \\ \sin. 4A &= (4s - 8s^3)\sqrt{R^2 - s^2} \\ \sin. 5A &= 5s - 20s^3 + 16s^5 \\ \sin. 6A &= (6s - 32s^3 + 32s^5)\sqrt{R^2 - s^2} \\ &\text{\&c. \&c.} \end{aligned}\right\} \text{(X.)}$$

cos.

$$\left.\begin{array}{lcl}
\text{Cos. A} & = & c \\
\text{cos. 2A} & = & 2c^2 - 1 \\
\text{cos. 3A} & = & 4c^3 - 3c \\
\text{cos. 4A} & = & 8c^4 - 8c^2 + 1 \\
\text{cos. 5A} & = & 16c^5 - 20c^3 + 5c \\
\text{cos. 6A} & = & 32c^6 - 48c^4 + 18c^2 - 1 \\
 & & \text{\&c. \&c.*}
\end{array}\right\} \text{(XI.)}$$

Other very convenient expressions for multiple arcs may be obtained thus:

Add together the expanded expressions for sin. $(B + A)$, sin. $(B - A)$, that is,

add - - sin. $(B + A) =$ sin. B . cos. A + cos. B . sin. A,
to - - sin. $(B - A) =$ sin. B . cos. A − cos. B . sin. A;
there results sin. $(B + A) +$ sin. $(B - A) = 2$ cos A . sin B:
whence, - sin. $(B + A) = 2$ cos. A . sin. B − sin $(B - A)$.

Thus again, by adding together the expressions for cos $(B+A)$ and cos. $(B - A)$, we have

cos. $(B + A) +$ cos. $(B - A) = 2$ cos. A . cos B;
whence, cos. $(B + A) = 2$ cos. A . cos. B − cos. $(B - A)$.

Substituting in these expressions for the sine and cosine of $B + A$, the successive values A, 2A, 3A, &c, instead of B; the following series will be produced.

$$\left.\begin{array}{l}
\text{sin. 2A} = 2 \text{ cos. A . sin. A.} \\
\text{sin. 3A} = 2 \text{ cos. A . sin. 2A} - \text{sin. A.} \\
\text{sin. 4A} = 2 \text{ cos. A . sin. 3A} - \text{sin. 2A.} \\
\text{sin. } n\text{A} = 2 \text{ cos. A . sin. } (n-1)\text{ A} - \text{sin. } (n-2)\text{ A.}
\end{array}\right\} (x.)$$

$$\left.\begin{array}{l}
\text{cos. 2A} = 2 \text{ cos. A . cos. A} - \text{cos. } 0\ (=1). \\
\text{cos. 3A} = 2 \text{ cos. A . cos. 2A} - \text{cos. A.} \\
\text{cos. 4A} = 2 \text{ cos. A . cos. 3A} - \text{cos. 2A.} \\
\text{cos. } n\text{A} = 2 \text{ cos. A . cos. } (n-1)\text{ A} - \text{cos. } (n-2)\text{A.}
\end{array}\right\} (xi.)$$

Several other expressions for the sines and cosines of multiple arcs, might readily be found: but the above are the most useful and commodious.

20. From the equation sin. $2A = \frac{2 \sin A \,.\, \cos A}{R}$, it will be easy, when the sine of an arc is known, to find that of its half. For, substituting for cos A its value $\sqrt{(R^2 - \sin^2 A)}$, there will arise sin $2A = \frac{2 \sin A \sqrt{(R^2 - \sin^2 A)}}{R}$. This squared

gives $R^2 \sin^2 2A = 4R^2 \sin^2 A - 4 \sin^4 A$.

Here taking sin A for the unknown quantity, we have a quad-

* Here we have omitted the powers of R that were necessary to render all the terms homologous, merely that the expressions might be brought in upon the page; but they may easily be supplied, when needed, by the rule in art. 18.

ratic

ratic equation, which solved after the usual manner, gives

$$\sin A = \pm \sqrt{\tfrac{1}{2}R^2 \pm \tfrac{1}{2}R\sqrt{R^2 - \sin^2 2A}}.$$

If we make $2A = A'$, then will $A = \frac{1}{2}A'$, and consequently, the last equation becomes

$$\left.\begin{array}{l} \sin \tfrac{1}{2}A' = \pm \sqrt{\tfrac{1}{2}R^2 \pm \tfrac{1}{2}R\sqrt{R^2 - \sin^2 A'}} \\ \text{or } \sin \tfrac{1}{2}A' = \pm \tfrac{1}{2}\sqrt{2R^2 \pm 2R \cos A'}: \end{array}\right\} \text{(XII.)}$$

by putting $\cos A'$ for its value $\sqrt{R^2 - \sin^2 A'}$, multiplying the quantities under the radical by 4, and dividing the whole second number by 2. Both these expressions for the sine of half an arc or angle will be of use to us as we proceed.

21. If the values of $\sin(A + B)$ and sine $(A - B)$, given by equa. v, be added together, there will result

$$\sin(A + B) + \sin(A - B) = \frac{2 \sin A \cdot \cos B}{R}; \text{ whence,}$$

$$\sin A \cdot \cos B = \tfrac{1}{2}R \sin(A + B) + \tfrac{1}{2}R \sin(A - B) \,.\, \text{(XIII.)}$$

Also, taking $\sin(A - B)$ from $\sin(A + B)$ gives

$$\sin(A + B) - \sin(A - B) = \frac{2 \sin B \cdot \cos A}{R}; \text{ whence,}$$

$$\sin B \cdot \cos A = \tfrac{1}{2}R \sin(A + B) - \tfrac{1}{2}R \cdot \sin(A - B) \,..\, \text{(XIV.)}$$

When $A = B$, both equa. XIII and XIV, become

$$\cos A \cdot \sin A = \tfrac{1}{2}R \sin 2A \,..\, \text{(XV.)}$$

22. In like manner, by adding together the primitive expressions for $\cos(A + B)$, $\cos(A - B)$, there will arise

$$\cos(A + B) + \cos(A - B) = \frac{2 \cos A \cdot \cos B}{R}; \text{ whence,}$$

$$\cos A \cdot \cos B = \tfrac{1}{2}R \cdot \cos(A + B) + \tfrac{1}{2}R \cdot \cos(A - B) \text{ (XVI.)}$$

And here, when $A = B$, recollecting that when the arc is nothing the cosine is equal to radius, we shall have

$$\cos^2 A = \tfrac{1}{2}R \cdot \cos 2A + \tfrac{1}{2}R^2 \ldots \text{(XVII.)}$$

23. Deducting $\cos(A + B)$ from $\cos(A - B)$, there will remain

$$\cos(A - B) - \cos(A + B) = \frac{2 \sin A \cdot \sin B}{R}; \text{ whence,}$$

$$\sin A \cdot \sin B = \tfrac{1}{2}R \cdot \cos(A - B) - \tfrac{1}{2}R \cdot \cos(A + B) \text{(XVIII.)}$$

When $A = B$, this formula becomes

$$\sin^2 A = \tfrac{1}{2}R^2 - \tfrac{1}{2}R \cdot \cos. 2A \ldots \text{(XIX.)}$$

24 Multiplying together the expressions for $\sin(A + B)$ and $\sin(A - B)$, equa. v, and reducing, there results

$$\sin(A + B) \cdot \sin(A - B) = \sin^2 A - \sin^2 B.$$

And, in like manner, multiplying together the values of $\cos(A + B)$ and $\cos(A - B)$, there is produced

$$\cos(A + B) \cdot \cos(A - B) = \cos^2 A - \cos^2 B.$$

Here, since $\sin^2 A - \sin^2 B$, is equal to $(\sin A + \sin B) \times (\sin A - \sin B)$, that is, to the rectangle of the sum and difference

ference of the sines; it follows, that the first of these equations converted into an analogy, becomes

$\sin(A - B) : \sin A - \sin B :: \sin A + \sin B : \sin(A + B)$ (XX.)

That is to say, *the sine of the difference of any two arcs or angles, is to the difference of their sines, as the sum of those sines is to the sine of their sum.*

If A and B be to each other as $n + 1$ to n, then the preceding proportion will be converted into $\sin A : \sin(n + 1)A - \sin nA :: \sin(n + 1)A + \sin nA : \sin(2n + 1)A$. . . . (XXI.)

These two proportions are highly useful in computing a table of sines; as will be shown in the practical examples at the end of this chapter.

25. Let us suppose $A + B = A'$, and $A - B = B'$; then the half sum and the half difference of these equations will give respectively $A = \frac{1}{2}(A' + B')$, and $B = \frac{1}{2}(A' - B')$. Putting these values of A and B, in the expressions of $\sin A . \cos B$, $\sin B . \cos A$, $\cos A . \cos B$, $\sin A . \sin B$, obtained in arts. 21, 22, 23, there would arise the following formulæ:

$$\sin\tfrac{1}{2}(A' + B') . \cos\tfrac{1}{2}(A' - B') = \tfrac{1}{2}R(\sin A' + \sin B'),$$
$$\sin\tfrac{1}{2}(A' - B') . \cos\tfrac{1}{2}(A' + B') = \tfrac{1}{2}R(\sin A' - \sin B'),$$
$$\cos\tfrac{1}{2}(A' + B') . \cos\tfrac{1}{2}(A' - B') = \tfrac{1}{2}R(\cos A' + \cos B'),$$
$$\sin\tfrac{1}{2}(A' + B') . \sin\tfrac{1}{2}(A' - B') = \tfrac{1}{2}R(\cos B' - \cos A').$$

Dividing the second of these formulæ by the first, there will be had

$$\frac{\sin\frac{1}{2}(A' - B') . \cos\frac{1}{2}(A' + B')}{\sin\frac{1}{2}(A' + B') . \cos\frac{1}{2}(A' - B')} = \frac{\sin\frac{1}{2}(A' - B')}{\cos\frac{1}{2}(A' - B')} . \frac{\cos\frac{1}{2}(A' + B')}{\sin\frac{1}{2}(A' + B')} = \frac{\sin A' - \sin B'}{\sin A' + \sin B'}$$

But since $\frac{\sin}{\cos} = \frac{\tan}{R}$, and $\frac{\cos}{\sin} = \frac{R}{\tan}$, it follows that the two factors of the first member of this equation, are $\frac{\tan\frac{1}{2}(A' - B')}{R}$, and $\frac{R}{\tan\frac{1}{2}(A' + B')}$, respectively; so that the equation manifestly becomes $\frac{\tan\frac{1}{2}(A' - B')}{\tan\frac{1}{2}(A' + B')} = \frac{\sin A' - \sin B'}{\sin A' + \sin B'}$ . . . (XXII.)

This equation is readily converted into a very useful proportion, viz, *The sum of the sines of two arcs or angles, is to their difference, as the tangent of half the sum of those arcs or angles, is to the tangent of half their difference.*

26. Operating with the third and fourth formulæ of the preceding article, as we have already done with the first and second, we shall obtain

$$\frac{\tan\frac{1}{2}(A' + B') . \tan\frac{1}{2}(A' - B')}{R^2} = \frac{\cos B' - \cos A'}{\cos A' + \cos B'}$$

In like manner, we have by division,

$$\frac{\sin A' + \sin B'}{\cos A' + \cos B'} = \frac{\sin\frac{1}{2}(A' + B')}{\cos\frac{1}{2}(A' + B')} = \tan\tfrac{1}{2}(A' + B'); \quad \frac{\sin A' + \sin B'}{\cos B' - \cos A'} = \cot\tfrac{1}{2}(A' - B');$$

$$\frac{\sin A' - \sin B'}{\cos A' + \cos B'} = \tan\tfrac{1}{2}(A' - B') \ . \ . \ . \ \frac{\sin A' - \sin B'}{\cos B' - \cos A'} = \cot\tfrac{1}{2}(A' + B'),$$

cos

$$\frac{\cos A' + \cos B'}{\cos B' - \cos A'} = \frac{\cot \frac{1}{2}(A' + B')}{\tan \frac{1}{2}(A' - B')}.$$

Making $B = 0$, in one or other of these expressions, there results,

$$\left.\begin{aligned} \frac{\sin A'}{1 + \cos A'} &= \tan \tfrac{1}{2}A' = \frac{1}{\cot \frac{1}{2}A'}. \\ \frac{\sin A'}{1 - \cos A'} &= \cot \tfrac{1}{2}A' = \frac{1}{\tan \frac{1}{2}A'}. \\ \frac{1 + \cos A'}{1 - \cos A'} &= \frac{\cot \frac{1}{2}A'}{\tan \frac{1}{2}A'} = \cot^2 \tfrac{1}{2}A' = \frac{1}{\tan^2 \frac{1}{2}A'}. \end{aligned}\right\} (xxii.)$$

These theorems will find their application in some of the investigations of spherical trigonometry.

27. Once more, dividing the expression for $\sin(A \pm B)$ by that for $\cos(A \pm B)$, there results

$$\frac{\sin(A \pm B)}{\cos(A \pm B)} = \frac{\sin A \cdot \cos B \pm \sin B \cdot \cos A}{\cos A \cdot \cos B \mp \sin A \cdot \sin B}:$$

then dividing both numerator and denominator of the second fraction, by $\cos A \cdot \cos B$, and recollecting that $\frac{\sin}{\cos} = \frac{\tan}{R}$, we shall thus obtain

$$\frac{\tan(A \pm B)}{R} = \frac{R(\tan A \pm \tan B)}{R^2 \mp \tan A \cdot \tan B};$$

or, lastly, $\tan(A \pm B) = \dfrac{R^2(\tan A \pm \tan B)}{R^2 \mp \tan A \cdot \tan B}$. . . . (XXIII.

Also, since $\cot = \dfrac{R^2}{\tan}$, we shall have

$$\cot(A \pm B) = \frac{R^2}{\tan(A \pm B)} = \frac{R^2 \mp \tan A \cdot \tan B}{\tan A \pm \tan B};$$

which, after a little reduction, becomes

$$\cot(A \pm B) = \frac{\cot A \cdot \cot B \mp R^2}{\cot B \pm \cot A} \quad \ldots . \text{(XXIV.)}$$

28. We might now proceed to deduce expressions for the tangents, cotangents, secants, &c, of multiple arcs, as well as some of the usual formulæ of verification in the construction of tables, such as

$$\sin(54° + A) + \sin(54° - A) - \sin(18° + A) - \sin(18° - A) = \sin(90° - A);$$
$$\sin A + \sin(36° - A) + \sin(72° + A) = \sin(36° + A) + \sin(72° - A).$$

&c. &c.

But, as these enquiries would extend this chapter to too great a length, we shall pass them by; and merely investigate a few properties where *more* than two arcs or angles are concerned, and which may be of use in some subsequent part of this volume.

29. Let

29. Let A, B, C, be in any three arcs or angles, and suppose radius to be unity; then

$$\sin(B+C) = \frac{\sin A \cdot \sin C + \sin B \cdot \sin(A+B+C)}{\sin(A+B)}.$$

For, by equa. V, $\sin(A+B+C) = \sin A \cdot \cos(B+C) + \cos A \cdot \sin(B+C)$, which, (putting $\cos B \cdot \cos C - \sin B \cdot \sin C$ for $\cos(B+C)$), is $= \sin A \cdot \cos B \cdot \cos C - \sin A \cdot \sin B \cdot \sin C + \cos A \cdot \sin(B+C)$; and, multiplying by $\sin B$, and adding $\sin A \cdot \sin C$, there results $\sin A \cdot \sin C + \sin B \cdot \sin(A+B+C) = \sin A \cdot \cos B \cdot \cos C \cdot \sin B + \sin A \cdot \sin C \cdot \cos^2 B + \cos A \cdot \sin B \cdot \sin(B+C) = \sin A \cdot \cos B \cdot (\sin B \cdot \cos C + \cos B \cdot \sin C) + \cos A \cdot \sin B \cdot \sin(B+C) = (\sin A \cdot \cos B + \cos A \cdot \sin B) \times \sin(B+C) = \sin(A+B) \cdot \sin(B+C)$. Consequently, by dividing by $\sin(A+B)$, we obtain the expression above given.

In a similar manner it may be shown, that

$$\sin(B-C) = \frac{\sin A \cdot \sin C - \sin B \cdot \sin(A-B+C)}{\sin(A-B)}.$$

30. If A, B, C, D, represent four arcs or angles, then writing $C+D$ for C in the preceding investigation, there will result,

$$\sin(B+C+D) = \frac{\sin A . \sin(C+D) + \sin B . \sin(A+B+C+D)}{\sin(A+B)}.$$

A like process for five arcs or angles will give

$$\sin(B+C+D+E) = \frac{\sin A \sin(C+D+E) + \sin B \sin(A+B+C+D+E)}{\sin(A+B)}.$$

And for any number, A, B, C, &c, to L,

$$\sin(B+C+\ldots L) = \frac{\sin A . \sin(C+D+\ldots L) + \sin B . \sin(A+B+C+\ldots L)}{\sin(A+B)}.$$

31. Taking again the three A, B, C, we have

$$\sin(B-C) = \sin B \cdot \cos C - \sin C \cdot \cos B,$$
$$\sin(C-A) = \sin C \cdot \cos A - \sin A \cdot \cos C,$$
$$\sin(A-B) = \sin A \cdot \cos B - \sin B \cdot \cos A.$$

Multiplying the first of these equations by $\sin A$, the second by $\sin B$, the third by $\sin C$; then adding together the equations thus transformed, and reducing; there will result,

$$\sin A \cdot \sin(B-C) + \sin B \cdot \sin(C-A) + \sin C \cdot \sin(A-B) = 0,$$
$$\cos A \cdot \sin(B-C) + \cos B \cdot \sin(C-A) + \cos C \cdot \sin(A-B) = 0.$$

These two equations obtaining for any three angles whatever, apply evidently to the three angles of any triangle.

32. Let the series of arcs or angles A, B, C, D . . . . L, be contemplated, then we have (art. 24),

sin

$$\sin(\text{A}+\text{B}) \cdot \sin(\text{A}-\text{B}) = \sin^2\text{A} - \sin^2\text{B},$$
$$\sin(\text{B}+\text{C}) \cdot \sin(\text{B}-\text{C}) = \sin^2\text{B} - \sin^2\text{C},$$
$$\sin(\text{C}+\text{D}) \cdot \sin(\text{C}-\text{D}) = \sin^2\text{C} - \sin^2\text{D},$$
&c. &c. &c.
$$\sin(\text{L}+\text{A}) \cdot \sin(\text{L}-\text{A}) = \sin^2\text{L} - \sin^2\text{A}.$$

If all these equations be added together, the second member of the equation will vanish, and of consequence we shall have

$$\sin(\text{A}+\text{B}) \cdot \sin(\text{A}-\text{B}) + \sin(\text{B}+\text{C}) \cdot \sin(\text{B}-\text{C}) + \&\text{c} \ldots$$
$$\ldots\ldots + \sin(\text{L}+\text{A}) + \sin(\text{L}-\text{A}) = 0.$$

Proceeding in a similar manner with $\sin(\text{A}-\text{B})$, $\cos(\text{A}+\text{B})$, $\sin(\text{B}-\text{C})$, $\cos(\text{B}+\text{C})$, &c, there will at length be obtained

$$\cos(\text{A}+\text{B}) \cdot \sin(\text{A}-\text{B}) + \cos(\text{B}+\text{C}) \cdot \sin(\text{B}-\text{C}) + \&\text{c} \ldots$$
$$\ldots\ldots + \cos(\text{L}+\text{A}) \cdot \sin(\text{L}-\text{A}) = 0.$$

33. If the arcs A, B, C, &c .... L form an arithmetical progression, of which the first term is 0, the common difference D', and the last term L any number n of circumferences; then will $\text{B}-\text{A}=\text{D}'$, $\text{C}-\text{B}=\text{D}'$, &c, $\text{A}+\text{B}=\text{D}'$, $\text{B}+\text{C}=3\text{D}'$, &c : and dividing the whole by $\sin\text{D}'$, the preceding equations will become

$$\left.\begin{array}{l}\sin\text{D}' + \sin 3\text{D}' + \sin 5\text{D}' + \&\text{c} = 0,\\ \cos\text{D}' + \cos 3\text{D}' + \cos 5\text{D}' + \&\text{c} = 0.\end{array}\right\} \text{(XXV.)}$$

If E' were equal $2\text{D}'$, these equations would become

$$\sin\text{D}' + \sin(\text{D}'+\text{E}') + \sin(\text{D}'+2\text{E}') + \sin(\text{D}'+3\text{E}') + \&\text{c} = 0,$$
$$\cos\text{D}' + \cos(\text{D}'+\text{E}') + \cos(\text{D}'+2\text{E}') + \cos(\text{D}'+3\text{E}') + \&\text{c} = 0.$$

34. The last equation, however, only shows the sums of sines and cosines of arcs or angles in arithmetical progression, when the common difference is to the first term in the ratio of 2 to 1. To investigate a *general* expression for an infinite series of this kind, let

$$s = \sin\text{A} + \sin(\text{A}+\text{B}) + \sin(\text{A}+2\text{B}) + \sin(\text{A}+3\text{B}) + \&\text{c}.$$

Then, since this series is a recurring series, whose scale of relation is $2\cos\text{B} - 1$, it will arise from the developement of a fraction whose denominator is $1 - 2z \cdot \cos\text{B} + z^2$, making $z = 1$.

Now this fraction will be $= \dfrac{\sin\text{A} + z[\sin(\text{A}+\text{B}) - 2\sin\text{A} \cdot \cos\text{B}]}{1 - 2z \cdot \cos\text{B} + z^2}.$

Therefore, when $z = 1$, we have

$s = \dfrac{\sin\text{A} + \sin(\text{A}+\text{B}) - 2\sin\text{A} \cdot \cos\text{B}}{2 - 2\cos\text{B}}$; and this, because $2\sin\text{A} \cdot \cos\text{B} = \sin(\text{A}+\text{B}) + \sin(\text{A}-\text{B})$ (art. 21), is equal to $\dfrac{\sin\text{A} - \sin(\text{A}-\text{B})}{2(1-\cos\text{B})}$. But, since $\sin\text{A}' - \sin\text{B}' = 2\cos\frac{1}{2}(\text{A}'+\text{B}')$.

sin.

$\sin \frac{1}{2}(A'-B')$, by art. 25, it follows, that $\sin A - \sin (A-B) = 2 \cos (A - \frac{1}{2}B) \sin \frac{1}{2}B$; besides which, we have $1 - \cos B = 2 \sin^2 \frac{1}{2}B$. Consequently the preceding expression becomes $s = \sin A + \sin (A+B) + \sin (A+2B) + \sin (A+3B) +$ &c, *ad infinitum* $= \frac{\cos (A - \frac{1}{2}B)}{2 \sin \frac{1}{2}B}$. . . . (XXVI.)

35. To find the sum of $n+1$ terms of this series, we have simply to consider that the sum of the terms past the $(n+1)$th, that is, the sum of $\sin[A+(n+1)B] + \sin[A+(n+2)B] + \sin[A+(n+3)B] +$ &c, *ad infinitum*, is, by the preceding theorem, $= \frac{\cos[A+(n+\frac{1}{2})B]}{2 \sin \frac{1}{2}B}$. Deducting this, therefore, from the former expression, there will remain, $\sin A + \sin (A+B) + \sin (A+2B) + \sin (A+3B) + \ldots . \sin (A+nB) = \frac{\cos(A-\frac{1}{2}B) - \cos[A+(n+\frac{1}{2})B]}{2 \sin \frac{1}{2}B} = \frac{\sin(A+\frac{1}{2}nB).\sin \frac{1}{2}(n+1)B}{\sin \frac{1}{2}B}$.(XXVII.)

By like means it will be found, that, the sums of the cosines of arcs or angles in arithmetical progression will be $\cos A + \cos (A+B) + \cos (A+2B) + \cos (A+3B) +$ &c, *ad infinitum* $= -\frac{\sin (A-\frac{1}{2}B)}{2 \sin \frac{1}{2}B}$. . . . (XXVIII.)

Also,

$\cos A + \cos (A+B) + \cos (A+2B) + \cos (A+3B) + \ldots \ldots \ldots (\cos A + nB) = \frac{\cos (A+\frac{1}{2}.B) . \sin \frac{1}{2}(n+1)B}{\sin \frac{1}{2}B}$. . . (XXIX.)

36. With regard to the tangents of more than two arcs, the following property (the only one we shall here deduce) is a very curious one, which has not yet been inserted in works of Trigonometry, though it has been long known to mathematicians. Let the three arcs A, B, C, together make up the whole circumference, O: then since $\tan (A+B) = \frac{R^2 (\tan A + \tan B)}{R^2 - \tan A . \tan B}$ (by equa. XXIII), we have $R^2 \times (\tan A + \tan B + \tan C) = R^2 \times [\tan A + \tan B - \tan (A+B)] = R^2 \times (\tan A + \tan B - \frac{R^2 (\tan A + \tan B)}{R^2 - \tan A . \tan B}) =$, by actual multiplication and reduction, to $\tan A . \tan B . \tan C$, since $\tan C = \tan [O - (A+B)] = -\tan (A-B) = -\frac{R^2 (\tan A + \tan B)}{R^2 - \tan A . \tan B}$, by what has preceded in this article. The result therefore is, that *the sum of the tangents of any three arcs which together constitute a circle, multiplied by the square of the radius, is equal to the product of those tangents*. . . . . (XXX.)

Since both arcs in the second and fourth quadrants have their tangents considered negative, the above property will apply to arcs any way trisecting a semicircle; and it will there-

 fore

fore apply to the angles of a plane triangle, which are, together, measured by arcs constituting a semicircle. So that if radius be considered as unity, we shall find that, *the sum of tangents of the three angles of any plane triangle, is equal to the continued product of those tangents.* (XXXI.)

37. Having thus given the chief properties of the sines, tangents, &c, of arcs, their sines, products, and powers, we shall merely subjoin investigations of theorems for the 2d and 3d cases in the solutions of plane triangles. Thus, with respect to the second case, where two sides and their included angle are given:

By equa. IV, $a : b :: \sin A : \sin B$.

By compos. and division $\left.\right\}$ $a + b : a - b :: \sin A + \sin B : \sin A - \sin B$.

But, eq. XXII, $\tan \frac{1}{2}(A + B) : \tan \frac{1}{2}(A - B) :: \sin A + \sin B : \sin A - \sin B$; whence, ex equal $a + b : a - b :: \tan \frac{1}{2}(A + B) : \tan \frac{1}{2}(A - B)$. . . . (XXXII.)

Agreeing with the result of the geometrical investigation, at pa. 386, vol. i.

38. If, instead of having the two sides a, b, given, we know their *logarithms*, as frequently happens in geodesic operations, $\tan \frac{1}{2}(A - B)$ may be readily determined without first finding the number corresponding to the logs. of a and b. For if a and b were considered as the sides of a right-angled triangle, in which φ denotes the angle opposite the side a, then would $\tan \varphi = \frac{a}{b}$. Now, since a is supposed greater than b, this angle will be greater than half a right angle, or it will be measured by an arc greater than $\frac{1}{8}$ of the circumference, or than $\frac{1}{8}\bigcirc$. Then, because $\tan(\phi - \frac{1}{8}\bigcirc) = \frac{\tan \phi - \tan \frac{1}{8}\bigcirc}{1 + \tan\phi \tan \frac{1}{8}\bigcirc}$ and because $\tan \frac{1}{8}\bigcirc = R = 1$, we have

$$\tan(\varphi - \tfrac{1}{8}\bigcirc) = (\tfrac{a}{b} - 1) \div (1 + \tfrac{a}{b}) = \frac{a-b}{a+b}.$$

And, from the preceding article,

$$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)} = \frac{\tan\frac{1}{2}(A-B)}{\cot \frac{1}{2}C} \text{: consequently,}$$

$$\tan \tfrac{1}{2}(A - B) = \cot \tfrac{1}{2}C \,.\, \tan(\varphi - \tfrac{1}{8}\bigcirc) \;.\;.\;. \text{ (XXXIII.)}$$

From this equation we have the following practical rule: Subtract the less from the greater of the given logs, the remainder will be the log tan of an angle: from this angle take 45 degrees, and to the log tan of the remainder add the log cotan of half the given angle; the sum will be the log tan of half the *difference* of the other two angles of the plane triangle.

39. The

39. The remaining case is that in which the three sides of the triangle are known, and for which indeed we have already obtained expressions for the angles in arts. 6 and 8. But, as neither of these is best suited for logarithmic computation, (however well fitted they are for instruments of investigation), another may be deduced thus: in the equation for cos A, (given equation II), viz, $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$, if we substitute, instead of cos A, its value, $1 - 2 \sin^2 \frac{1}{2}A$, change the signs of all the terms, transpose the 1, and divide by 2, we shall have $\sin^2 \frac{1}{2}A = \frac{a^2 - b^2 - c^2 + 2bc}{4bc} = \frac{a^2 - (b-c)^2}{4bc}$. Here, the numerator of the second member being the product of the two factors $(a + b - c)$ and $(a - b + c)$, the equation will become $\sin^2 \frac{1}{2}A = \frac{\frac{1}{2}(a+b-c)\ \frac{1}{2}(a-b+c)}{4bc}$. But, since $\frac{1}{2}(a+b-c) = \frac{1}{2}(a+b+c) - c$, and $\frac{1}{2}(a-b+c) = \frac{1}{2}(a+b+c) - b$; if we put $s = a + b + c$, and extract the square root, there will result,

$$\left.\begin{aligned} \sin \tfrac{1}{2}A &= \sqrt{\frac{(\frac{1}{2}s-b)\,.\,(\frac{1}{2}s-c)}{bc}}. \\ \text{In like manner}\left\{\ \sin \tfrac{1}{2}B \right. &= \sqrt{\frac{(\frac{1}{2}s-a)\,.\,(\frac{1}{2}s-c)}{ac}}. \\ \sin \tfrac{1}{2}C &= \sqrt{\frac{(\frac{1}{2}s-a)\,.\,(\frac{1}{2}s-b)}{ab}}. \end{aligned}\right\} \text{(XXXIV.)}$$

These expressions, besides their convenience for logarithmic computation, have the further advantage of being perfectly free from ambiguity, because the half of any angle of a plane triangle will always be *less* than a right angle.

40. The student will find it advantageous to collect into one place all those formulæ which relate to the computation of sines, tangents, &c*; and, in another place, those which are of use in the solutions of plane triangles: the former of these are equations V, VIII, IX, X, XI, *x*, *xi*, XII, XIII, XIV, XV, XVI, XVII, XVIII, XIX, XX, XXII, *xxii*, XXIII, XXIV, XXVII; the latter are equa. II, III, IV, VII, XXXII, XXXIII, XXXIV.

To exemplify the use of some of these formulæ, the following exercises are subjoined.

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* What is here given being only a brief sketch of an inexhaustible subject; the reader who wishes to pursue it further is referred to the copious Introduction to our Mathematical Tables, and the comprehensive treatises on Trigonometry, by Emerson and many other modern writers on the same subject, where he will find his curiosity richly gratified.

EXERCISES.

Ex. 1. Find the sines and tangents of 15°, 30°, 45°, 60°, and 75°: and show how from thence to find the sines and tangents of several of their submultiples.

First, with regard to the arc of 45°, the sine and cosine are manifestly equal; or they form the perpendicular and base of a right-angled triangle whose hypothenuse is equal to the assumed radius. Thus, if radius be R, the sine and cosine of 45°, will each be $= \sqrt{\frac{1}{2}R^2} = R\sqrt{\frac{1}{2}} = \frac{1}{2}R\sqrt{2}$. If R be equal to 1, as is the case with the tables in use, then

$$\sin 45° = \cos 45° = \tfrac{1}{2}\sqrt{2} = \cdot 7071068.$$

$$\tan 45° = \frac{\sin}{\cos} = 1\cdot = \frac{\cos}{\sin} = \text{cotangent } 45°.$$

Secondly, for the sines of 60° and of 30°: since each angle in an equilateral triangle contains 60°, if a perpendicular be demitted from any one angle of such a triangle on the opposite side, considered as a base, that perpendicular will be the sine of 60°, and the half base the sine of 30°, the side of the triangle being the assumed radius. Thus, if it be R, we shall have $\frac{1}{2}R$ for the sine of 30°, and $\sqrt{R^2 - \frac{1}{4}R^2} = \frac{1}{2}R\sqrt{3}$, for the sine of 60°. When $R = 1$, these become

$$\sin 30° = \cdot 5 \ldots\ldots \sin 60° = \cos 30° = \cdot 8660254.$$

$$\text{Hence, } \tan 30° = \frac{\cdot 5}{\frac{1}{2}\sqrt{3}} = \frac{1}{\sqrt{3}} = \tfrac{1}{3}\sqrt{3} = \cdot 5773503,$$

$$\tan 60° = \frac{\frac{1}{2}\sqrt{3}}{\frac{1}{2}} = \sqrt{3} = \ldots\ldots 1\cdot 7320508.$$

Consequently, $\tan 60° = 3 \tan 30°$.

Thirdly, for the sines of 15° and 75°, the former arc is the half of 30°, and the latter is the compliment of that half arc. Hence, substituting 1 for R and $\frac{1}{2}\sqrt{3}$, for $\cos A$, in the expression $\sin \frac{1}{2}A = \pm \frac{1}{2}\sqrt{2R^2 \pm 2R \cos A}$ . . . (equa. XII), it becomes $\sin 15° = \frac{1}{2}\sqrt{2 - \sqrt{3}} = \cdot 2588190$.

Hence, $\sin 75° = \cos 15° = \sqrt{1 - \frac{1}{4}(2 - \sqrt{3})} = \frac{1}{2}\sqrt{2 + \sqrt{3}} = \dfrac{\sqrt{6} + \sqrt{2}}{4} = \cdot 9659258$.

$$\text{Consequently, } \tan 15° = \frac{\sin}{\cos} = \frac{\cdot 2588190}{\cdot 9659258} = \cdot 2679492.$$

$$\text{And, } \tan 75° = \frac{\cdot 9659258}{\cdot 2588190} = \cdot 3\cdot 7320508.$$

Now, from the sine of 30°, those of 6°, 2°, and 1°, may easily be found. For, if $5A = 30°$, we shall have, from equation X, $\sin 5A = 5 \sin A - 20 \sin^3 A + 16 \sin^5 A$: or, if $\sin A = x$, this will become $16x^5 - 20x^3 + 5x = \cdot 5$. This equation solved by any of the approximating rules for such equations, will give $x = \cdot 1045285$, which is the sine of 6°.

Next

Next, to find the sine of 2°, we have again, from equation x, $\sin 3A = 3 \sin A - 4 \sin^3 A$: that is, if x be put for $\sin 2°$, $3x - 4x^3 = \cdot 1045285$. This cubic solved, gives $x = \cdot 0348995 = \sin 2°$.

Then, if $s = \sin 1°$, we shall, from the second of the equations marked x, have $2s\sqrt{1-s^2} = \cdot 0348995$; whence s is found $= \cdot 0174524 = \sin 1°$.

Had the expression for the sines of bisected arcs been applied successively from sin 15°, to sin 7°30′, sin 3°45′, sin 1°52½′, sin 56¼′, &c, a different series of values might have been obtained: or, if we had proceeded from the quinquisection of 45°, to the trisection of 9°, the bisection of 3°, and so on, a different series still would have been found. But what has been done above, is sufficient to illustrate *this* method. The next example will exhibit a very simple and compendious way of ascending from the sines of smaller to those of larger arcs.

Ex. 2. Given the sine of 1°, to find the sine of 2°, and then the sines of 3°, 4°, 5°, 6°, 7°, 8°, 9°, and 10°, each by a single proportion.

Here, taking first the expression for the sine of a double arc, equa. x, we have $\sin 2° = 2 \sin 1° \sqrt{1-\sin^2 1°} = \cdot 034895$.

Then it follows from the rule in equa. xx, that

$\sin 1° : \sin 2° - \sin 1° :: \sin 2° + \sin 1° : \sin 3° = \cdot 0523360$
$\sin 2° : \sin 3° - \sin 1° :: \sin 3° + \sin 1° : \sin 4° = \cdot 0697565$
$\sin 3° : \sin 4° - \sin 1° :: \sin 4° + \sin 1° : \sin 5° = \cdot 0871557$
$\sin 4° : \sin 5° - \sin 1° :: \sin 5° + \sin 1° : \sin 6° = \cdot 1045285$
$\sin 5° : \sin 6° - \sin 1° :: \sin 6° + \sin 1° : \sin 7° = \cdot 1218693$
$\sin 6° : \sin 7° - \sin 1° :: \sin 7° + \sin 1° : \sin 8° = \cdot 1391731$
$\sin 7° : \sin 8° - \sin 1° :: \sin 8° + \sin 1° : \sin 9° = \cdot 1564375$
$\sin 8° : \sin 9° - \sin 1° :: \sin 9° + \sin 1° : \sin 10° = \cdot 1736482$

To check and verify operations like these, the proportions should be changed at certain stages. Thus,

$\sin 1° : \sin 3° - \sin 2° :: \sin 3° + \sin 2° : \sin 5°$,
$\sin 1° : \sin 4° - \sin 3° :: \sin 4° + \sin 3° : \sin 7°$,
$\sin 4° : \sin 7° - \sin 3° :: \sin 7° + \sin 3° : \sin 10°$.

The coincidence of the results of these operations with the analogous results in the preceding, will manifestly establish the correctness of both.

Cor. By the same method, knowing the sines of 5°, 10°, and 15°, the sines of 20°, 25°, 35°, 55°, 65°, &c, may be found, each by a single proportion. And the sines of 1°, 9°, and 10°, will lead to those of 19°, 29°, 39°, &c. So that the sines may be computed to any arc: and the tangents and other trigonometrical lines, by means of the expressions in art. 4, &c.

Ex.

Ex. 3. Find the sum of all the natural sines to every minute in the quadrant, radius $= 1$.

In this problem the actual addition of all the terms would be a most tiresome labour: but the solution by means of equation XXVII, is rendered very easy. Applying that theorem to the present case, we have $\sin(A + \frac{1}{2}nB) = \sin 45°$, $\sin \frac{1}{2}(n + 1)B = \sin 45°0'30''$, and $\sin \frac{1}{2}B = \sin 30''$. Therefore $\frac{\sin 45° \times \sin 45°\ 0'\ 30''}{\sin 30''} = 3438\ 2467465$ the same sum required.

From another method, the investigation of which is omitted here, it appears that the same sum is equal to $\frac{1}{2}(\cot 30'' + 1)$.

Ex. 4. Explain the method of finding the *logarithmic* sines, cosines, tangents, secants, &c, the natural sines, cosines, &c, being known.

The *natural* sines and cosines being computed to the radius unity, are all proper fractions, or quantities less than unity, so that their logarithms would be negative. To avoid this, the tables of logarithmic sines, cosines, &c, are computed to a radius of 10000000000, or 10^{10}; in which case the logarithm of the radius is 10 times the log of 10, that is, it is 10.

Hence, if s represent any sine to radius 1, then $10^{10} \times s =$ sine of the same arc or angle to rad 10^{10}. And this, in logs is, $\log 10^{10} s = 10 \log 10 + \log s = 10 + \log s$.

The log cosines are found by the same process, since the cosines are the sines of the complements.

The logarithmic expressions for the tangents, &c, are deduced thus:

$\text{Tan} = \text{rad}\,\frac{\sin}{\cos}$. Therf. $\log\tan = \log\text{rad} + \log\sin - \log\cos = 10 + \log\sin - \log\cos$.

$\text{Cot} = \frac{\text{rad}^2}{\tan}$. Therf. $\log\cot = 2\log\text{rad} - \log\tan = 20 - \log\tan$.

$\text{Sec} = \frac{\text{rad}^2}{\cos}$. Therf. $\log\sec = 2\log\text{rad} - \log\cos = 20 - \log\cos$.

$\text{Cosec} = \frac{\text{rad}^2}{\sin}$. Therf. $\text{l.}\,\text{cosec} = 2\log\text{rad} - \log\sin = 20 - \log\sin$.

Versed sine $= \frac{\text{chord}^2}{\text{diam}} = \frac{(2\sin\frac{1}{2}\text{arc})^2}{2\,\text{rad}} = \frac{2 \times \sin^2\frac{1}{2}\text{arc}}{\text{rad}}$.

Therefore, $\log\text{vers}\sin = \log 2 + 2\log\sin\frac{1}{2}\text{arc} - 10$.

Ex. 5. Given the sum of the natural tangents of the angles A and B of a plane triangle $= 3{\cdot}1601988$, the sum of the tangents of the angles B and C $= 31{\cdot}8765577$, and the continued product, $\tan A . \tan B . \tan C = 5{\cdot}3047057$: to find the angles A, B, and C.

It

It has been demonstrated in art. 36, that when radius is unity, the product of the natural tangents of the three angles of a plane triangle is equal to their continued product. Hence the process is this:

From tan A + tan B + tan C	=	5·3047057
Take tan A + tan B	=	3 1601988
Remains tan C	=	2·1445069 = tan 65°.
From tan A + tan B + tan C	=	5·3047057
Take tan B + tan C	=	3·8765577
Remains tan A	=	1·4281480 = tan 55°.

Consequently, the three angles are 55°, 60°, and 65°.

Ex. 6. There is a plane triangle, whose sides are three consecutive terms in the natural series of integer numbers, and whose largest angle is just double the smallest. Required the sides and angles of that triangle?

If A, B, C, be three angles of a plane triangle, a, b, c, the sides respectively opposite to A, B, C; and $s = a + b + c$. Then from equa, III and XXXIV, we have

$$\sin A = \frac{2}{bc}\sqrt{\tfrac{1}{2}s\ (\tfrac{1}{2}s - a)\,.\,(\tfrac{1}{2}s - b)\,.\,(\tfrac{1}{2}s - c)}.$$

$$\text{and} \sin \tfrac{1}{2}C = \sqrt{\frac{(\tfrac{1}{2}s - a)\,.\,(\tfrac{1}{2}s - b)}{ab}}.$$

Let the three sides of the required triangle be represented by x, $x + 1$, and $x + 2$; the angle A being supposed opposite to the side x, and C opposite to the side $x + 2$: then the preceding expressions will become

$$\sin A = \frac{2}{(x+)\,(1+2)}\sqrt{\frac{3x+3}{2}\,.\,\frac{x+3}{2}\,.\,\frac{x+1}{2}\,.\,\frac{x-1}{2}}.$$

$$\sin \tfrac{1}{2}C = \sqrt{\frac{(x+1).(x+3)}{4x(x+1)}}.$$

Assuming these two expressions equal to each other, as they ought to be, by the question; there results, after a little reduction, $x^3 - \frac{5}{2}x^2 - \frac{11}{2}x - 2 = 0$, a cubic equation, with one positive integer root $x = 4$. Hence 4, 5, and 6, are the sides of the triangle.

$$\sin A = \frac{2}{5\cdot6}\sqrt{\frac{15}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}} = \frac{2}{5\cdot6}\sqrt{\frac{15}{4}\cdot\frac{15}{4}\cdot7} = \frac{2\cdot15}{4\cdot5\cdot6}\sqrt{7} = \frac{1}{4}\sqrt{7}.$$

$$\sin B = \frac{5}{16}\sqrt{7};\ \sin C = \frac{6}{16}\sqrt{7};\ \sin\tfrac{1}{2}C = \sqrt{2.\frac{7\cdot5}{2\cdot4\cdot5}} = \frac{1}{4}\sqrt{7}.$$

The angles are, A = 41°·409603 = 41°24′ 34″ 34‴,

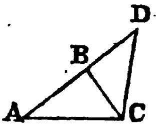
B = 55°·771191 = 55 46 16 18,

C = 82°·819206 = 82 49 9 8.

Any *direct* solution to this curious problem, except by means of the analytical formulæ employed above, would be exceedingly tedious and operose.

Solution

Solution to the same by R. Adrain.



Let ABC be the triangle, having the angle ABC double the angle A, produce AB to D, making BD = BC, and join CD; and the triangles CBD ACD are evidently isosceles and equiangular; therefore BD or BC is to CD or AC as AC to AD. Now let AB $= x$, BC $= x - 1$, AC $= x + 1$, then AD $= 2x - 1$, and the preceding stating becomes $x - 1 : x + 1 :: x + 1 : 2x - 1$, which by multiplying extremes and means gives $2x^2 - 3x + 1 = x^2 + 2x + 1$, and by subtraction $x^2 = 5x$, or dividing by x, simply $x = 5$, hence the sides are 4, 5, 6.

The same conclusion is also readily obtained without the use of algebra.

Ex. 7. Demonstrate that sin 18° = cos 72° is = $\frac{1}{4}$R $(-1 + \sqrt{5})$, and sin 54° = cos 36° is = $\frac{1}{4}$R $(1 + \sqrt{5})$.

Ex. 8. Demonstrate that the sum of the sines of two arcs which together make 60°, is equal to the sine of an arc which is greater than 60, by either of the two arcs: Ex. gr. sin 3′ + sin 59° 57′ = sin 60° 3′; and thus that the tables may be continued by addition only.

Ex. 9. Show the truth of the following proportion: As the sine of half the difference of two arcs, which together make 60°, or 90°, respectively, is to the difference of their sines; so is 1 to $\sqrt{2}$, or $\sqrt{3}$, respectively.

Ex. 10. Demonstrate that the sum of the square of the sine and versed sine of an arc, is equal to the square of double the sine of half the arc.

Ex. 11. Demonstrate that the sine of an arc is a mean proportional between half the radius and the versed sine of double the arc.

Ex. 12. Show that the secant of an arc is equal to the sum of its tangent and the tangent of half its complement.

Ex. 13. Prove that, in any plane triangle, the base is to the difference of the other two sides, as the sine of half the sum of the angles at the base, to the sine of half their difference: also, that the base is to the sum of the other two sides, as the cosine of half the sum of the angles at the base, to the cosine of half their difference.

Ex.

Ex. 14. How must three trees, A, B, C, be planted, so that the angle at A may be double the angle at B, the angle at B double that at C; and so that a line of 400 yards may just go round them?

Ex. 15. In a certain triangle, the sines of the three angles are as the numbers 17, 15, and 8, and the perimeter is 160. What are the sides and angles?

Ex. 16. The logarithms of two sides of a triangle are 2·2407293 and 2·5378191, and the included angle, is 37° 20′. It is required to determine the other angles, without first finding any of the sides?

Ex. 17. The sides of a triangle are to each other as the fractions $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$: what are the angles?

Ex. 18. Show that the secant of 60°, is double the tangent of 45°, and that the secant of 45° is a mean proportional between the tangent of 45° and the secant of 60°.

Ex. 19. Demonstrate that 4 times the rectangle of the sines of two arcs, is equal to the difference of the squares of the chords of the sum and difference of those arcs.

Ex. 20. Convert the equations marked XXXIV into their equivalent logarithmic expressions; and by means of them and equa. IV, find the angles of a triangle whose sides are 5, 6, and 7.

SPHERICAL TRIGONOMETRY.

SECTION I.

General Properties of Spherical Triangles.

ART. 1. *Def.* 1. Any portion of a spherical surface bounded by three arcs of great circles is called a *Spherical Triangle.*

Def. 2. Spherical Trigonometry is the art of computing the measures of the sides and angles of spherical triangles.

Def. 3. A *right-angled* spherical triangle has one right angle: the sides about the right angle are called *legs*; the side opposite to the right angle is called the *hypothenuse*.

Def. 4. A *quadrantal* spherical triangle has one side equal to 90° or a quarter of a great circle.

Def. 5. Two arcs or angles, when compared together, are said to be *alike*, or of the *same affection*, when both are less than 90°, or both are greater than 90°. But when one is greater and the other less than 90°, they are said to be *unlike*, or of *different affections*.

Art. 2. The small circles of the sphere do not fall under consideration in Spherical Trigonometry; but such only as have the same centre with the sphere itself. And hence it is that spherical trigonometry is of so much use in Practical Astronomy, the apparent heavens assuming the shape of a concave sphere, whose centre is the same as the centre of the earth.

3. Every spherical triangle has three sides, and three angles: and if any three of these six parts, be given, the remaining three may be found, by some of the rules which will be investigated in this chapter.

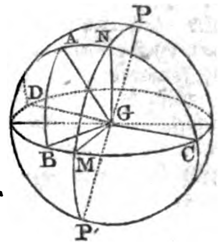
4. In *plane* trigonometry, the knowledge of the three angles is not sufficient for ascertaining the sides: for in that case the *relations* only of the three sides can be obtained, and not their absolute values: whereas, in *spherical* trigonometry, where the sides are circular arcs, whose values depend on their proportion to the whole circle, that is, on the number of degrees they contain, the sides may always be determined when the three angles are known. Other remarkable differences between plane and spherical triangles are, 1st. That in the former, two angles always determine the third; while in the latter they never do. 2dly. The surface of a plane triangle cannot be determined from a knowledge of the angles alone; while that of a spherical triangle always can.

6. The *sides* of a spherical triangle are all arcs of great circles, which, by their intersection on the surface of the sphere, constitute that triangle.

6. The *angle* which is contained between the arcs of two great circles, intersecting each other on the surface of the sphere, is called a spherical angle; and its measure is the same as the measure of the plane angle which is formed by two lines issuing from the same point of, and perpendicular to, the common section of the planes which determine the containing

taining sides: that is to say, it is the same as the angle made by those planes. Or, it is equal to the plane angle formed by the tangents to those arcs at their point of intersection.

7. Hence it follows, that the surface of a spherical triangle BAC, and the three planes which determine it form a kind of triangular pyramid, BCGA, of which the vertex G is at the centre of the sphere, the base ABC a portion of the spherical surface, and the faces AGC, AGB, BGC, sectors of the great circles whose intersections determine the sides of the triangle.



Def. 6. A line perpendicular to the plane of a great circle, passing through the centre of the sphere, and terminated by two points, diametrically opposite, at its surface, is called the *axis* of such circle; and the extremities of the axis, or the points where it meets the surface, are called the *poles* of that circle. Thus, PGP′ is the axis, and P, P′, are the poles, of the great circle CND.

If we conceive any number of less circles, each parallel to the said great circle, this axis will be perpendicular to them likewise; and the points P, P′, will be their poles also.

8 Hence, each pole of a great circle is 90° distant from every point in its circumference; and all the arcs drawn from either pole of a little circle to its circumference, are equal to each other.

9. It likewise follows, that all the arcs of great circles drawn through the poles of another great circle, are perpendicular to it: for since they are great circles by the supposition, they all pass through the centre of the sphere, and consequently through the axis of the said circle. The same thing may be affirmed with regard to small circles.

10 Hence, in order to find the *poles* of any circle, it is merely necessary to describe, upon the surface of the sphere, two great circles perpendicular to the plane of the former; the points where these circles intersect each other will be the poles required.

11. It may be inferred also, from the preceding, that if it were proposed to draw, from any point assumed on the surface of the sphere, an arc of a circle which may measure the shortest distance from that point, to the circumference of any given circle; this arc must be so described, that its prolongation may pass through the poles of the given circle. And conversely, if an arc pass through the poles of a given circle,

circle, it will measure the shortest distance from any assumed point to the circumference of that circle.

12. Hence again, if upon the sides, AC and BC, (produced if necessary) of a spherical triangle BCA, we take the arcs CN, CM, each equal 90°, and through the radii GN, GM (figure to art. 7) draw the plane NGM, it is manifest that the point C will be the pole of the circle coinciding with the plane NGM: so that, as the lines GM, GN, are both perpendicular to the common section GC, of the planes AGC, BGC, they measure, by their inclination, the angle of these planes; or the arc NM measures that angle, and consequently the spherical angle BCA.

13. It is also evident that every arc of a little circle, described from the pole C as centre, and containing the same number of degrees as the arc MN, is equally proper for measuring the angle BCA; though it is customary to use only arcs of great circles for this purpose.

14. Lastly, we infer, that if a spherical angle be a right angle, the arcs of the great circles which form it, will pass mutually through the poles of each other: and that, if the planes of two great circles contain each the axis of the other, or pass through the poles of each other, the angle which they include is a right angle.

These obvious truths being premised and comprehended, the student may pass to the consideration of the following theorems.

THEOREM I.

Any Two Sides of a Spherical Triangle are together Greater than the Third.

This proposition is a necessary consequence of the truth, that the shortest distance between any two points, measured on the surface of the sphere, is the arc of a great circle passing through these points.

THEOREM II.

The Sum of the Three Sides of any Spherical Triangle is Less than 360 degrees.

For, let the sides AC, BC, (fig. to art. 7) containing any angle A, be produced till they meet again in D: then will the arcs DAC, DBC, be each 180°, because all great circles cut each other into two equal parts: consequently DAC + DBC = 360°. But (theorem 1) DA and DB are together greater than the third

third side AB of the triangle DAB; and therefore, since CA + CB + DA + DB = 360°, the sum CA + CB + AB is less than 360°. Q. E. D.

THEOREM III.

The Sum of the Three Angles of any Spherical Triangle is always Greater than Two Right Angles, but less than Six.

1. The first part of this theorem is demonstrated in cor. 2 of THE. IV. following.

2. The angle of inclination of no two of the planes can be so great as two right angles; because, in that case, the two planes would become but one continued plane, and the arcs, instead of being arcs of distinct circles, would be joint arcs of one and the same circle. Therefore, each of the three spherical angles must be less than two right angles; and consequently their sum less than six right angles. Q. E. D.

Cor. 1. Hence it follows, that a spherical triangle may have all its angles either right or obtuse; and therefore the knowledge of any two right angles is not sufficient for the determination of the third.

Cor. 2. If the three angles of a spherical triangle be right or obtuse, the three sides are likewise each equal to, or greater than 90°: and, if each of the angles be acute, each of the sides is also less than 90°; and conversely.

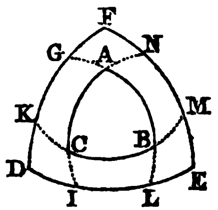
Scholium. From the preceding theorem the student may clearly perceive what is the essential difference between plane and spherical triangles, and how absurd it would be to apply the rules of plane trigonometry to the solution of cases in spherical trigonometry. Yet, though the difference between the two kinds of triangles be really so great, still there are various properties which are common to both, and which may be demonstrated exactly in the same manner. Thus, for example, it might be demonstrated here, (as well as with regard to plane triangles in the elements of Geometry, vol. 1) that two spherical triangles are equal to each other, 1st. When the three sides of the one are respectively equal to the three sides of the other. 2dly. When each of them has an equal angle contained between equal sides: and, 3dly. When they have each two equal angles at the extremities of equal bases. It might also be shown, that a spherial triangle is equilateral, isosceles, or scalene, according as it hath three equal, two equal, or three unequal angles: and again, that the greatest side is always opposite to the greatest angle, and the least side to

to the least angle. But the brevity that our plan requires, compels us merely to *mention* these particulars. It may be added, however, that a spherical triangle may be at once *right-angled* and *equilateral*; which can never be the case with a plane triangle.

THEOREM IV.

If from the Angles of a Spherical Triangle, as Poles, there be described, on the Surface of the Sphere, Three Arcs of Great Circles, which by their Intersections form another Spherical Triangle; Each Side of this New Triangle will be the Supplement to the Measure of the Angle which is at its Pole, and the Measure of each of its Angles the Supplement to that Side of the Primitive Triangle to which it is Opposite.

From B, A, and C, as poles, let the arcs DF, DE, FE, be described, and by their intersections form another spherical triangle DEF; either side, as DE, of this triangle, is the supplement of the measure of the angle A at its pole; and either angle, as D, has for its measure the supplement of the side AB.



Let the sides AB, AC, BC, of the primitive triangle, be produced till they meet those of the triangle DEF, in the points I, L, M, N, G, K: then, since the point A is the pole of the arc DILE, the distance of the points A and E (measured on an arc of a great circle) will be 90°; also, since C is the pole of the arc EF, the points C and E will be 90° distant: consequently (art. 8) the point E is the pole of the arc AC. In like manner it may be shown, that F is the pole of BC, and D that of AB.

This being premised, we shall have $DL = 90°$, and $IE = 90°$: whence $DL + IE = DL + EL + IL = DE + IL = 180°$. Therefore $DE = 180° - IL$: that is, since IL is the measure of the angle BAC, the arc DE is = the supplement of that measure. Thus also may it be demonstrated that EF is equal the supplement to MN, the measure of the angle BCA, and that DF is equal the supplement to GK, the measure of the angle ABC: which constitutes the first part of the proposition.

2dly. The respective measures of the angles of the triangle DEF are supplemental to the opposite sides of the triangles ABC. For, since the arcs AL and BG are each 90°, therefore is

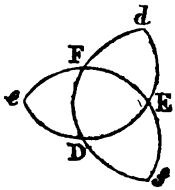
is $AL + BG = GL + AB = 180°$; whence $GL = 180° - AB$; that is the measure of the angle D is equal to the supplement to AB. So likewise may it be shown that AC, BC, are equal to the supplements to the measures of the respectively opposite angles E and F. Consequently, the measures of the angles of the triangle DEF are supplemental to the several opposite sides of the triangle ABC. Q. E. D.

Cor. 1. Hence these two triangles are called *supplemental* or *polar* triangles.

Cor. 2. Since the three sides DE, EF, DF, are supplements to the measures of the three angles A, B, C; it results that $DE + EF + DF + A + B + C = 3 \times 180° = 540°$. But (th. 2), $DE + EF + DF < 360°$: consequently $A + B + C > 180°$. Thus the first part of theorem 3 is very compendiously demonstrated.

Cor. 3. This theorem suggests mutations that are sometimes of use in computation.—Thus, if three angles of a spherical triangle are given, to find the sides: the student may subtract each of the angles from 180°, and the three remainders will be the three sides of a new triangle; the angles of this new triangle being found, if their measures be each taken from 180°, the three remainders will be the respective sides of the primitive triangle, whose angles were given.

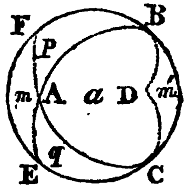
Scholium. The invention of the preceding theorem is due to *Philip Langsberg*. Vide, Simon Stevin, liv. 3, de la Cosmographie, prop. 31 and Alb. Girard in loc. It is often however treated very loosely by authors on trigonometry: some of them speaking of sides as the supplements of angles, and scarcely any of them remarking which of the several triangles formed by the intersection of the arcs DE, EF, DF, is the one in question. Besides the triangle DEF, three others may be formed by the intersection of the semicircles, and if the *whole* circles be considered, there will be *seven* other triangles formed. But the proposition only obtains with regard to the central triangle (of each hemisphere), which is distinguished from the three others in this, that the two angles A and F are situated on the same side of BC, the two B and E on the same side of AC, and the two C and D on the same side of AB.



THEOREM V.

In Every Spherical Triangle, the following proportion obtains, viz, As Four Right Angles (or 360°) to the surface of a Hemisphere;

Hemisphere; or, as Two Right Angles (or 180°) to a Great Circle of the Sphere; so is the Excess of the three angles of the triangle above Two Right Angles, to the Area of the triangle.



Let ABC be the spherical triangle. Complete one of its sides as BC into the circle BCEF, which may be supposed to bound the upper hemisphere. Prolong also, at both ends, the two sides AB, AC, until they form semicircles estimated from each angle, that is, until BAE = ABD = CAF = ACD = 180°. Then will CBF = 180° = BFE; and consequently the triangle AEF, on the anterior hemisphere will be equal to the triangle BCD on the opposite hemisphere. Putting m, m' to represent the surface of these triangles, p for that of the triangle BAF, q for that of CAE, and a for that of the proposed triangle ABC. Then a and m' together (or their equal a and m together) make up the surface of a spheric lune comprehended between the two semicircles ACD, ABD, inclined in the angle A : a and p together make up the lune included between the semicircles CAF, CBF, making the angle C : a and q together make up the spheric lune included between the semicircles BCE, BAE, making the angle B. And the surface of each of these lunes, is to that of the hemisphere, as the angle made by the comprehending semicircles, to two right angles. Therefore, putting $\frac{1}{2}$S for the surface of the hemisphere, we have

$$180^\circ : \text{A} :: \tfrac{1}{2}\text{S} : a + m.$$
$$180^\circ : \text{B} :: \tfrac{1}{2}\text{S} : a + q.$$
$$180^\circ : \text{C} :: \tfrac{1}{2}\text{S} : a + p.$$

Whence, $180^\circ : \text{A} + \text{B} + \text{C} :: \frac{1}{2}\text{S} : 3a + m + p + q = 2a + \frac{1}{2}\text{S}$;
and consequently, by division of proportion,

as $180^\circ : \text{A} + \text{B} + \text{C} - 180^\circ :: \frac{1}{2}\text{S} : 2a + \frac{1}{2}\text{S} - \frac{1}{2}\text{S} = 2a$;

or, $180^\circ : \text{A} + \text{B} + \text{C} - 180^\circ :: \frac{1}{4}\text{S} : a = \frac{1}{2}\text{S} \cdot \dfrac{\text{A}+\text{B}+\text{C}-180^\circ}{360^\circ}$

Q. E. D.*

Cor. 1. Hence the excess of the three angles of any spherical triangle above two right angles, termed technically the

* This determination of the area of a spherical triangle is due to *Albert Girard* (who died about 1653). But the demonstration now commonly given of the rule was first published by Dr. Wallis. It was considered as a mere speculative truth, until General Roy, in 1787, employed it very judiciously in the great Trigonometrical Survey, to correct the errors of spherical angles. See Phil. Trans. vol. 80, and the next chapter of this volume.

spherical

spherical excess, furnishes a correct measure of the surface of that triangle.

Cor 2. If $\pi = 3{\cdot}141593$, and d the diameter of the sphere, then is $\pi d^2 \cdot \frac{A+B+C-180°}{720°}$ = the area of the spherical triangle.

Cor. 3. Since the length of the radius, in any circle, is equal to the length of 57·2957795 degrees, measured on the circumference of that circle; if the *spherical excess* be multiplied by 57·297795, the product will express the surface of the triangle in square degrees.

Cor. 4. When $a = 0$, then $A + B + C = 180°$: and when $a = \frac{1}{2}s$, then $A + B + C = 540°$. Consequently the sum of the three angles of a spherical triangle, is always between 2 and 6 right angles: which is another confirmation of th. 3.

Cor. 5. When *two* of the angles of a spherical triangle are right angles, the surface of the triangle varies with its third angle. And when a spherical triangle has *three* right angles its surface is one eighth of the surface of the sphere.

Remark. Some of the uses of the spherical excess, in the more extensive geodesic operations, will be shown in the following chapter. The mode of finding it, and thence the area when the three angles of a spherical triangle are given, is obvious enough; but it is often requisite to ascertain it by means of other data, as, when two sides and the included angle are given, or when all the three sides are given. In the former case, let a and b be the two sides, C the included angle. and E the spherical excess: then is $\cot \frac{1}{2}E = \frac{\cot \frac{1}{2}a \cdot \cot \frac{1}{2}b + \cos C}{\sin C}$. When the three sides a, b, c, are given, the spherical excess may be found by the following very elegant theorem, discovered by Simon Lhuillier:

$$\tan \tfrac{1}{4}E = \sqrt{\left(\tan \frac{a+b+c}{4} \cdot \tan \frac{a+b-c}{4} \cdot \tan \frac{a-b+c}{4} \cdot \tan \frac{-a+b+c}{4}\right)}.$$

The investigation of these theorems would occupy more space than can be allotted to them in the present volume.

THEOREM VI.

In every Spherical Polygon, or surface included by any number of intersecting great circles, the subjoined proportion obtains, viz, As Four Right Angles, or 360°, to the Surface of a Hemisphere; or, as Two Right Angles, or 180°, to a Great Circle of the Sphere; so is the Excess of the Sum of the Angles above the Product of 180° and Two Less than the number of Angles of the spherical polygon, to its Area.

 For,

For, if the polygon be supposed to be divided into as many triangles as it has sides, by great circles drawn from all the angles through any point within it, forming at that point the vertical angles of all the triangles. Then, by th. 5, it will be as $360° : \frac{1}{2}s :: A + B + C - 180°$: its area. Therefore, putting P for the sum of all the angles of the polygon, n for their number, and V for the sum of all the vertical angles of its constituent triangles, it will be, by composition,
as $360° : \frac{1}{2}s :: P + V - 180°.n$: surface of the polygon. But V is manifestly equal to $360°$ or $180° \times 2$. Therefore, as $360° : \frac{1}{2}s :: P - (n-2)\ 180° : \frac{1}{2}s. \frac{P-(n-2)\ 180°}{360°}$, the area of the polygon. Q. E. D.

Cor. 1. If π and d represent the same quantities as in theor. 5 cor. 2, then the surface of the polygon will be expressed by $\pi d^2 . \frac{P-n-2)\ 180°}{720°}$.

Cor. 2. If $R° = 57 \cdot 2957795$, then will the surface of the polygon in square degrees be $= R° . (P - (n-2)180°)$.

Cor. 3. When the surface of the polygon is 0, then $P = (n-2)\ 180°$; and when it is a maximum, that is, when it is equal to the surface of the hemisphere, then $P = (n-2)\ 180° + 360° = n . 180°$: Consequently P, the sum of all the angles of any spheric polygon, is always *less* than $2n$ right angles, but *greater* than $(2n-4)$ right angles, n denoting the number of angles of the polygon.

GENERAL SCHOLIUM.

On the Nature and Measure of Solid Angles.

A *Solid Angle* is defined by Euclid, that which is made by the meeting of more than two plane angles, which are not in the same plane, in one point.

Others define it the angular space comprized between several planes meeting in one point.

It may be defined still more generally, the *angular space* included between several plane surfaces or one or more curved surfaces, meeting in the point which forms the summit of the angle.

According to this definition, solid angles bear just the same relation to the surfaces which comprize them, as plane angles do to the lines by which they are included: so that, as in the latter, it is not the magnitude of the lines, but their mutual inclination, which determines the angle; just so, in the former it

it is not the magnitude of the planes, but *their* mutual inclinations which determine the angles. And hence all those geometers, from the time of Euclid down to the present period, who have confined their attention principally to the magnitude of the plane angels, instead of their relative positions, have never been able to develope the properties of this class of geometrical quantities; but have affirmed that no solid angle can be said to be the half or the double of another, and have spoken of the bisection and trisection of solid angles, even in the simplest cases, as impossible problems.

But all this supposed difficulty vanishes, and the doctrine of solid angles becomes simple, satisfactory, and universal in its application, by assuming *spherical surfaces* for their measure; just as circular arcs are assumed for the measures of plane angles*. Imagine, that from the summit of a solid angle (formed by the meeting of three planes) as a centre, any sphere be described, and that those planes are produced till they cut the surface of the sphere; then will the surface of the spherical triangle, included between those planes, be a proper measure of the solid angle made by the planes at their common point of meeting; for no change can be conceived in the relative position of those planes, that is, in the magnitude of the solid angle, without a corresponding and proportional mutation in the surface of the spherical triangle. If, in like manner, the three or more surfaces, which by their meeting constitute another solid angle, be produced till they cut the surface of the same or an equal sphere, whose centre coincides with the summit of the angle; the surface of the spheric triangle or polygon, included between the planes which

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* It may be proper to anticipate here the only objection which *can* be made to this assumption; which is founded on the principal, *that quantities should always be measured by quantities of the same kind.* But this, often and positively as it is affirmed, is by no means necessary; nor in many cases is it possible. To measure is to *compare* mathematically: and if by comparing two quantities, whose ratio we know or can ascertain, with two other quantities whose ratio we wish to know, the point in question becomes determined: it signifies not at all whether the magnitudes which constitute one ratio, are like or unlike the magnitudes which constitute the other ratio. It is thus that mathematicians, with perfect safety and correctness, make use of space as a measure of velocity, mass as a measure of inertia, mass and velocity conjointly as a measure of force, space as a measure of time, weight as a measure of density, expansion as a measure of heat, a certain function of planetary velocity as a measure of distance from the central body, arcs of the same circle as measures of plane angles; and it is in conformity with this general procedure that we adopt surfaces, of the same sphere, as measures of solid angles.

deter-

determine the angle, will be a correct measure of *that* angle. And the ratio which subsists between the areas of the spheric triangles, polygons, or other surfaces thus formed, will be accurately the ratio which subsists between the solid angles, constituted by the meeting of the several planes or surfaces, at the centre of the sphere.

Hence, the comparison of solid angles becomes a matter of great ease and simplicity: for, since the areas of spherical triangles are measured by the excess of the sums of their angles each above two right angles (th. 5); and the areas of spherical polygons of n sides, by the excess of the sum of their angles above $2n-4$ right angles (th. 6); it follows, that the magnitude of a trilateral solid angle, will be measured by the excess of the sum of the three angles, made respectively by its bounding planes, above 2 right angles; and the magnitudes of solid angles formed by n bounding planes, by the excess of the sum of the angles of inclination of the several planes above $2n-4$ right angles.

As to solid angles limited by curve surfaces, such as the angles at the vertices of cones; they will manifestly be measured by the spheric surfaces cut off by the prolongation of their bounding surfaces, in the same manner as angles determined by planes are measured by the triangles or polygons, they mark out upon the same, or an equal sphere. In all cases, the maximum limit of solid angles, will be the *plane* towards which the various planes determining such angles approach, as they diverge further from each other about the same summit: just as a right line is the maximum limit of plane angles, being formed by the two bounding lines when they make an angle of 180°. The maximum limit of solid angles is measured by the surface of a hemisphere, in like manner as the maximum limit of plane angles is measured by the arc of a semicircle. The solid right angle (either angle, for example, of a cube) is $\frac{1}{4}(=\frac{1}{2}^2)$ of the maximum solid angle: while the plane right angle is half the maximum plane angle.

The analogy between plane and solid angles being thus traced, we may proceed to exemplify this theory by a few instances; assuming 1000 as the numeral measure of the maximum solid angle = 4 times 90° solid = 360° solid.

1. The solid angles of right prisms are compared with great facility. For, of the three angles made by the three planes which, by their meeting, constitute every such solid angle, two are right angles: and the third is the same as the corresponding plane angle of the polygonal base; on which, therefore, the measure of the solid angle depends. Thus, with respect

respect to the right prism with an equilateral triangular base, each solid angle is formed by planes which respectively make angles of 90°, 90°, and 60°. Consequently 90° + 90° + 60° − 180° = 60°, is the measure of such angle, compared with 360° the maximum angle. It is, therefore, one-sixth of the maximum angle. A right prism with a square base, has, in like manner, each, solid angle measured by 90°+90°+90° − 180° = 90°, which is $\frac{1}{4}$ of the maximum angle. And thus may be found, that each solid angle of a right prism, with an equilateral

triangular base	is	$\frac{1}{6}$ max. angle	=	$\frac{1}{6}$ .1000.
square base	is	$\frac{1}{4}$	=	$\frac{2}{8}$ .1000.
pentagonal base	is	. . . . .	=	$\frac{3}{10}$ .1000.
hexagonal	is	$\frac{1}{3}$	=	$\frac{4}{12}$ .1000.
heptagonal	is	. . . . .	=	$\frac{5}{14}$ .1000.
octagonal	is	$\frac{3}{8}$	=	$\frac{6}{16}$ .1000.
nonagonal	is	. . . . .	=	$\frac{7}{18}$ .1000.
decagonal	is	$\frac{2}{5}$	=	$\frac{8}{20}$ .1000.
undecagonal	is	. . . . .	=	$\frac{9}{22}$ .1000.
duodecagonal	is	$\frac{5}{12}$	=	$\frac{10}{24}$ .1000.
m gonal	is	. . . . .	=	$\frac{m-2}{2m}$ .1000.

Hence it may be deduced, that each solid angle of a regular prism, with triangular base, is *half* each solid angle of a prism with a regular hexagonal base. Each with regular

square base	$= \frac{2}{3}$ of each, with regular	octagonal base,
pentagonal	$= \frac{3}{4}$	decagonal.
haxagonal	$= \frac{4}{5}$	duodecagonal,
$\frac{1}{2}m$ gonal	$= \frac{m-4}{m-2}$	m gonal base.

Hence again we may infer, that the sum of all the solid angles of any prism of triangular base, whether that base be regular or irregular, is *half* the sum of the solid angles of a prism of quadrangular base, regular or irregular. And, the sum of the solid angles of any prism of

tetragonal base is	$= \frac{2}{3}$ sum of angles in prism of	pentag. base,
pentagonal . . .	$= \frac{3}{4}$	haxagonal,
haxagonal . . .	$= \frac{4}{5}$	heptagonal,
m gonal	$= \frac{m-2}{m-1}$	$(m+1)$gonal.

2. Let us compare the solid angles of the five regular bodies. In these bodies, if m be the number of sides of each face; n the number of planes which meet at each solid angle; $\frac{1}{2}$O = half the circumference or 180°; and A the plane angle made by two adjacent faces: then we have $\sin \frac{1}{2}A = \frac{\cos \frac{1}{2n} O}{\sin \frac{1}{2m} O}$.

This

This theorem gives, for the plane angle formed by every two contiguous faces of the tetraëdron, 70°31′42″; of the hexaëdron, 90°; of the octaëdron, 109°28′18″; of the dodecaëdron, 116°33′54″; of the icosaëdron, 138°11′23″. But in these polyedræ, the number of faces meeting about each solid angle, 3, 3, 4, 3, 5 respectively. Consequently the several solid angles will be determined by the subjoined proportions:

		Solid Angle.	
360° : 3.70°31′42″ — 180°	: : 1000 :	87·73611	Tetraëdron.
360° : 3.90° — 180°	: : 1000 :	250·	Haxaëdron.
360° : 4.109°28′18″—360°	: : 1000 :	216·35185	Octaëdron.
360° : 3.116°33′54″—180°	: : 1000 :	471·395	Dodecaëdron.
360° : 5.138°11′23″—540°	: : 1000 :	419·30169	Icosaëdron.

3. The solid angles at the vertices of cones, will be determined by means of the spheric segments cut off at the bases of those cones; that is, if right cones, instead of having plane bases, had bases formed of the segments of equal spheres, whose centres were the vertices of the cones, the surfaces of those segments would be measures of the solid angles at the respective vertices. Now, the surfaces of spheric segments, are to the surface of the hemisphere, as their altitudes, to the radius of the sphere; and therefore the solid angles at the vertices of right cones will be to the maximum solid angle, as the excess of the slant side above the axis of the cone, to the slant side of the cone. Thus, if we wish to ascertain the solid angles at the vertices of the equilateral and the right-angled cones; the axis of the former is $\frac{1}{2}\sqrt{3}$, of the latter, $\frac{1}{2}\sqrt{2}$, the slant side of each being unity. Hence,

Angle at vertex.

$1 : 1 - \frac{1}{2}\sqrt{3} : : 1000 :$ 133·97464, equilateral cone,

$1 : 1 - \frac{1}{2}\sqrt{2} : : 1000 :$ 292·89322, right-angled cone.

4. From what has been said, the mode of determining the solid angles at the vertices of pyramids will be sufficiently obvious. If the pyramids be regular ones, if N be the number of faces meeting about the vertical angle in one, and A the angle of inclination of each two of its plane faces; if n be the number of planes meeting about the vertex of the other, and a the angle of inclination of each two of its faces: then will the vertical angle of the former, be to the vertical angle of the latter pyramid, as $NA - (N - 2)\ 180°$, to $na - (n - 2)\ 180°$.

If a cube be cut by diagonal planes, into 6 equal pyramids with square bases, their vertices all meeting at the centre of the circumscribing sphere; then each of the solid angles, made by the four planes meeting at each vertex, will be $\frac{1}{3}$ of the maximum solid angle; and each of the solid angles at the bases of the pyramids, will be $\frac{1}{12}$ of the maximum solid angle

angle. Therefore, each solid angle at the base of such pyramid, is *one-fourth* of the solid angle at its vertex : and, if the angle at the vertex be bisected, as described below, either of the solid angles arising from the bisection, will be double of either solid angle at the base. Hence also, and from the first subdivision of this scholium, each solid angle of a prism, with equilateral triangular base, will be *half* each vertical angle of these pyramids, and *double* each solid angle at their bases.

The angles made by one plane with another, must be ascertained, either by measurement or by computation, according to circumstances. But, the general theory being thus explained, and illustrated, the further application of it is left to the skill and ingenuity of geometers; the following simple example, merely, being added here.

Ex. Let the solid angle at the vertex of a square pyramid be bisected.

1st. Let a plane be drawn through the vertex and any two opposite angles of the base, that plane will bisect the solid angle at the vertex ; forming two trilateral angles, each equal to half the original quadrilateral angle.

2dly. Bisect either diagonal of the base, and draw *any* plane to pass through the point of bisection and the vertex of the pyramid ; such plane, if it do *not* coincide with the former, will divide the quadrilateral solid angle into two equal quadrilateral solid angles. For this plane, produced, will bisect the great circle diagonal of the spherical parallelogram cut off by the base of the pyramid ; and any great circle bisecting such diagonal is known to bisect the spherical parallelogram, or square ; the plane, therefore, bisects the solid angle.

Cor. Hence an indefinite number of planes may be drawn, each to bisect a given quadrilateral solid angle.

SECTION II.

Resolution of Spherical Triangles.

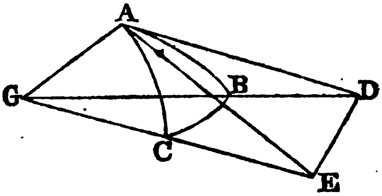
The different cases of spherical trigonometry, like those in plane trigonometry, may be solved either geometrically or algebraically. We shall here adopt the analytical method, as well on account of its being more compatible with brevity, as because of its correspondence and connexion with the substance

stance of the preceding chapter*. The whole doctrine may be comprehended in the subsequent problems and theorems.

PROBLEM. I.

To Find Equations, from which may be deduced the solution of all the Cases of Spherical Triangles.

Let ABC be a spherical triangle; AD the tangent, and GD the secant, of the arc AB; AE the tangent, and GE the secant, of the arc AC; let the capital letters A, B, C, denote the angles of the triangle, and the small letters a, b, c, the opposite sides BC, AC, AB. Then the first equations in art. 6 Pl. Trig.



applied to the two triangles ADE, GDE, give, for the former, $DE^2 = \tan^2 b + \tan^2 c - \tan b \,.\, \tan c \,.\, \cos A$; for the latter, $DE^2 = \sec^2 b + \sec^2 c - \sec b \,.\, \sec c \,.\, \cos a$. Subtracting the first of these equations from the second, and observing that $\sec^2 b - \tan^2 b = R^2 = 1$, we shall have, after a little reduction, $1 + \frac{\sin b \,.\, \cos c}{\cos b \,.\, \cos c} \cos A - \frac{\cos a}{\cos b \,.\, \cos c} = 0$. Whence the three folowing symmetrical equations are obtained:

$$\left.\begin{aligned} \cos a &= \cos b \,.\, \cos c + \sin b \,.\, \sin c \,.\, \cos A \\ \cos b &= \cos a \,.\, \cos c + \sin a \,.\, \sin c \,.\, \cos B \\ \cos c &= \cos a \,.\, \cos b + \sin a \,.\, \sin b \,.\, \cos C \end{aligned}\right\} \quad (\text{I.})$$

THEOREM VII.

In Every Spherical Triangle, the Sines of the Angles are Proportional to the Sines of their Opposite sides.

If, from the first of the equations marked I, the value of cos A be drawn, and substituted for it in the equation $\sin^2 A = 1 - \cos^2 A$, we shall have

$$\sin^2 A = 1 - \frac{\cos^2 a + \cos^2 b \,.\, \cos^2 c - 2 \cos a \,.\, \cos b \,.\, \cos c}{\sin^2 b \,.\, \sin^2 c}$$

Reducing the terms of the second side of this equation to a common denominator, multiplying both numerator and denominator by $\sin^2 a$ and extracting the sq. root there will result

$$\sin A = \sin a . \frac{\sqrt{(1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \,.\, \cos b \,.\, \cos c)}}{\sin a \,.\, \sin b \,.\, \sin c}$$

* For the geometrical method, the reader may consult Simson's or Playfair's Euclid, or Bishop Horsley's Elementary Treatises on Practical Mathematics.

Here,

Here, if the whole fraction which multiplies sin a, be denoted by K (see art. 8 chap. iii), we may write sin A = K . sin a. And, since the fractional factor, in the above equation, contains terms in which the sides a, b, c, are alike affected, we have similar equations for sin B, and sin C, That is to say, we have

sin A = K . sin a . . . sin B = K . sin b . . . sin C = K . sin c.

Consequently, $\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$ . . . (II.) which is the algebraical expression of the theorem.

THEOREM VIII.

In Every Right-Angled Spherical Triangle, the Cosine of the Hypothenuse, is equal to the Product of the Cosines of the Sides Including the right angle.

For, if A be measured by $\frac{1}{4}\odot$, its cosine becomes nothing, and the first of the equations I becomes $\cos a = \cos b \cdot \cos c$.

Q. E. D.

THEOREM IX.

In Every Right-Angled Spherical Triangle, the Cosine of either Oblique Angle, is equal to the Quotient of the Tangent of the Adjacent Side divided by the Tangent of the Hypothenuse.

If, in the second of the equations I, the preceding value of cos a be substituted for it, and for sin a its value tan a . cos a= cos a . cos b . cos c; then recollecting that $1-\cos^2 c = \sin^2 c$, there will result, tan a . cos c . cos B = sin c : whence it follows that,

$$\tan a \cdot \cos B = \tan c, \text{ or } \cos B = \frac{\tan c}{\tan a}.$$

Thus also it is found that $\cos C = \frac{\tan b}{\tan a}$.

THEOREM X.

In Any Right-Angled Spherical Triangle, the Cosine of one of the Sides about the right angle, is equal to the Quotient of the Cosine of the Opposite angle divided by the sine of the Adjacent angle.

From th. 7, we have $\frac{\sin B}{\sin A} = \frac{\sin b}{\sin a}$; which, when A is a right angle, becomes simply $\sin B = \frac{\sin b}{\sin a}$. Again, from th. 9, we have $\cos C = \frac{\tan b}{\tan a}$. Hence by division,

$$\frac{\cos C}{\sin B} = \frac{\tan b}{\sin b} = \frac{\sin a}{\tan a} = \frac{\cos a}{\cos b}.$$

Now, th. 8 gives $\frac{\cos a}{\cos c} = \cos c$. Therefore $\frac{\cos C}{\sin B} = \cos b$; and in like manner, $\frac{\cos B}{\sin C} = \cos b$. Q. E. D.

THEOREM XI.

In Every Right-Angled Spherical Triangle, the Tangent of either of the Oblique Angles, is equal to the Quotient of the Tangent of the Opposite Side, divided by the sine of the Other Side about the right angle.

For, since $\sin B = \frac{\sin b}{\sin a}$, and $\cos B = \frac{\tan c}{\tan a}$,

we have $\frac{\sin B}{\cos B} = \frac{\sin b}{\sin a} \cdot \frac{\tan a}{\tan c}$.

Whence, because (th. 8) $\cos a = \cos b \,.\, \cos c$, and since $\sin a = \cos a \,.\, \tan a$, we have

$$\tan B = \frac{\sin b}{\cos a \,.\, \tan c} = \frac{\sin b}{\cos b \,.\, \cos c \,.\, \tan c} = \frac{\sin b}{\cos b} \cdot \frac{1}{\cos c \,.\, \tan c} = \frac{\tan b}{\sin c}.$$

In like manner, $\tan C = \frac{\tan c}{\sin b}$. Q. E. D.

THEOREM XII.

In Every Right-Angled Spherical Triangle, the Cosine of the Hypothenuse, is equal to the Quotient of the Cotangent of one of the Oblique Angles, divided by the Tangent of the Other Angle.

For, multiplying together the resulting equations of the preceding theorem, we have

$$\tan B \,.\, \tan C = \frac{\tan b}{\sin b} \cdot \frac{\tan c}{\sin c} = \frac{1}{\cos b \,.\, \cos c}.$$

But, by th. 8, $\cos b \,.\, \cos c = \cos a$.

Therefore $\tan B \,.\, \tan C = \frac{1}{\cos a}$, or $\cos a = \frac{\cot C}{\tan B}$. Q. E. D.

THEOREM XIII.

In Every Right-Angled Spherical Triangle, the Sine of the Difference between the Hypothenuse and Base, is equal to the Continued Product of the Sine of the Perpendicular, Cosine of the Base, and Tangent of Half the Angle Opposite to the Perpendicular; or equal to the Continued Product of the Tangent of the Perpendicular, Cosine of the Hypothenuse, and Tangent of Half the Angle Opposite to the Perpendicular*.

* This theorem is due to M. Prony, who published it without demonstration in the *Connaissance des Temps* for the year 1808, and made use of it in the construction of a chart of the course of the Po.

Here,

Here, retaining the same notation, since we have $\sin a = \frac{\sin b}{\sin B}$, and $\cos B = \frac{\tan c}{\tan a}$; if for the tangents there be substituted their values in sines and cosines, there will arise,

$$\sin c . \cos a = \cos B . \cos c . \sin a = \cos B . \cos c . \frac{\sin b}{\sin B}.$$

Then substituting for sin A, and $\sin c . \cos a$, their values in the known formula (equ. v chap. iii) viz,

$$\text{in } \sin(a-c) = \sin a . \cos c - \cos a . \sin c,$$

and recollecting that $\frac{1-\cos B}{\sin B} = \tan \frac{1}{2}B$,

it will become, $\sin(a-c) = \sin b . \cos c . \tan \frac{1}{2}B$, which is the first part of the theorem: and, if in this result we introduce, instead of cos c, its value $\frac{\cos a}{\cos b}$ (th. 8), it will be transformed into $\sin(a-c) = \tan b . \cos a . \tan \frac{1}{2}B$; which is the second part of the theorem. Q. E. D.

Cor. This theorem leads manifestly to an analogous one with regard to rectilinear triangles, which, if h, b, and p denote the hypothenuse, base, and perpendicular, and B, P, the angles respectively opposite to b, p; may be expressed thus:

$$h - b = p . \tan \tfrac{1}{2}P \; . \; . \; . \; . \; . \; h - p = b . \tan \tfrac{1}{2}B.$$

These theorems may be found useful in reducing inclined lines to the plane of the horizon.

PROBLEM II.

Given the Three Sides of a Spherical Triangle; it is required to find Expressions for the Determination of the Angles.

Retaining the notation of prob. 1, in all its generality, we soon deduce from the equations marked I in that problem, the following; viz,

$$\left.\begin{aligned} \cos A &= \frac{\cos a - \cos b . \cos c}{\sin b . \sin c} \\ \cos B &= \frac{\cos b - \cos a . \cos c}{\sin a . \sin c} \\ \cos C &= \frac{\cos c - \cos a . \cos b}{\sin a . \sin b} \end{aligned}\right\}$$

As these equations, however, are not well suited for logarithmic computation; they must be so transformed, that their second members will resolve into factors. In order to this, substitute in the known equation $1 - \cos A = 2 \sin^2 \frac{1}{2}A$, the preceding value of cos A, and there will result

$$2 \sin^2 \tfrac{1}{2}A = \frac{\cos(b-c) - \cos a}{\sin b . \sin c}.$$

But, because $\cos B' - \cos A' = 2 \sin \frac{1}{2}(A'+B') . \sin \frac{1}{2}(A'-B')$ (art. 25 ch. iii), and consequently,

cos

$$\cos(b-c) - \cos a = 2\sin\frac{a+b-c}{2} \cdot \sin\frac{a+c-b}{2}:$$

we have, obviously,

$$\sin^2 \tfrac{1}{2}A = \frac{\sin\frac{1}{2}(a+b-c) \cdot \sin\frac{1}{2}(a+c-b)}{\sin b \cdot \sin c}.$$

Whence, making $s = a + b + c$, there results

$$\left.\begin{aligned} \sin\tfrac{1}{2}A &= \sqrt{\frac{\sin(\frac{1}{2}s-b) \cdot \sin(\frac{1}{2}s-c)}{\sin b \cdot \sin c}}. \\ \text{So also, } \sin\tfrac{1}{2}B &= \sqrt{\frac{\sin(\frac{1}{2}s-a) \cdot \sin(\frac{1}{2}s-c)}{\sin a \cdot \sin c}}. \\ \text{And, } \sin\tfrac{1}{2}C &= \sqrt{\frac{\sin(\frac{1}{2}s-a) \cdot \sin(\frac{1}{2}s-b)}{\sin a \cdot \sin b}}. \end{aligned}\right\} \text{(III.)}$$

The expressions for the tangents of the half angles, might have been deduced with equal facility; and we should have obtained, for example,

$$\tan\tfrac{1}{2}A = \sqrt{\frac{\sin(\frac{1}{2}s-b) \cdot \sin(\frac{1}{2}s-c)}{\sin\frac{1}{2}s \cdot \sin\frac{1}{2}(s-a)}}. \quad (iii.)$$

Thus again, the expressions for the cosine and cotangent of half one of the angles, are

$$\cos\tfrac{1}{2}A = \sqrt{\frac{\sin\frac{1}{2}s \cdot \sin\frac{1}{2}(s-a)}{\sin b \cdot \sin c}}.$$

$$\cot\tfrac{1}{2}A = \sqrt{\frac{\sin\frac{1}{2}s \cdot \sin\frac{1}{2}(s-a)}{\sin(\frac{1}{2}s-b) \cdot \sin(\frac{1}{2}s-c)}}.$$

The three latter flowing naturally from the former, by means of the values $\tan = \frac{\sin}{\cos}$, $\cot = \frac{\cos}{\sin}$. (art. 4 ch. iii.)

Cor. 1. When two of the sides, as b and c, become equal, then the expression for $\sin\frac{1}{2}A$ becomes

$$\sin\tfrac{1}{2}A = \frac{\sin(\frac{1}{2}s-b)}{\sin b} = \frac{\sin\frac{1}{2}a}{\sin b}.$$

Cor. 2. When all the three sides are equal, or $a = b = c$, then $\sin\frac{1}{2}A = \frac{\sin\frac{1}{2}a}{\sin a}$.

Cor. 3. In this case, if $a = b = c = 90°$; then $\sin\frac{1}{2}A = \frac{\frac{1}{2}\surd 2}{1} = \frac{1}{2}\surd 2 = \sin 45°$: and $A = B = C = 90°$.

Cor. 4. If $a = b = c = 60°$: then $\sin\frac{1}{2}A = \frac{\frac{1}{2}}{\frac{1}{2}\surd 3} = \frac{1}{3}\surd 3 = \sin 35°15'51''$: and $A = B = C = 70°31'42''$, the same as the angle between two contiguous planes of a tetraedron.

Cor. 5. If $a = b = c$ were assumed $= 120°$: then $\sin\frac{1}{2}A = \frac{\sin 60°}{\sin 120°} = \frac{\frac{1}{2}\surd 3}{\frac{1}{2}\surd 3} = 1$; and $A = B = C = 180°$: which shows that no such triangle can be constructed (conformably to th. 2); but that the three sides would, in such case, form three continued arcs completing a great circle of the sphere.

PROBLEM

PROBLEM III.

Given the Three Angles of a Spherical Triangle, to find Expressions for the Sides.

If from the first and third of the equations marked I (prob. 1), cos c be exterminated, there will result,

$$\cos A \cdot \sin c + \cos C \cdot \sin a \cdot \cos b = \cos a \cdot \sin b.$$

But, it follows from th. 7, that $\sin c = \frac{\sin a \cdot \sin C}{\sin A}$. Substituting for sin c this value of it, and for $\frac{\cos A}{\sin A}, \frac{\cos a}{\sin a}$, their equivalents cot A, cot a, we shall have,

$$\cot A \cdot \sin C + \cos C \cdot \cos b = \cot a \cdot \sin b.$$

Now, $\cot a \cdot \sin b = \frac{\cos a}{\sin a} \cdot \sin b = \cos a \cdot \frac{\sin b}{\sin a} = \cos a \cdot \frac{\sin B}{\sin A}$, (th. 7). So that the preceding equation at length becomes,

$$\cos A \cdot \sin C = \cos a \cdot \sin B - \sin A \cdot \cos C \cdot \cos b.$$

In like manner, we have,

$$\cos B \cdot \sin C = \cos b \cdot \sin A - \sin B \cdot \cos C \cdot \cos a.$$

Exterminating cos b from these, there results

$$\left.\begin{array}{ll} & \cos A = \cos a \cdot \sin B \sin C - \cos B \cdot \cos C. \\ \text{So like-} & \cos B = \cos b \cdot \sin A \sin C - \cos A \cdot \cos C. \\ \text{wise} & \cos C = \cos c \cdot \sin A \sin B - \cos A \cdot \cos B. \end{array}\right\} \text{(IV.)}$$

This systém of equations is manifestly analogous to equation I; and if they be reduced in the manner adopted in the last problem, they will give

$$\left.\begin{array}{l} \sin \frac{1}{2}a = \sqrt{-\frac{\cos \frac{1}{2}(A+B+C) \cos \frac{1}{2}(B+C-A)}{\sin B \cdot \sin C}}. \\ \sin \frac{1}{2}b = \sqrt{-\frac{\cos \frac{1}{2}(A+B+C) \cdot \cos \frac{1}{2}(A+C-B)}{\sin A \sin C}}. \\ \sin \frac{1}{2}c = \sqrt{-\frac{\cos \frac{1}{2}(A+B+C) \cdot \cos \frac{1}{2}(A+B-C)}{\sin A \cdot \sin B}}. \end{array}\right\} \text{(V).}$$

The expression for the tangent of half a side is

$$\tan \frac{1}{2}a = \sqrt{-\frac{\cos \frac{1}{2}(A+B+C) \cdot \cos \frac{1}{2}(B+C-A)}{\cos \frac{1}{2}(A+C-B) \cdot \cos \frac{1}{2}(A+B-C)}}.$$

The values of the cosines and cotangents are omitted, to save room; but are easily deduced by the student.

Cor. 1. When two of the angles, as B and C, become equal, then the value of $\cos \frac{1}{2}a$ becomes $\cos \frac{1}{2}a = \frac{\cos \frac{1}{2}A}{\sin B}$.

Cor. 2. When $A = B = C$; then $\cos \frac{1}{2}a = \frac{\cos \frac{1}{2}A}{\sin A}$.

Cor. 3. When $A = B = C = 90°$, then $a = b = c = 90°$.

Cor. 4. If $A = B = C = 60°$; then $\cos \frac{1}{2}a = \frac{\sin 60°}{\sin 60°} = 1$.

So that $a = b = c = 0$. Consequently no such triangle can be constructed: conformably to th. 3.

Cor

Cor. 5. If $A=B=C=120°$: then $\cos \frac{1}{2}a = \frac{\cos 60°}{\sin 120°} = \frac{\frac{1}{2}}{\frac{1}{2}\sqrt{3}} = \frac{1}{3}\sqrt{3} = \cos 54° 44' 9''$. Hence $a = b = c = 109° 28' 18''$.

Schol. If, in the preceding values of $\sin \frac{1}{2}a$, $\sin \frac{1}{2}b$, &c, the quantities under the radical were negative in reality, as they are in appearance, it would obviously be impossible to determine the value of $\sin \frac{1}{2}a$, &c. But this value is in fact always real. For, in general, $\sin (x - \frac{1}{4}\bigcirc) = -\cos x$: therefore $\sin (\frac{A+B+C}{2} - \frac{1}{4}\bigcirc) = -\cos \frac{1}{2}(A+B+C)$; a quantity which is always positive, because, as $A+B+C$ is necessarily comprised between $\frac{1}{2}\bigcirc$ and $\frac{3}{2}\bigcirc$, we have $\frac{1}{2}(A+B+C) - \frac{1}{4}\bigcirc$ greater than nothing, and less than $\frac{1}{2}\bigcirc$. Further, any one side of a spherical triangle being smaller than the sum of the other two, we have, by the property of the polar triangle (theorem 4), $\frac{1}{2}\bigcirc - A$ less than $\frac{1}{2}\bigcirc - B + \frac{1}{2}\bigcirc - C$; whence $\frac{1}{2}(B+C-A)$ is less than $\frac{1}{4}\bigcirc$; and of course its cosine is positive.

PROBLEM IV.

Given Two Sides of a Spherical Triangle, and the Included Angle to obtain Expressions for the Other Angles.

1. In the investigation of the last problem, we had

$$\cos A . \sin c = \cos a . \sin b - \cos C . \sin a . \cos b:$$

and by a simple permutation of letters, we have

$$\cos B . \sin c = \cos b . \sin a - \cos C . \sin b . \cos a:$$

adding together these two equations, and reducing, we have

$$\sin c (\cos A + \cos B) = (1 - \cos C) \sin (a+b).$$

Now we have from theor. 7,

$$\frac{\sin a}{\sin A} = \frac{\sin c}{\sin C}, \text{ and } \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

Freeing these equations from their denominators, and respectively adding and subtracting them, there results

$$\sin c (\sin A + \sin B) = \sin C (\sin a + \sin b),$$

$$\text{and } \sin c (\sin A - \sin B) = \sin C (\sin a - \sin b).$$

Dividing each of these two equations by the preceding, there will be obtained

$$\frac{\sin A + \sin B}{\cos A + \cos B} = \frac{\sin C}{1 - \cos C} . \frac{\sin a + \sin b}{\sin (a+b)},$$

$$\frac{\sin A - \sin B}{\cos A + \cos B} = \frac{\sin C}{1 - \cos C} . \frac{\sin a - \sin b}{\sin (a+b)}.$$

Comparing these with the equations in arts. 25, 26, 27, ch. iii, there will at length result

$$\left.\begin{aligned} \tan \tfrac{1}{2}(A+B) &= \cot \tfrac{1}{2}C . \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)}. \\ \tan \tfrac{1}{2}(A-B) &= \cot \tfrac{1}{2}C . \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)}. \end{aligned}\right\} \ldots \text{(VI.)}$$

Cor.

Cor. When $a = b$, the first of the above equations becomes $\tan A = \tan B = \cot \frac{1}{2}C \,.\, \sec a$.

And in this case it will be, as rad : $\sin \frac{1}{2}C$: : $\sin a$ or $\sin b$: $\sin \frac{1}{2}c$

And, as rad : $\cos A$ or $\cos B$: : $\tan a$ or $\tan b$: $\tan \frac{1}{2}c$.

2. The preceding values of $\tan \frac{1}{2}(A + B)$, $\tan \frac{1}{2}(A - B)$ are very well fitted for logarithmic computation : it may, notwithstanding, be proper to investigate a theorem which will at once lead to one of the angles, by means of a subsidiary angle. In order to this, we deduce immediately from the second equation in the investigation of prob. 3,

$$\cot A = \frac{\cot a \quad \sin b}{\sin C} - \cot C \,.\, \cos b.$$

Then, choosing the subsidiary angle ϕ so that

$$\tan \phi = \tan a \,.\, \cos C,$$

that is, finding the angle ϕ, whose tangent is equal to the product $\tan a \,.\, \cos C$, which is equivalent to dividing the original triangle into two right-angled triangles, the preceding equation will become

$$\cot A = \cot C(\cot \phi \,.\, \sin b - \cos b) = \frac{\cot C}{\sin \phi}(\cos \phi \,.\, \sin b - \sin \phi \,.\cos b).$$

And this, since $\sin(b - \phi) = \cos \phi \,.\, \sin b - \sin \phi \,.\cos b$ becomes

$$\cot A = \frac{\cot C}{\sin \phi} \cdot \sin (b - \phi).$$

Which is a very simple and convenient expression.

PROBLEM V.

Given Two Angles of a Spherical Triangle, and the Side Comprehended between them; to find Expressions for the Other Two Sides.

1. Here, a similar analysis to that employed in the preceding problem, being pursued with respect to the equations IV, in prob. 3, will produce the following formulæ :

$$\frac{\sin a + \sin b}{\cos a + \cos b} = \frac{\sin c}{1 + \cos c} \cdot \frac{\sin A + \sin B}{\sin (A + B)},$$

$$\frac{\sin a - \sin b}{\cos a + \cos b} = \frac{\sin c}{1 + \cos c} \cdot \frac{\sin A - \sin B}{\sin (A + B)}.$$

Whence, as in prob. 4, we obtain

$$\left.\begin{aligned} \tan \tfrac{1}{2}(a+b) &= \tan \tfrac{1}{2}c \,.\, \frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)} \\ \tan \tfrac{1}{2}(a-b) &= \tan \tfrac{1}{2}c \,.\, \frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} \end{aligned}\right\} (\text{VII}^*)$$

* The formulæ marked VI, and VII, converted into analogies, by making the denominator of the second member the first term, the other two factors the second and third terms, and the first member of the equation, the fourth term of the proportion, as

2. If

2. If it be wished to obtain a side at once, by means of a subsidiary angle; then, find ϕ so that $\frac{\cot A}{\cos c} = \tan \phi$; then will $\cot a = \frac{\cot c}{\cos \phi} . \cos (B - \phi)$.

PROBLEM VI.

Given Two Sides of a Spherical Triangle, and an Angle Opposite to one of them; to find the Other Opposite Angle.

Suppose the sides given are a, b, and the given angle B: then from theor. 7, we have $\sin A = \frac{\sin a . \sin B}{\sin b}$; or, $\sin A$, a fourth proportional to $\sin b$, $\sin B$, and $\sin a$.

PROBLEM VII.

Given Two Angles of a Spherical Triangle, and a Side Opposite to one of them; to find the Side Opposite to the other.

Suppose the given angles are A, and B, and b the given side: then th. 7, gives $\sin a = \frac{\sin b . \sin A}{\sin B}$; or, $\sin a$, a fourth proportional to $\sin B$ $\sin b$, and $\sin A$.

Scholium.

In problems 2 and 3, if the circumstances of the question leave any doubt, whether the arcs or the angles sought, are greater or less than a quadrant, or than a right angle, the difficulty will be entirely removed by means of the table of mutations of signs of trigonometrical quantities, in different quadrants, marked VII in chap. 3. In the 6th and 7th problems, the question proposed will often be susceptible of two solutions: by means of the subjoined table the student may always tell when this will or will not be the case.

1. With the data a, b, and B, there can only be one solution when $B = \frac{1}{4} O$ (a right angle),

or, when $B < \frac{1}{4} O \ldots . a < \frac{1}{4} O \ldots . b > a$,

$B < \frac{1}{4} O \ldots . a > \frac{1}{4} O \ldots . b > \frac{1}{2} O - a$,

$B > \frac{1}{4} O \ldots . a < \frac{1}{4} O \ldots . b < \frac{1}{2} O - a$,

$B > \frac{1}{4} O \ldots . a > \frac{1}{4} O \ldots . b < a$.

$\cos \frac{1}{2}(a+b) : \cos \frac{1}{2}(a-b) :: \cot \frac{1}{2}C : \tan \frac{1}{2}(A+B)$,
$\sin \frac{1}{2}(a+b) : \sin \frac{1}{2}(a-b) :: \cot \frac{1}{2}C : \tan \frac{1}{2}(A-B)$. &c. &c.
are called *the Analogies of Napier*, being invented by that celebrated geometer. He likewise invented other rules for spherical trigonometry, known by the name of *Napier's Rules for the circular parts*; but these, notwithstanding their ingenuity, are not inserted here; because they are too artificial to be applied by a young computist, to every case that may occur, without considerable danger of misapprehension and error.

The

The triangle is susceptible of two forms and solutions

when $B < \frac{1}{4}O$ .... $a < \frac{1}{4}O$ .... $b < a$,

$B < \frac{1}{4}O$ .... $a > \frac{1}{4}O$ .... $b < \frac{1}{2}O - a$,

$B > \frac{1}{4}O$ .... $a < \frac{1}{4}O$ .... $b > \frac{1}{2}O - a$,

$B > \frac{1}{4}O$ .... $a > \frac{1}{4}O$ .... $b > a$,

$B <$ or $> \frac{1}{4}O$ .... $a = \frac{1}{4}O$.

2. With the data A, B, and b, the triangle can exist, but in one form,

when $b = \frac{1}{4}O$ (one quadrant),

$b > \frac{1}{4}O$ .... $A > \frac{1}{4}O$ .... $B < A$,

$b > \frac{1}{4}O$ .... $A < \frac{1}{4}O$ .... $B < \frac{1}{2}O - A$,

$b < \frac{1}{4}O$ .... $A > \frac{1}{4}O$ .... $B > \frac{1}{2}O - A$,

$b < \frac{1}{4}O$ .... $A < \frac{1}{4}O$ .... $B > A$.

It is susceptible of two forms,

when $b > \frac{1}{4}O$ .... $A > \frac{1}{4}O$ .... $B > A$,

$b > \frac{1}{4}O$ .... $A < \frac{1}{4}O$ .... $B > \frac{1}{2}O - A$,

$b < \frac{1}{4}O$ .... $A > \frac{1}{4}O$ .... $B < \frac{1}{2}O - A$,

$b < \frac{1}{4}O$ .... $A < \frac{1}{4}O$ .... $B < A$,

$b <$ or $> \frac{1}{4}O$ .... $A = \frac{1}{4}O$.

It may here be observed, that all the analogies and formulæ, of spherical trigonometry, in which *cosines* or *cotangents* are not concerned, may be applied to *plane* trigonometry; taking care to use only a *side* instead of the *sine* or the *tangent of a side*; or the *sum* or *difference* of the sides instead of the *sine* or *tangent* of such sum or difference. The reason of this is obvious: for analogies or theorems raised, not only from the consideration of a triangular figure, but the curvature of the sides also, are of consequence more general; and therefore, though the curvature should be deemed evanescent, by reason of a diminution of the surface, yet what depends on the *triangle* alone will remain, notwithstanding

We have now deduced all the rules that are essential in the opperations of spherical trigonometry; and explained under what limitations ambiguities may exist. That the student, however, may want nothing further to direct his practice in this branch of science, we shall add three tables, in which the several formulæ, already given, are respectively applied to the solution of all the cases of right and oblique-angled spherical triangles, that can possibly occur.

TABLE I.

For the Solution of all the cases of Right-Angled Spherical Triangles.

Given.	Required.	Values of the terms required.	Cases in which the terms required are less than 90°.
I. Hypothenuse, and one leg.	Angle opposite to the given leg.	Its sin $= \frac{\text{sin given leg}}{\text{sin hypoth.}}$.	If the given leg be less than 90°.
	Angle adjacent to the given leg.	Its cos $= \frac{\text{tan given leg}}{\text{tan hypoth.}}$.	If the things given be of the same affection.
	Other leg.	Its cos $= \frac{\text{cos hypoth.}}{\text{cos given leg}}$.	Idem.
II. One leg and its opposite angle.	Hypothenuse.	Its sin $= \frac{\text{sin given leg}}{\text{sin given ang.}}$	Ambiguous.
	Other leg.	Its sin $= \frac{\text{tan given leg}}{\text{tan given ang.}}$.	Idem.
	Other angle.	Its sin $= \frac{\text{cos given ang.}}{\text{cos given leg}}$.	Idem.
III. One leg, and the adjacent angle.	Hypothenuse.	Its tan $= \frac{\text{tan given leg}}{\text{cos given ang.}}$.	If the things given be of like affection.
	Other angle.	Its cos $=$ cos giv. leg $\times$ sin giv. ang.	If the given leg be less than 90°.
	Other leg.	Its tan $=$ sin giv. leg. tan giv. ang.	If the given angle be less than 90°.

IV. Hypothenuse, and one angle.	Adjacent leg.	Its tan = tan hyp. × cos giv. ang.	If the things given be of like affection.
	Leg opp. to the given angle.	Its sin = sin hyp. × sin giv. ang.	If the given angle be acute.
	Other angle.	Its tan = $\frac{\text{cot giv. angle}}{\text{cos hypothen.}}$.	If the things given be of like affection.
V. The two legs.	Hypothenuse.	Its cos = rectan. cos giv. legs.	If the given legs be of like affection.
	Either of the angles.	Its tan = $\frac{\text{tan oppos. leg}}{\text{sin adjac. leg}}$.	If the opposite leg be less than 90°.
VI. The two angles.	Hypothenuse.	Its cos = rect. cot giv. angles.	If the angles be of like affection.
	Either of the legs.	Its cos = $\frac{\text{cos opposite angle}}{\text{sin adjacent angle}}$.	If the opposite angle be acute.

In working by the logarithms, the student must observe that when the resulting logarithm is the log. of a quotient, 10 must be *added* to the index; when it is the log. of a product, 10 must be *subtracted* from the index. Thus when the two angles are given,

Log. cos hypothen. = log. cos one angle + log. cos other angle − 10:
Log. cos either leg = log. cos opp. angle − log. sin adjac. angle + 10.

In a quadrantal triangle, if the quadrantal side be called radius, the supplement of the angle opposite to that side be called hypothenuse, the other sides be called angles, and their opposite angles be called legs: then the solutions of all the cases will be as in this table; merely changing *like* for *unlike* in the determinations.

TABLE II.—*For the Solution of Oblique-Angled Spherical Triangles.*

An angle or a side being divided by a perpendicular, the first and second segments are denoted by 1 seg. and 2 seg.

Given.	Required.		Values of the Quantities Required.
I. Two angles and a side opposite to one of them.	The side opp. to other angle.	By the common analogy.	Sines of angles are as sines of oppos. sides.
	Third side.	Let fall a per. on the side contained between the given angles.	Tan 1 seg. of this side = cos adj. angle × tan given side. Sin 2 seg. = $\frac{\text{sin 1 seg.} \times \text{tan ang. adj. given side}}{\text{tan ang. opp. given side}}$.
	Third angle	Let fall a per. as before.	Cot 1 seg. of this ang. = cos giv. side × tan adj. angle. Sin 2 seg. = $\frac{\text{sin 1 seg.} \times \text{cos ang. opp. given side}}{\text{cos ang adj. given side}}$.
II. Two sides and an angle opposite to one of them.	The angle opp to the other side.	By the common analogy.	Sines of sides are as sines of their opposite angles.
	Angle included between the given sides.	Let fall a perpendicular from the included angle.	Cot 1 seg. ang. req. = tan giv. ang. × cos. adj. side. Cos 2 seg. = $\frac{\text{cos 1 seg.} \times \text{tan giv. side adj. giv. angle}}{\text{tan side opp. given angle}}$.
	Third side.	Let fall a perpendicular as before.	Tan 1 seg. side req. = cos given ang. × tan adj. side. Cos 2 seg. = $\frac{\text{cos 1 seg.} \times \text{cos side opp. given angle}}{\text{cos side adj. given angle}}$.

III. Two sides and the includ. angle.	An angle oppos. to one of the giv. sides.	Let fall a perpen. from the third angle.	Tan 1 seg. of div. side = cos giv. ang. × tan side opp. ang. sought. Tan ang. sought = $\frac{\text{tan giv. ang.} \times \text{sin 1 seg.}}{\text{sin 2 seg. of div. side}}$.
	Third side.	Let fall a perpen. on one of the giv. sides.	Tan 1 seg. of div. side = cos giv. ang. × tan other given side. Cos. side sought = $\frac{\text{cos side not div.} \times \text{cos 2 seg.}}{\text{cos 1 seg. of side divided}}$.
IV. A side and the two adjacent angles.	A side oppos. to one of the given angles.	Let fall a perpendicular on the third side.	Cot 1 seg. of div. ang. = cos giv. side × tan ang. opp. side sought. Tan side sought = $\frac{\text{tan giv. side} \times \text{cos 1 seg. div. ang.}}{\text{cos 2 seg. of divided angle}}$.
	Third angle.	Let fall a perpen. from one of the giv. angles.	Cot 1 seg. div. ang. = cos giv. side × tan other giv. angle. Cos angle sought = $\frac{\text{cos ang. not div.} \times \text{sin 2 seg.}}{\text{sin 1 seg. div. angle}}$.
V. The three sides.	An angle by the sine or cosine of its half.	Let a, b, c, be the sides; A, B, C, the angles, b and c including the angle sought, and $s = a + b + c$. Then, $\sin \frac{1}{2} A. = \sqrt{\frac{\sin(\frac{1}{2}s - b) . \sin(\frac{1}{2}s - c)}{\sin b . \sin c}}$ ...... $\cos \frac{1}{2} A = \sqrt{\frac{\sin \frac{1}{2}s . \sin(\frac{1}{2}s - a)}{\sin b . \sin c}}$.	
VI. The three angles.	A side by the sine or cosine of its half.	Let s be the sum of the angles A, B, and C; and let B and C be adjacent to a the side required. Then, $\sin \frac{1}{2} a = \sqrt{-\frac{\cos \frac{1}{2}s . \cos(\frac{1}{2}s - A)}{\sin B . \sin C}}$ .... $\cos \frac{1}{2} a = \sqrt{-\frac{\sin(\frac{1}{2}s - B) . \sin(\frac{1}{2}s - C)}{\sin B . \sin C}}$	

TABLE III.

For the Solution of all the cases of Oblique-Angled Spherical Triangles, by the Analogies of Napier.

Given.	Required.	Values of the Terms required.
I. Two angles and one of their opposite sides.	Side opp. to the other given angle.	By the common analogy, sines of angles as sines of opp. sides.
	Third side.	Tan of its half $= \frac{\text{tan } \frac{1}{2} \text{ diff. giv. sides} \times \text{sin } \frac{1}{2} \text{ sum opp. angles}}{\text{sin } \frac{1}{2} \text{ diff. of those angles}}$. $= \frac{\text{tan } \frac{1}{2} \text{ sum giv. sides} \times \text{cos } \frac{1}{2} \text{ sum opp. angles}}{\text{cos } \frac{1}{2} \text{ diff. of those angles}}$.
	Third angle.	By the common analogy.
II. Two sides, and an opposite angle.	Angle opposite to the other known side.	By the common analogy.
	Third angle.	Cot of its half $= \frac{\text{tan } \frac{1}{2} \text{ diff. other two ang.} \times \text{sin } \frac{1}{2} \text{ sum giv. sides}}{\text{sin } \frac{1}{2} \text{ diff. those sides}}$. $= \frac{\text{tan } \frac{1}{2} \text{ sum of other two ang.} \times \text{cos } \frac{1}{2} \text{ sum giv. sides}}{\text{cos } \frac{1}{2} \text{ diff. of these sides}}$.
	Third side.	By the common analogy.

III. Two sides, and the included angle.	The other two angles.	Tan $\frac{1}{2}$ their diff. $= \frac{\text{cot } \frac{1}{2} \text{ giv. ang.} \times \text{sin } \frac{1}{2} \text{ diff. giv. sides}}{\text{sin } \frac{1}{2} \text{ sum of those sides}}$. Tan $\frac{1}{2}$ their sum $= \frac{\text{cot } \frac{1}{2} \text{ giv. ang.} \times \text{cos } \frac{1}{2} \text{ diff. giv. sides}}{\text{cos } \frac{1}{2} \text{ sum of those sides}}$.
	Third side.	By the common analogy.
IV. Two angles, and the side between them.	The other two sides.	Tan $\frac{1}{2}$ their diff. $= \frac{\text{tan } \frac{1}{2} \text{ giv. side} \times \text{sin } \frac{1}{2} \text{ diff. giv. angles.}}{\text{sin } \frac{1}{2} \text{ sum of those angles.}}$. Tan $\frac{1}{2}$ their sum $= \frac{\text{tan } \frac{1}{2} \text{ giv side} \times \text{cos } \frac{1}{2} \text{ diff. giv. angles}}{\text{cos } \frac{1}{2} \text{ sum of those angles}}$.
	Third angle.	By the common analogy.
V. The three sides.	Either of the angles.	Let fall a perpen. on the side adjacent to the angle sought. Tan $\frac{1}{2}$ sum or $\frac{1}{2}$ diff. of the seg. of the base $= \frac{\text{tan } \frac{1}{2} \text{ sum} \times \text{tan } \frac{1}{2} \text{ diff. of the sides}}{\text{tan } \frac{1}{2} \text{ base}}$. Cos angle sought = tan adj. seg. $\times$ cot adja. side.
VI. The three angles.	Either of the sides.	Will be obtained by finding its correspondent angle, in a triangle which has all its parts supplemental to those of the triangle whose three angles are given.

Questions for Exercise in Spherical Trigonometry.

Ex. 1. In the right-angled spherical triangle BAC, right-angled at A, the hypothenuse $a = 78°20'$, and one leg $c = 76°52'$, are given: to find the angles B, and C, and the other leg b.

Here, by table I case 1, $\sin C = \frac{\sin c}{\sin a}$;

$\cos B = \frac{\tan c}{\tan a}$; .... $\cos b = \frac{\cos a}{\cos c}$.

Or, $\log \sin C = \log \sin c - \log \sin a + 10.$
$\log \cos B = \log \tan c - \log \tan a + 10.$
$\log \cos b = \log \cos a - \log \cos c + 10.$

Hence, $10 + \log \sin c = 10 + \log \sin 76°52' = 19{\cdot}9884894$
$\log \sin a = \log \sin 78°20' = 9{\cdot}9909338$

Remains, $\log \sin C = \log \sin 83°56' = 9{\cdot}9975556$

Here C is acute, because the given leg is less than 90°.

Again, $10 + \log \tan c = 10 + \log \tan 76°52' = 20{\cdot}6320468$
$\log \tan a = \log \tan 78°20' = 10{\cdot}6851149$

Remains, $\log \cos B = \log \cos 27°45' = 9{\cdot}9469319$

B is here acute, because a and c are of like affection.

Lastly, $10 + \log \cos a = 10 + \log \cos 78°20' = 19{\cdot}3058189$
$\log \cos c = \log \cos 76°52' = 9{\cdot}3564426$

Remains, $\log \cos b = \log \cos 27°\ 8' = 9{\cdot}9493763$

where b is less than 90°, because a and c both are so.

Ex. 2. In a right-angled spherical triangle, denoted as above, are given $a = 78°20'$, $B = 27°45'$; to find the other sides and angle.

Ans. $b = 27°\ 8'$, $c = 76°52'$, $C = 83°56'$.

Ex. 3. In a spherical triangle, with A a right angle, given $b = 117°34'$, $c = 31°51'$; to find the other parts.

Ans. $a = 113°55'$, $C = 28°51'$, $B = 104°\ 8'$.

Ex. 4. Given $b = 27°6'$, $c = 76°52'$; to find the other parts. Ans. $a = 78°20'$, $B = 27°45'$, $C = 83°56'$.

Ex. 5. Given $b = 42°12'$, $B = 48°$; to find the other parts.

Ans. $a = 64°40'\frac{1}{2}$, or its supplement,
$c = 54°44'$, or its supplement,
$C = 64°35'$, or its supplement.

Ex. 6. Given $B = 48°$, $C = 64°35'$; required the other parts? Ans. $b = 42°12'$, $c = 54°44'$, $a = 64°40'\frac{1}{2}$.

Ex.

Ex. 7. In the quadrantal triangle ABC, given the quadrantal side $a = 90°$, an adjacent angle $C = 42° \ 12'$, and the opposite angle $A = 64° \ 40'$; required the other parts of the triangle?

Ex. 8. In an oblique-angled spherical triangle are given the three sides, viz, $a = 56° \ 40'$, $b = 83° \ 13'$, $c = 114° \ 30'$: to find the angles.

Here, by the fifth case of table 2, we have

$$\sin \tfrac{1}{2}A = \sqrt{\frac{\sin(\frac{1}{2}s - b) \cdot \sin(\frac{1}{2}s - c)}{\sin b \cdot \sin c}}:$$

Or, log sin $\frac{1}{2}$A = log sin $(\frac{1}{2}s - b)$ + log sin $(\frac{1}{2}s - c)$ + ar. comp. log sin b + ar. comp. log sin c: where $s = a + b + c$.

log sin $(\frac{1}{2}s - b)$ =	log sin 43° 58′$\frac{1}{2}$	= 9·8415749
log sin $(\frac{1}{2}s - c)$ =	log sin 12° 41′$\frac{1}{2}$	= 9·3418385
A. C. log sin b =	A . C . log sin 83° 13′	= 0·0030508
A. C. log sin c =	A . C . log sin 114° 30′	= 0·0409771
Sum of the four logs		19·2274413
Half sum = log sin $\frac{1}{2}$A =	log sin 24° 15′$\frac{1}{2}$	= 9·6137206

Consequently the angle A is 48°31′

Then, by the common analogy,

As, sin a . . .	sin 56°40′ . . .	log = 9·9219401
To, sin A . . .	sin 48°31′ . . .	log = 9·8745679
So is, sin b . . .	sin 83°13′ . . .	log = 9·9969492
To, sin B . . .	sin 62°56′ . . .	log = 9·9495770
And so is, sin c . . .	sin 114°30′ . . .	log = 9·9590229
To, sin C . . .	sin 125°19′ . . .	log = 9·9116507.

So that the remaining angles are, B = 62°56′, and C = 125°19′.

2dly. By way of comparison of methods, let us find the angle A, by the analogies of Napier, according to case 5 table 3. In order to which, suppose a perpendicular demitted from the angle C on the opposite side c. Then shall we have tan $\frac{1}{2}$ diff seg. of $c = \frac{\tan \frac{1}{2}(b+a) \cdot \tan \frac{1}{2}(b-a)}{\tan \frac{1}{2}c}$.

This in logarithms, is

log tan $\frac{1}{2}(b + a)$ =	log tan 69°56′$\frac{1}{2}$ =	10·4375601
log tan $\frac{1}{2}(b - a)$ =	log tan 13°16′$\frac{1}{2}$ =	9·3727819
	Their sum =	19·8103420.
Subtract log tan $\frac{1}{2}c$ =	log tan 57°15′ =	10·1916394
Rem. log cos dif. seg. =	log cos 22°34′ =	9·6187026

Hence, the segments of the base are 79°49′ and 34°41′.

 Therefore,

Therefore, since $\cos A = \tan 79°49' \times \cot b$:

To log tan adja. seg.	= log tan 79°49′ =	10·7456257
Add log tan side b	= log tan 83°13′ =	9·0753563
The sum rejecting 10 from the index log cos A = log cos 48°32′	=	9·8209820

The other two angles may be found as before. The preference is, in this case, manifestly due to the former method.

Ex. 9. In an oblique-angled spherical triangle, are given two sides equal to 114°40′ and 56°30′ respectively, and the angle opposite the former equal to 125°20′; to find the other parts. Ans. Angles 48°30′, and 62°55′; side, 83°12′.

Ex 10 Given, in a spherical triangle, two angles, equal to 48°30′, and 125°20′, and the side opposite the latter; to find the other parts.

Ans. Side opposite first angle, 56°40′; other side, 83°12′; third angle, 62°54′.

Ex. 11. Given two sides, equal 114°30′, and 56°40′; and their included angle 62°54′: to find the rest.

Ex. 12. Given two angles, 125°20′ and 48°30′, and the side comprehended between them 83°12′: to find the other parts.

Ex. 13. In a spherical triangle, the angles are 48°31′, 62°56′, and 125°20′: required the sides?

Ex. 14. Given two angles, 50° 12′, and 58° 8′; and a side opposite the former, 62° 42′; to find the other parts.

Ans. The third angle is either 130°56 or 156°14′.
Side betw. giv. angles, either 119°4′ or 152°14′
Side opp. 58°8′, either 79°12′ or 100°48′.

Ex. 15. The excess of the three angles of a triangle, measured on the earth's surface, above two right angles, is 1 second; what is its area, taking the earth's diameter at 7957¾ miles?

Ans. 76·75299, or nearly 76¾ square miles.

Ex. 16. Determine the solid angles of a regular pyramid, with hexagonal base, the altitude of the pyramid being to each side of the base as 2 to 1.

Ans. Plane angle between each two lateral faces 126°52′11″½.
between the base and each face 66°35′12″½.
Solid angle at the vertex 114·49768 } The max. angle
Each ditto at the base 222·34298 } being 1000.

CHAPTER V.

ON GEODESIC OPERATIONS, AND THE FIGURE OF THE EARTH.

SECTION I.

General Account of this kind of Surveying.

ART. 1. In the treatise on Land Surveying in the first volume of this Course of Mathematics, the directions were restricted to the necessary operations for surveying fields, farms, lordships, or at most counties; these being the only operations in which the generality of persons, who practise this kind of measurement, are likely to be engaged: but there are especial occasions when it is requisite to apply the principles of plane and spherical geometry, and the practices of surveying, to much more extensive portions of the earth's surface; and when of course much care and judgment are called into exercise, both with regard to the direction of the practical operations, and the management of the computations. The extensive processes which we are now about to consider, and which are characterised by the terms *Geodesic Operations* and *Trigonometrical Surveying*, are usually undertaken for the accomplishment of one of these three objects. 1. The finding the difference of longitude, between two moderately distant and noted meridians; as the meridians of the observatories at Greenwich and Oxford, or of those at Greenwich and Paris. 2. The accurate determination of the geographical positions of the principal places, whether on the coast or inland, in an island or kingdom; with a view to give greater accuracy to maps, and to accommodate the navigator with the actual position, as to latitude and longitude, of the principal promontories, havens, and ports. These have, till lately, been desiderata, even in this country: the position of some important points, as the Lizard, not being known within seven minutes of a degree; and, until the publication of the board of Ordnance maps, the best county maps being so erroneous, as in some cases to exhibit *blunders of three miles in distances of less than twenty.*

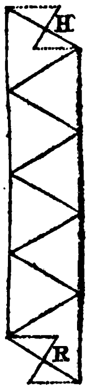
3. The

3. The measurement of a degree in various situations; and thence the determination of the figure and magnitude of the earth.

When objects so important as these are to be attained, it is manifest that, in order to ensure the desirable degree of correctness in the results, the instruments employed, the operations performed, and the computations required, must each have the greatest possible degree of accuracy. Of these, the first depend on the artist; the second on the surveyor, or engineer, who conducts them; and the latter on the theorist and calculator: they are these last which will chiefly engage our attention in the present chapter.

2. In the determination of distances of many miles, whether for the survey of a kingdom, or for the measurement of a degree, the whole line intervening between two extreme points is not *absolutely measured*; for this, on account of the inequalities of the earth's surface, would be always very difficult, and often impossible But, a line of a few miles in length is very carefully measured on some plain, heath, or marsh, which is so nearly level as to facilitate the measurement of an actually horizontal line; and this line being assumed as the base of the operations, a variety of hills and elevated spots are selected, at which signals can be placed, suitably distant and visible one from another: the straight lines joining these points constitute a double series of triangles, of which the assumed base forms the first side; the angles of these, that is, the angles made at each station or signal staff, by two other signal staffs, are carefully measured by a theodolite, which is carried successively from one station to another. In such a series of triangles, care being always taken that one side is common to two of them, all the angles are known from the observations at the several stations, and a side of one of them being given, namely, that of the base measured, the sides of all the rest, as well as the distance from the first angle of the first triangle, to any part of the last triangle, may be found by the rules of trigonometry. And so again, the bearing of any one of the sides, with respect to the meridian, being determined by observation, the bearings of any of the rest, with respect to the same meridian, will be known by computation. In these operations, it is always adviseable, when circumstances will admit of it, to measure another base (called a base of verification) at or near the ulterior extremity of the series: for the length of this base, *computed* as one of the sides of the chain of triangles, compared with its length determined by *actual admeasurement*, will be a test of the accuracy of all the operations made in the series between the two bases.

3. Now



3. Now, in every series of triangles, where each angle is to be ascertained with the same instrument, they should, as nearly as circumstances will permit, be equilateral. For, if it were possible to choose the stations in such manner, that each angle should be exactly 60 degrees; then, the half number of triangles in the series, multiplied into the length of one side of either triangle, would, as in the annexed figure, give at once the total distance; and then also, not only the sides of the scale or ladder, constituted by this series of triangles, would be perfectly parallel, but the diagonal steps, marking the progress from one extremity to the other, would be alternately parallel throughout the whole length. Here too, the first side might be found by a base crossing it perpendicularly of about half its length, as at H; and the last side verified by another such base, R, at the opposite extremity. If the respective sides of the series of triangles were 12 or 18 miles, these bases might advantageously be between 6 and 7, or between 9 and 10 miles respectively; according to circumstances. It may also be remarked, (and the reason of it will be seen in the next section) that whenever only two angles of a triangle can be actually observed, each of them should be as nearly as possible 45°, or the sum of them about 90°; for the less the third or computed angle differs from 90°, the less probability there will be of any considerable error. See prob. 1 sect. 2, of this chapter.

4. The student may obtain a general notion of the method, employed in measuring an arc of the meridian, from the following brief sketch and introductory illustrations.

The earth, it is well known, is nearly spherical. It may be either an ellipsoid of revolution, that is, a body formed by the rotation of an ellipse, the ratio of whose axes is nearly that of equality, on one of those axes; or it may approach nearly to the form of such an ellipsoid or spheroid, while its deviations from that form, though small *relatively*, may still be sufficiently great in themselves, to prevent its being called a spheroid with much more propriety than it is called a sphere. One of the methods made use of to determine this point, is by means of extensive Geodosic operations.

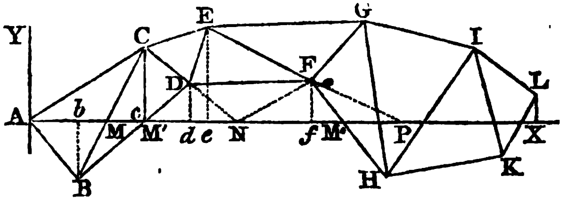
The earth however, be its exact form what it may, is a planet, which not only revolves in an orbit, but turns upon an axis. Now, if we conceive a plane to pass through the axis of rotation of the earth, and through the zenith of any place on its surface, this plane, if prolonged to the limits of the

the apparent celestial sphere, would there trace the circumference of a great circle, which would be the *meridian* of that place. All the points of the earth's surface, which have their zenith in that circumference, will be under the same celestial meridian, and will form the corresponding *terrestrial meridian.* If the earth be an irregular spheroid, this meridian will be a curve of double curvature; but if the earth be a solid of revolution, the terrestrial meridian will be a plane curve.

5. If the earth were a sphere, then every point upon a terrestrial meridian would be at an equal distance from the centre, and of consequence every degree upon that meridian would be of equal length. But if the earth be an ellipsoid of revolution slightly flattened at its poles, and protuberant at the equator; then, as will be shown soon, the degrees of the terrestrial meridian, in receding from the equator towards the poles, will be increased in the duplicate ratio of the right sine of the latitude; and the ratio of the earth's axes, as well as their actual magnitude, may be ascertained by comparing the lengths of a degree on the meridian in different latitudes. Hence appears the great importance of measuring a degree.

6. Now, instead of actually tracing a meridian on the surface of the earth,—a measure which is prevented by the interposition of mountains, woods, rivers, and seas,—a construction is employed which furnishes the same result. It consists in this.

Let ABCDEF, &c, be a series of triangles, carried on, as nearly as may be, in the direction of the meridian, according



to the observations in art. 3. These triangles are really *spherical* or spheroidal triangles; but as their curvature is extremely small, they are treated the same as rectilinear triangles, either by reducing them to the *chords* of the respective terrestrial arcs AC, AB, BC, &c, or by deducting *a third* of the excess, of the sum of the three angles of each triangle above two right angles, from each angle of that triangle, and working with the remainders, and the three sides, as the dimensions of a plane triangle; the proper reductions to the centre of the station, to the horizon, and to the level of the sea, having been previously made. These computations being made throughout

throughout the series, the sides of the successive triangles are contemplated as arcs of the terrestrial spheriod. Suppose that we know, by observation, and the computations which will be explained in this chapter, the *azimuth*, or the inclination of the side AC to the first portion AM of the measured meridian, and that we find by trigonometry, the point M where that curve will cut the side BC. The points A, B, C, being in the same horizontal plane, the line AM will also be in that plane: but, because of the curvature of the earth, the prolongation MM′, of that line, will be found *above* the plane of the second horizontal triangle BCD: if, therefore, without changing the angle CMM, the line MM′ be brought down to coincide with the plane of this second triangle, by being turned about BC as an axis, the point M′ will describe an arc of a circle, which will be so very small, that it may be regarded as a right line perpendicular to the plane BCD: whence it follows, that the operation is reduced to bending down the side MM′ in the plane of the meridian, and calculating the distance AMM′, to find the position of the point M′. By bending down thus in imagination, one after another, the parts of the meridian on the corresponding horizontal triangles, we may obtain, by the aid of the computation, the direction and the length of such meridian, from one extremity of the series of triangles, to the other.

A line traced in the manner we have now been describing, or deduced from trigonometrical measures, by the means we have indicated, is called a *geodetic* or *geodesic line*; it has the property of being the shortest which can be drawn between its two extremities on the surface of the earth; and it is therefore the proper itinerary measure of the distance between those two points. Speaking rigorously, this curve differs a *little* from the terrestrial meridian, when the earth is not a solid of revolution: yet, in the real state of things, the difference between the two curves is so extremely minute, that it may safely be disregarded.

7. If now we conceive a circle perpendicular to the celestial meridian, and passing through the vertical of the place of the observer, it will represent the prime vertical of that place. The series of all the points of the earth's surface which have their zenith in the circumference of this circle, will form the *perpendicular* to the meridian, which may be traced in like manner as the meridian itself.

In the sphere the perpendiculars to the meridian are great circles which all intersect mutually, on the equator, in two points diametrically opposite: but in the ellipsoid of revolution,

tion, and *a fortiori* in the irregular spheroid, these concurring perpendiculars are curves of double curvature. Whatever be the nature of the terrestrial spheroid, the parallels to the equator are curves of which all the points are at the same latitude: on an ellipsoid of revolution, these curves are plane and circular.

8. The situation of a place is determined, when we know either the individual perpendicular to the meridian, or the individual parallel to the equator, on which it is found, and its position on such perpendicular, or on such parallel. Therefore, when all the triangles, which constitute such a series as we have spoken of, have been computed, according to the principles just sketched, the respective positions of their angular points, either by means of their longitudes and latitudes, or of their distances from the first meridian, and from the perpendicular to it. The following is the method of computing these distances.

Suppose that the triangles ABC, BCD, &c, (see the fig. to art. 6) make part of a chain of triangles, of which the sides are arcs of great circles of a sphere, whose radius is the distance from the level or surface of the sea to the centre of the earth; and that we know by observation the angle CAX which measures the *azimuth* of the side AC, or its inclination to the meridian AX. Then, having found the excess E, of the three angles of the triangle ACc (Cc being perpendicular to the meridian) above two right angles, by reason of a theorem which will be demonstrated in prob 8 of this chapter, subtract a third of this excess from each angle of the triangle, and thus, by means of the following proportions find Ac, and Cc.

$$\sin (90° - \tfrac{1}{3}E : \cos (CAc - \tfrac{2}{3}E) :: AC : Ac;$$
$$\sin (90° - \tfrac{1}{3}E : \sin (CAc - \tfrac{1}{3}E) :: AC : Cc.$$

The azimuth of AB is known immediately, because $BAX = CAB - CAX$; and if the spherical excess proper to the triangle ABM′ be computed, we shall have

$$AM'B = 180' - M'AB - ABM' + E'.$$

To determine the sides AM′, BM′, a third of E must be deducted from each of the angles of the triangle ABM′; and then these proportions will obtain: viz,

$$\sin (180° - M'AB - ABM' + \tfrac{2}{3}E') : \sin (ABM' - \tfrac{1}{3}E') :: AB : AM',$$
$$\sin (180° - M'AB - ABM' + \tfrac{2}{3}E) : \sin (M'AB - \tfrac{1}{3}E') :: AB : BM'.$$

In each of the right-angled triangles AbB, M′dD, are known two angles and the hypothenuse, which is all that is necessary to determine the sides Ab, bB, and M′d, dD. Therefore the distances of the points B, D, from the meridian and from the perpendicular, are known.

9. Pro-

9. Proceeding in the same manner with the triangle ACN, or M'DN, to obtain AN and DN, the prolongation of CD; and then with the triangle DNF to find the side NF and the angles DNF, DFN, it will be easy to calculate the rectangular co-ordinates of the point F.

The distance fF and the angles DEN, NFf, being thus known, we shall have (th. 6 cor. 3 Geom.)

$$f\text{FP} = 180° - \text{EFD} - \text{DFN} - \text{NF}f.$$

So that, in the right-angled triangle fFP, two angles and one side are known; and therefore the appropriate spherical excess may be computed, and thence the angle FPf and the sides fP, FP. Resolving next the right-angled triangle eEP, we shall in like manner obtain the position of the point E, with respect to the meridian AX, and to its perpendicular AY; that is to say, the distances Ee, and Ae=AP—eP. And thus may the computist proceed through the whole of the series. It is requisite however, previous to these calculations, to draw, by any suitable scale, the chain of triangles observed, in order to see whether any of the subsidiary triangles ACN, NFP, &c, formed to facilitate the computation of the distances from the meridian, and from the perpendicular to it, are too obtuse or too acute.

Such, in few words, is the method to be followed, when we have principally in view the finding the length of the portion of the meridian comprised between any two points, as A and X. It is obvious that, in the course of the computations, the azimuths of a great number of the sides of triangles in the series is determined; it will be easy therefore to check and verify the work in its process, by comparing the azimuths found by observation, with those resulting from the calculations. The amplitude of the whole arc of the meridian measured, is found by ascertaining the *latitude* at each of its extremities; that is, commonly by finding the differences of the zenith distances of some known fixed star, at both those extremities.

10. Some mathematicians, employed in this kind of operations, have adopted different means from the above. They draw through the summits of all the triangles, parallels to the meridian and to its perpendicular; by these means, the sides of the triangles become the hypothenuses of right-angled triangles, which they compute in order, proceeding from some known azimuth, and without regarding the spherical excess, considering all the triangles of the chain as described on a plane surface. This method, however, is manifestly defective in point of accuracy.

Others have computed the sides and angles of all the triangles, by the rules of spherical trigonometry. Others again,

reduce the observed angles to angles of the chords of the respective arches; and calculate by plane trigonometry, from such reduced angles and their chords. Either of these two methods is equally correct as that by means of the spherical excess: so that the principal reason for preferring one of these to the other, must be derived from its relative facility. As to the methods in which the several triangles are contemplated as spheroidal, they are abstruse and difficult, and may, happily, be safely disregarded: for M. Legendre has demonstrated, in *Mémoires de la Classe des Sciences Physiques et Mathématiques de l' Institut*, 1806, pa. 130, that the difference between spherical and sphěroidal angles, is less than *one sixtieth* of a second, in the greatest of the triangles which occurred in the late measurement of an arc of a meridian, between the parallels of Dunkirk and Barcelona.

11. Trigonometrical surveys for the purpose of measuring a degree of a meridian in different latitudes, and thence inferring the figure of the earth, have been undertaken by different philosophers, under the patronage of different governments. As by M. Maupertius, Clairaut, &c, in Lapland, 1736: by M. Bouguer and Condamine, at the equator, 1736—1743; by Cassini, in lat. 45°, 1739—40; by Boscovich and Lemaire, lat. 43°, 1752; by Beccaria, lat. 44° 44′, 1768; by Mason and Dixon in America, 1764—8; by Major Lambton, in the East Indies, 1803; by Mechain, Delambre, &c, France, &c, 1790—1805; by Swanberg, Ofverbom, &c, in Lapland, 1802; and by General Roy, Colonel Williams, Mr. Dalby, and Colonel Mudge, in England, from 1784 to the present time. The three last mentioned of these surveys are doubtless the most accurate and important.

The trigonometrical survey in England was first commenced, in conjunction with similar operations in France, in order to determine the difference of longitude between the meridians of the Greenwich and Paris observatories: for this purpose, three of the French Academecians, M. M. Cassini, Mechain, and Legendre, met General Roy and Dr. (now Sir Charles) Blagden, at Dover, to adjust their plans of operation. In the course of the survey, however, the English philosophers, selected from the Royal Artillery officers, expanded their views, and pursued their operations, under the patronage, and at the expence of the Honourable Board of Ordnance, in order to perfect the geography of England, and to determine the lengths of as many degrees on the meridian as fell within the compass of their labours.

12. It is not our province to enter into the history of these surveys

surveys: but it may be interesting and instructive to speak a little of the instruments employed, and of the extreme accuracy of some of the results obtained by them.

These instruments are, besides the signals, those for measuring distances, and those for measuring angles. The French philosophers used for the former purpose, in their measurement to determine the length of the *metre*, rulers of platina and of copper, forming metallic thermometers. The Swedish mathematicians, Swanberg and Ofverbom, employed iron bars, covered towards each extremity with plates of silver. General Roy commenced his measurement of the base at Hounslow Heath with *deal* rods, each of 20 feet in length. Though they, however, were made of the best seasoned timber, were perfectly straight, and were secured from bending in the most effectual manner; yet the changes in their lengths, occasioned by the variable moisture and dryness of the air, were so great, as to take away all confidence in the results deduced from them. Afterwards, in consequence of having found by experiments, that a solid bar of glass is more dilatable than a tube of the same matter, glass tubes were substituted for the deal rods. They were each 20 feet long, inclosed in wooden frames, so as to allow only of expansion or contraction in length, from heat or cold, according to a law ascertained by experiments. The base measured with these was found to be 27404·08 feet, or about 5·19 miles. Several years afterwards the same base was remeasured by Colonel Mudge, with a steel-chain of 100 feet long, constructed by Ramsden, and jointed somewhat like a watch-chain. This chain was always stretched to the same tension, supported on troughs laid horizontally, and allowances were made for changes in its length by reason of variations of temperature, at the rate of ·0075 of an inch for each degree of heat from 62° of Fahrenheit: the result of the measurement by this chain was found not to differ more than $2\frac{3}{4}$ inches, from General Roy's determination by means of the glass tubes: a minute difference in a distance of more than 5 miles; which, considering that the measurements were effected by different persons, and with different instruments, is a remarkable confirmation of the accuracy of both operations. And further, as steel chains can be used with more facility and convenience than glass rods, this remeasurement determines the question of the comparative fitness of these two kinds of instruments.

13. For the determination of angles, the French and Swedish philosophers employed *repeating circles* of Borda's construction: instruments which are extremely portable, and with which, though they are not above 14 inches in diameter, the observer

observers can take angles to within 1″ or 2″ of the truth. But this kind of instrument, however great its ingenuity in theory, has the accuracy of its observations necessarily limited by the imperfections of the *small* telescope which must be attached to it. General Roy and Colonel Mudge made use of a very excellent theodolite constructed by Ramsden, which, having both an altitude and an azimuth circle, combines the powers of a theodolite, a quadrant, and a transit instrument, and is capable of measuring horizontal angles to fractions of a second. This instrument, besides, has a telescope of a much higher magnifying power than had ever before been applied to observations purely terrestrial; and this is one of the superiorities in its construction, to which is to be ascribed the extreme accuracy in the results of this trigonometrical survey.

Another circumstance which has augmented the accuracy of the English measures, arises from the mode of fixing and using this theodolite. In the method pursued by the Continental mathematicians, a reduction is necessary to the plane of the horizon, and another to bring the observed angles to the true angles at the centres of the signals: these reductions, of course, require formulæ of computation, the actual employment of which *may* lead to error. But, in the trigonometrical survey of England, great care has always been taken to place the centre of the theodolite exactly in the vertical line, previously or subsequently occupied by the centre of the signal: the theodolite is also placed in a perfectly horizontal position. Indeed, as has been observed by a competent judge, "In no other survey has the work in the field been conducted so much with a view to save that in the closet, and at the same time to avoid all those causes of error, however minute, that are not essentially involved in the nature of the problem. The French mathematicians trust to the *correction* of those errors; the English endeavour to *cut them off* entirely; and it can hardly be doubted that the latter, though perhaps the slower and more expensive, is by far the safest proceeding."

14. In proof of the great correctness of the English survey, we shall state a very few particulars, besides what is already mentioned in art. 12.

General Roy, who first measured the base on Hounslow-Heath, measured another on the flat ground of Romney-Marsh in Kent, near the southern extremity of the first series of triangles, and at the distance of more than 60 miles from the first base. The length of this base of verification, as actually measured, compared with that resulting from the computation through the whole series of triangles, differed only by 28 inches.

Colonel

Colonel Mudge measured another base of verification on Salisbury plain. Its length was 36574·4 feet, or more than 7 miles; the measurement did not differ more than *one inch* from the computation carried through the series of triangles from Hounslow Heath to Salisbury Plain. A most remarkable proof of the accuracy with which all the angles, as well as the two bases, were measured!

The distance between Beachy-Head in Sussex, and Dunnose in the Isle of Wight, as deduced from a mean of four series of triangles, is 339397 feet, or more than 64¼ miles. The extremes of the four determinations do not differ more than 7 feet, which is less than 1⅔ inches in a mile. Instances of this kind frequently occur in the English survey*. But we have not room to specify more. We must now proceed to discuss the most important problems connected with this subject; and refer those who are desirous to consider it more minutely, to Colonel Mudge's "Account of the Trigonometrical Survey;" Mechain and Delambre, "Base du Systéme Métrique Décimal;" Swanberg, "Exposition des Opérations faites en Lapponie;" and Puissant's works entitled "Geodesie" and "Traite de Topographie, d'Arpentage, &c."

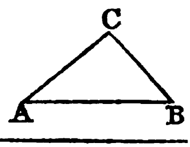
SECTION II.

Problems connected with the detail of Operations in Extensive Trigonometrical Surveys.

PROBLEM I.

It is required to determine the Most Advantageous Conditions of Triangles.

1. In any rectilinear triangle ABC, it is, from the proportionality of sides to the sines of their opposite angles, AB : BC : : sin C : sin A, and consequently AB. sin A = BC. sin C. Let AB be the base, which is supposed to be measured without perceptible error, and which therefore is assumed as constant; then finding the extremely



* Puissant, in his "Geodésie," after quoting some of them, says, "Neanmoins, jusqu'à présent, rienn'egale en exactitude les opérations géodesiques qui ont servi de fondement à *notre* systême métrique." He, however, gives no instances. We have no wish to depreciate the labours of the French measurers; but we cannot yield them the preference on mere assertion.

small

small variation or fluxion of the equation on this hypothesis, it is $AB \cdot \cos A \cdot \dot{A} = \sin C \cdot \dot{BC} + BC \cdot \cos C \cdot \dot{C}$. Here, since we are ignorant of the magnitude of the errors or variations expressed by $\dot{A}$ and $\dot{C}$, suppose them to be equal (a probable supposition, as they are both taken by the same instrument), and each denoted by v: then will

$$\dot{BC} = v \times \frac{AB \cos A - BC \cos C}{\sin C};$$

or, substituting $\frac{BC}{\sin A}$ for its equal $\frac{AB}{\sin C}$, the equation will become $\dot{BC} = v \times \left(BC \cdot \frac{\cos A}{\sin A} - BC \cdot \frac{\cos C}{\sin C}\right)$;
or, finally, $\dot{BC} = v \cdot BC\,(\cot A - \cot C)$.

This equation (in the use of which it must be recollected that v taken in seconds should be divided by R'', that is, by the length of the radius expressed in seconds) gives the error $\dot{BC}$ in the estimation of BC occasioned by the errors in the angles A and C. Hence, that these errors, supposing them to be equal, may have no influence on the determination of BC, we must have $A = C$, for in that case the second member of the equation will vanish.

2. But, as the two errors, denoted by $\dot{A}$, and $\dot{C}$, which we have supposed to be of the same kind, or in the same direction, may be committed in different directions, when the equation will be $\dot{BC} = \pm v \cdot BC\,(\cot A + \cot C)$; we must enquire what magnitude the angles A and C ought to have, so that the sum of their cotangents shall have the least value possible; for in this state it is manifest that $\dot{BC}$ will have its least value. But, by the formulæ in chap. 3, we have

$$\cot A + \cot C = \frac{\sin(A+C)}{\sin A \cdot \sin C} = \frac{\sin(A+C)}{\frac{1}{2}\cos(A \backsim C) - \frac{1}{2}\cos(A+C)} = \frac{2 \sin B}{\cos(A \backsim C) + \cos B}.$$

Consequently, $\dot{BC} = \pm v \cdot BC \cdot \frac{2 \sin B}{\cos(A \backsim) + \cos B}$.

And hence, whatever be the magnitude of the angle B, the error in the value of BC will be the least when $\cos(A \backsim C)$ is the greatest possible, which is when $A = C$.

We may therefore infer, for a general rule, that *the most advantageous state of a triangle, when we would determine one side only, is when the base is equal to the side sought.*

3. Since, by this rule, the base should be equal to the side sought, it is evident that *when we would determine two sides, the most advantageous condition of a triangle is that it be equilateral.*

4. It

4. It rarely happens, however, that a base can be commodiously measured which is as long as the sides sought. Supposing, therefore, that the length of the base is limited, but that its direction at least may be chosen at pleasure, we proceed to enquire what that direction should be, in the case where one only of the other two sides of the triangles is to be determined.

Let it be imagined, as before, that AB is the base of the triangle ABC, and BC the side required. It is proposed to find the least value of $\cot A \mp \cot C$, when we cannot have $A = C$.

Now, in the case where the negative sign obtains, we have

$$\cot A - \cot C = \frac{AB - BC \cdot \cos B}{BC \cdot \sin B} - \frac{BC - AB \cdot \cos B}{AB \cdot \sin B} = \frac{AB^2 - BC^2}{AB \cdot BC \cdot \sin B}.$$

This equation again manifestly indicates the equality of AB and BC, in circumstances where it is possible: but if AB and BC are constant, it is evident, from the form of the denominator of the last fraction, that the fraction itself will be the least, or $\cot A - \cot C$ the least, when $\sin B$ is a maximum, that is, when $B = 90°$.

5. When the positive sign obtains, we have $\cot A + \cot C =$

$$\cot A + \frac{\sqrt{(BC^2 - AB^2 \sin^2 A)}}{AB \sin A} = \cot A + \sqrt{\left(\frac{BC^2}{AB^2 \sin^2 A} - 1\right)}.$$

Here, the least value of the expression under the radical sign, is obviously when $A = 90°$. And in that case the first term, $\cot A$, would disappear. Therefore the least value of $\cot A + \cot C$, obtains when $A = 90°$; conformably to the rule given by M Bouguer (*Fig. de la Terre*, pa. 88). But we have already seen that in the case of $\cot A - \cot C$, we must have $B = 90$. Whence we conclude, since the conditions $A = 90°$, $B = 90°$, cannot obtain simultaneously, that a medium result would give $A = B$.

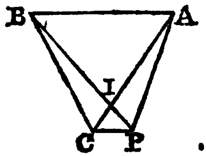
If we apply to the side AC the same reasoning as to BC, similar results will be obtained: therefore in general, *when the base cannot be equal to one or to both the sides required, the most advantageous condition of the triangle is, that the base be the longest possible, and that the two angles at the base be equal.* These equal angles however, should never, if possible, be less than 23 degrees.

PROBLEM II.

To deduce, from Angles measured Out of one of the stations, but Near it, the True Angles at the station.

When the centre of the instrument cannot be placed in the vertical line occupied by the axis of a signal, the angles observed must undergo a reduction, according to circumstances.

1. Let



1. Let C be the centre of the station, P the place of the centre of the instrument, or the summit of the observed angle APB: it is required to find C, the measure of ACB, supposing there to be known APB $=$ P, BPC $= p$, CP $= d$, BC $=$ L, AC $=$ R.

Since the exterior angle of a triangle is equal to the sum of the two interior opposite angles (th. 16 Geom.), we have, with respect to the triangle IAP, AIB $=$ P $+$ IAP; and with regard to the triangle BIC, AIB $=$ C $+$ CBP. Making these two values of AIB equal, and transposing IAP, there results.

$$\text{C} = \text{P} + \text{IAP} - \text{CBP}.$$

But the triangles CAP, CBP, give

$$\sin \text{CAP} = \sin \text{IAP} = \frac{\text{CP}}{\text{AC}} \sin \text{APC} = \frac{d \cdot \sin(\text{P}+p)}{\text{R}}.$$

$$\sin \text{CBP} = \frac{\text{CP}}{\text{BC}} \cdot \sin \text{BPC} = \frac{d \cdot \sin p}{\text{L}}.$$

And, as the angles CAP, CBP, are, by the hypothesis of the problem, always very small, their sines may be substituted for their arcs or measures: therefore

$$\text{C} - \text{P} = \frac{d \sin(\text{P}+p)}{\text{R}} - \frac{d \cdot \sin p}{\text{L}}.$$

Or, to have the reduction in seconds,

$$\text{C} - \text{P} = \frac{d}{\sin 1''} \left(\frac{\sin(\text{P}+p)}{\text{R}} - \frac{\sin p}{\text{L}}\right).$$

The use of this formula cannot in any case be embarrassing, provided the signs of $\sin p$, and $\sin(\text{P}+p)$ be attended to. Thus, the first term of the correction will be positive, if the angle $(\text{P}+p)$ is comprised between 0 and 180°; and it will become negative, if that angle surpass 180°. The contrary will obtain in the same circumstances with regard to the second term, which answers to the angle of direction p. The letter R denotes the distance of the object A to the right, L the distance of the object B situated to the left, and p the angle at the place of observation, between the centre of the station and the object to the left.

2. An approximate reduction to the centre may indeed be obtained by a single term; but it is not quite so correct as the form above. For, by reducing the two fractions in the second member of the last equation but one to a common denominator, the correction becomes

$$\text{C} - \text{P} = \frac{d\text{L} \cdot \sin(\text{P}+p) - d\text{R} \cdot \sin p}{\text{LR}}.$$

But the triangle ABC gives $\text{L} = \frac{\text{R} \cdot \sin \text{A}}{\sin \text{B}} = \frac{\text{R} \cdot \sin \text{A}}{\sin(\text{A}+\text{C})}$.

And

And because P is always very nearly equal to C, the sine of $A + P$ will differ extremely little from $\sin(A+C)$, and may therefore be substituted for it, making $L = \frac{R \sin A}{\sin(A+P)}$.

Hence we manifestly have

$$C - P = \frac{d \cdot \sin A \cdot \sin(P+p) - d \cdot \sin p \cdot \sin(A+P)}{R \cdot \sin A};$$

Which, by taking the expanded expressions, for $\sin(P+p)$, and $\sin(A+P)$, and reducing to seconds, gives

$$C - P = \frac{d}{\sin 1''} \cdot \frac{\sin P \cdot \sin(A-p)}{R \cdot \sin A}.$$

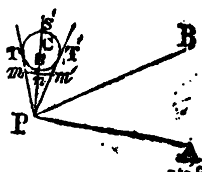
3 When either of the distances R, L, becomes infinite, with respect to d, the corresponding term in the expression art. 1 of this problem, vanishes, and we have accordingly

$$C - P = -\frac{d \cdot \sin p}{L \cdot \sin 1''}, \text{ or } C - P = \frac{d \cdot \sin(P+p)}{R \cdot \sin 1''}.$$

The first of these will apply when the object A is a heavenly body, the second when B is one. When both A and B are such, then $C - P = 0$.

But without supposing either A or B infinite, we may have $C - P = 0$, or $C = P$ in innumerable instances: that is, in all cases in which the centre P of the instrument is placed in the circumference of the circle that passes through the three points A, B, C; or when the angle BPC is equal to the angle BAC, or to BAC + 180°. Whence, though C should be inaccessible, the angle ACB may commonly be obtained by observation, without any computation. It may further be observed, that when P falls in the circumference of the circle passing through the three points A, B, C, the angles A, B, C, may be determined solely by measuring the angles APB and BPC. For, the opposite angles ABC, APC, of the quadrangle inscribed in a circle, are (theor. 54 Geom.) = 180°. Consequently, ABC = 180° — APC, and BAC = 180° — (ABC + ACB) = 180° — (ABC + APB).

4. If one of the objects, viewed from a further station, be a vane or staff in the centre of a steeple, it will frequently happen that such object, when the observer comes near it, is both invisible and inaccessible. Still there are various methods of finding the exact angle at C. Suppose, for example, the signal-staff be in the centre of a circular tower, and that the angle APB was taken at P near its base. Let the tangents PT, PT′, be marked, and on them two equal and arbitrary distances Pm, Pm', be measured. Bisect mm' at the point n; and, placing there a signal-

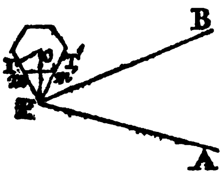


staff

staff, measure the angle npB, which, (since Pn prolonged obviously passes through C the centre,) will be the angle p of the preceding investigation. Also, the distance Ps added to the radius Cs of the tower, will give $PC = d$ in the former investigation.

If the circumference of the tower cannot be measured, and the radius thence inferred, proceed thus: Measure the angles BPT, BPT′, then will $BPC = \frac{1}{2}(BPT + BPT') = p$; and $CPT = BPT - BPC$: Measure PT, then $PC = PT \cdot \sec CPT = d$. With the values of p and d, thus obtained, proceed as before.

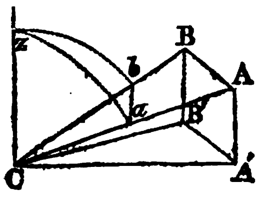
5. If the base of the tower be polygonal and *regular*, as most commonly happens; assume P in the point of intersection of two of the sides prolonged, and $BPC' = \frac{1}{2}(BPT + BPT')$ as before, PT = the distance from P to the middle of one of the sides whose prolongation passes through P; and hence PC is found, as above. If the figure be a regular hexagon, then the triangle Pmm' is equilateral, and $PC = m'm\sqrt{3}$.



PROBLEM III.

To Reduce Angles measured in a Plane Inclined to the Horizon, to the Corresponding Angles in the Horizontal Plane.

Let BCA be an angle measured in a plane inclined to the horizon, and let B′CA′ be the corresponding angle in the horizontal plane. Let d and d' be the zenith distances, or the complements of the angles of elevation ACA′, BCB′. Then from z the zenith of the observer, or of the angle C, draw the arcs za, zb, of vertical circles, measuring the zenith distances d, d', and draw the arc ab of another great circle to measure the angle C. It follows from this construction, that the angle z, of the spherical triangle zab, is equal to the horizontal angle A′CB′; and that, to find it, the three sides $za = d$, $zb = d'$, $ab = c$, are given. Call the sum of these s; then the resulting formulæ of prob. 2 ch. iv, applied to the present instance, becomes



$$\sin \tfrac{1}{2}z = \sin \tfrac{1}{2}C = \sqrt{\frac{\sin \frac{1}{2}(s-d) \cdot \sin \frac{1}{2}(s-d')}{\sin d \cdot \sin d'}}.$$

If

If h and h' represent the angles of altitude ACA′ BCB′, the preceding expression will become

$$\sin \tfrac{1}{2}c = \sqrt{\frac{\sin \frac{1}{2}(c + h - h') \cdot \sin \frac{1}{2}(c + h' - h)}{\cos h \cdot \cos h'}}.$$

Or, in logarithms,

$$\log \sin \tfrac{1}{2}c = \tfrac{1}{2}(20 + \log \sin \tfrac{1}{2}(c + h - h') + \log \sin \tfrac{1}{2}(c + h' - h) - \log \cos h - \log \cos h').$$

Cor. 1. If $h = h'$, then is $\sin \frac{1}{2}c = \frac{\sin \frac{1}{2}\text{ACB}}{\cos h}$; and
$\log \sin \frac{1}{2}\text{A}'\text{CB}' = 10 + \log \sin \frac{1}{2}\text{ACB} - \log \cos h$.

Cor. 2. If the angles h and h' be very small, and nearly equal; then, since the cosines of small angles vary extremely slowly, we may, without sensible error, take
$\log \sin \frac{1}{2}\text{A}'\text{CB}' = 10 + \log \sin \frac{1}{2}\text{ACB} - \log \cos \frac{1}{2}(h + h')$.

Cor. 3. In this case the correction $x = \text{A}'\text{CB}' - \text{ACB}$, may be found by the expression

$$x = \sin 1''(\tan \tfrac{1}{2}c(\tfrac{1}{2}\bigcirc - \tfrac{d + d'}{2})^2 - \cot \tfrac{1}{2}c(\tfrac{d - d'}{2})^2).$$

And in this formula, as well as the first given for $\sin \frac{1}{2}c$, d and d' may be either one or both greater or less than a quadrant; that is, the equations will obtain whether ACA′ and BCB′ be each an elevation or a depression.

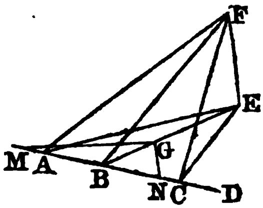
Scholium. By means of this problem, if the altitude of a hill be found barometrically, according to the method described in the 1st volume, or geometrically according to some of those described in heights and distances, or that given in the following problem; then, finding the angles formed at the place of observation, by any objects in the country below, and their respective angles of depression, their horizontal angles, and thence their distances may be found, and their relative places fixed in a map of the country; taking care to have a sufficient number of angles between intersecting lines, to verify the operations.

PROBLEM IV.

Given the Angles of Elevation of any Distant object, taken at Three places in a Horizontal Right Line, which does not pass through the point directly below the object; and the Respective Distances between the stations; to find the Height of the Object, and its Distance from either station.

Let AED be the horizontal plane: FE the perpendicular height of the object F above that plane; A, B, C, the three places of observation; FAE, FBE, FCE, the respective angles of

of elevation, and AB, BC, the given distances. Then, since the triangles AEF, BEF, CEF, are all right angled at E, the distances AE, BE CE, will manifestly be as the cotangents of the angles of elevation at A, B, and C: and we have to determine the point E, so that those lines may have that ratio. To effect this geometrically use the following



Construction. Take BM, on AC produced, equal to BC, BN equal to AB; and make

MG : BM (= BC) : : cot A : cot B,

and BN (= AB) : NG : : cot B : cot C.

With the lines MN, MG, NG, constitute the triangle MNG; and join BG. Draw AE so, that the angle EAB may be equal to MGB; this line will meet BG produced in E, the point in the horizontal plane falling perpendicularly below F.

Demonstration. By the similar triangles AEB, GMB, we have AE : BE : : MG : MB : : cot A : cot B,

and BE : BA(=BN) : : BM : BG.

Therefore the triangles BEC, BGN, are similar; consequently BE : EC : : BN : NG : : cot B : cot C. Whence it is obvious that AE, BE, CE, are respectively as cot A, cot B, cot C.

Calculation. In the triangle MGN, all the sides are given, to find the angle GMN = angle AEB. Then, in the triangle MGB, two sides and the included angle are given, to find the angle MGB = angle EAB. Hence, in the triangle AEB, are known AB and all the angles, to find AE, and BE. And then EF = AE . tan A = BE . tan B.

Otherwise, independent of the construction, thus.

Put AB = D, BC = d, EF = x; and then express algebraically the following theorem, given at p. 128 Simpson's Select Exercises:

$$\text{AE}^2 . \text{BC} + \text{CE}^2 . \text{AB} = \text{BE}^2 . \text{AC} + \text{AC} . \text{AB} . \text{BC},$$

the line EB being drawn from the vertex E of the triangle ACE, to any point B in the base. The equation thence originating is

$$dx^2 . \cot^2 \text{A} + \text{D}x^2 . \cot^2 \text{C} = (\text{D}+d)x^2 . \cot^2 \text{B} + (\text{D}+d)\text{D}d.$$

And from this, by transposing all the unknown terms to one side, and extracting the root, there results

$$x = \sqrt{\frac{(\text{D}+d)\text{D}d}{d . \cot^2 \text{A} + \text{D} . \cot^2 \text{C} - (\text{D}+d) \cot^2 \text{B}}}.$$

Whence

Whence EF is known, and the distances AE, BE, CE, are readily found.

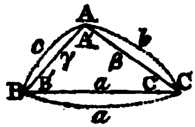
Cor. When D $= d$, or D $+ d =$ 2D $= 2d$, the expression becomes better suited for logarithmic computation, being then

$$x = d \div \sqrt{(\tfrac{1}{2}\cot^2 A + \tfrac{1}{2}\cot^2 C - \cot^2 B)}.$$

In this case, therefore, the rule is as follows: Double the log. cotangents of the angles of elevation of the extreme stations, find the natural numbers answering thereto, and take half their sum; from which subtract the natural number answering to twice the log. cotangent of the middle angle of elevation: then half the log. of this remainder subtracted from the log. of the measured distance between the 1st and 2d, or the 2d and 3d stations, will be the log. of the height of the object.

PROBLEM V.

In any Spherical Triangle, knowing Two Sides and the Included Angle; it is required to find the Angle Comprehended by the Chords of those two sides.



Let the angles of the spherical triangle be A, B, C, the corresponding angles included by the chords A′, B′, C′; the spherical sides opposite the former a, b, c, the chords respectively opposite the latter α, β, γ; then, there are given b, c, and A, to find A′.

Here, from prob. 1 equa. 1 chap. iv, we have

$$\cos a = \sin b \,.\, \sin c \,.\, \cos A + \cos b \,.\, \cos c.$$

But $\cos c = \cos(\frac{1}{2}c + \frac{1}{2}c) = \cos^2 \frac{1}{2}c - \sin^2 \frac{1}{2}c$ (by equa. v ch. iii) $= (1 - \sin^2 \frac{1}{2}c) - \sin^2 \frac{1}{2}c = 1 - 2\sin^2 \frac{1}{2}c$. And in like manner $\cos a = 1 - 2\sin^2 \frac{1}{2}c$, and $\cos b = 1 - 2\sin^2 \frac{1}{2}b$. Therefore the preceding equation becomes

$$1 - 2\sin^2 \tfrac{1}{2}a = 4\sin \tfrac{1}{2}b \,.\, \cos \tfrac{1}{2}b \,.\, \sin \tfrac{1}{2}c \,.\, \cos \tfrac{1}{2}c \,.\, \cos A + (1 - 2\sin^2 \tfrac{1}{2}b) \,.\, (1 - 2\sin^2 \tfrac{1}{2}c).$$

But $\sin \frac{1}{2}a = \frac{1}{2}\alpha$, $\sin \frac{1}{2}b = \frac{1}{2}\beta$, $\sin \frac{1}{2}c = \frac{1}{2}\gamma$: which values substituted in the equation, we obtain, after a little reduction,

$$2 \times \frac{\beta^2 + \gamma^2 - \alpha^2}{4} = \beta\gamma \,.\, \cos \tfrac{1}{2}b \,.\, \cos \tfrac{1}{2}c \,.\, \cos A + \tfrac{1}{4}\beta^2\gamma^2.$$

Now, (equa. II ch. iii), $\cos A' = \dfrac{\beta^2 + \gamma^2 - \alpha^2}{2\beta\gamma}$. Therefore, by substitution,

$$\beta\gamma \,.\, \cos A' = \beta\gamma \,.\, \cos \tfrac{1}{2}b \,.\, \cos \tfrac{1}{2}c \,.\, \cos A + \tfrac{1}{4}\beta^2\gamma^2;$$

whence, dividing by $\beta\gamma$, there results

$$\cos.\, A' = \tfrac{1}{2}b \,.\, \cos \tfrac{1}{2}c \,.\, \cos A + \tfrac{1}{2}\beta \,.\, \tfrac{1}{2}\gamma;$$

or, lastly, by restoring the values of $\frac{1}{2}\beta$, $\frac{1}{2}\gamma$, we have

$$\cos A' = \cos \tfrac{1}{2}b \,.\, \cos \tfrac{1}{2}c \,.\, \cos A + \sin \tfrac{1}{2}b \,.\, \sin \tfrac{1}{2}c \ldots \text{(I.)}$$

Cor.

Cor. 1. It follows evidently from this formula, that when the spherical angle is right or obtuse, it is always *greater* than the corresponding angle of the chords.

Cor. 2. The spherical angle, if acute, is *less* than the corresponding angle of the chords, when we have cos A greater than $\frac{\sin \frac{1}{2}b \cdot \sin \frac{1}{2}c}{1 - \cos \frac{1}{2}b \cdot \sin \frac{1}{2}c}$.

PROBLEM VI.

Knowing Two Sides and the Included Angle of a Rectilinear Triangle, it is required to find the Spherical Angle of the Two Arcs of which those two sides are the chords.

Here β, γ, and the angle A′ are given, to find A. Now, since in all cases, $\cos = \sqrt{(1 - \sin^2)}$, we have

$$\cos \tfrac{1}{2}b \cdot \cos \tfrac{1}{2}c = \sqrt{[(1 - \sin^2 \tfrac{1}{2}b) \cdot (1 - \sin^2 \tfrac{1}{2}c)]};$$

we have also, as above, $\sin \frac{1}{2}b = \frac{1}{2}\beta$, and $\sin \frac{1}{2}c = \frac{1}{2}\gamma$. Substituting these values in the equation I of the preceding problem, there will result, by reduction,

$$\cos A = \frac{\cos A' - \frac{1}{4}\beta\gamma}{\sqrt{(1-\frac{1}{2}\beta) \cdot (1+\frac{1}{2}\beta) \cdot (1-\frac{1}{2}\gamma) \cdot (1+\frac{1}{2}\gamma)}} \ldots \text{(II.)}$$

To compute by this formula, the values of the sides β, γ, must be reduced to the corresponding values of the chords of a circle whose radius is unity. This is easily effected by dividing the values of the sides given in feet, or toises, &c, by such a power of 10, that neither of the sides shall exceed 2, the value of the greatest chord, when radius is equal to unity.

From this investigation, and that of the preceding problem, the following corollaries may be drawn.

Cor. 1 If $c = b$, and of consequence $\gamma = \beta$, then will

$$\cos A' = \cos A \cdot \cos^2 \tfrac{1}{2}c + \sin^2 \tfrac{1}{2}c; \text{ and thence}$$

$$1-2 \sin^2 \tfrac{1}{2}A' = (1-2 \sin^2 \tfrac{1}{2}A) \cos^2 \tfrac{1}{2}c + (1-\cos^2 \tfrac{1}{2}c):$$

from which may be deduced

$$\sin \tfrac{1}{2}A' = \sin \tfrac{1}{2}A \cdot \cos \tfrac{1}{2}c \ldots \text{(III.)}$$

Cor. 2. Also, since $\cos \frac{1}{2}c = \sqrt{(1 - \sin^2 \frac{1}{2}c)} = \sqrt{(1-\frac{1}{4}\gamma^2)}$, equa. II will, in this case, reduce to

$$\sin \tfrac{1}{2}A = \frac{\sin \tfrac{1}{2}A'}{\sqrt{(1-\frac{1}{2}\gamma) \cdot (1+\frac{1}{2}\gamma)}} \ldots \text{(IV.)}$$

Cor. 3. From the equation III, it appears that the vertical angle of an isosceles spherical triangle, is always *greater* than the corresponding angle of the chords.

Cor. 4. If $A = 90°$. the formulæ I, II, give

$$\cos A' = \sin \tfrac{1}{2}b \cdot \sin \tfrac{1}{2}c = \tfrac{1}{4}\beta\gamma \ldots \text{(V.)}$$

These five formulæ are strict and rigorous, whatever be the magnitude of the triangle. But if the triangles be small, the arcs may be put instead of the sines in equa. V, then

Cor. 5. As $\cos A' = \sin (90° - A') =$ in this case, $90° - A'$; the small excess of the spherical right angle over the corresponding

sponding rectilinear angle, will, supposing the arcs b, c, taken in seconds, be given in seconds by the following expression

$$90° - A' = \frac{\frac{1}{4}bc}{R''} = \frac{bc}{4R''} \;\ldots\ldots\; \text{(VI.)}$$

The error in this formula will not amount to a second, when $b + c$ is less than $10°$, or than 700 miles measured on the earth's surface.

Cor. 6. If the hypothenuse does not exceed $1\frac{1}{2}°$, we may substitute $a \sin C$ instead of c, and $a \cos C$ instead of b; this will give $bc = a^2 . \sin C . \cos C = \frac{1}{2}a^2 . \sin 2(90° - B) = \frac{1}{2}a^2 . \sin 2B$: whence

$$(90° - A') = \frac{a^2 . \sin 2C}{8R''} = \frac{a^2 . \sin 2B}{8R''} \;\ldots\ldots\; \text{(VII.)}$$

If $a = 1\frac{1}{2}°$, and $B = C = 45°$ nearly; then will $90° - A' = 17''.7$.

Cor. 7. Retaining the same hypothesis of $A = 90°$, and $a =$ or $< 1\frac{1}{2}°$, we have

$$B - B' = \frac{b^2 \cot B}{8R''} = \frac{bc}{8R''} \;\ldots\ldots\; \text{(VIII.)}$$

$$\text{Also } C - C' = \frac{bc}{8R''} \;\ldots\ldots\ldots\ldots\; \text{(IX.)}$$

Cor. 8. Comparing formulæ VIII, IX, with VI, we have $B - B' = C - C' = \frac{1}{2}(90° - A'.)$ Whence it appears that the sum of the two excesses of the oblique spherical angles, over the corresponding angles of the chords, in a small right-angled triangle, is equal to the excess of the right angle over the corresponding angle of the chords. So that either of the formulæ VI, VII, VIII, IX, will suffice to determine the difference of each of the three angles of a small right-angled spherical triangle, from the corresponding angles of the chords. And hence *this* method may be applied to the measuring an arc of the meridian by means of a series of triangles. See arts. 8, 9, sect. 1 of this chapter.

PROBLEM VII.

In a Spherical Triangle ABC, Right Angled in A, Knowing the Hypothenuse BC (*less than* 4°) and the Angle B, it is required to find the Error e committed through finding by Plane Trigonometry, the Opposite Side AC.

Referring still to the diagram of prob. 5, where we now suppose the spherical angle A to be right, we have (theor. 10 chap. IV) $\sin b = \sin a . \sin B$. But it has been remarked at pa. 381 vol. i, that the sine of any arc A is equal to the sum of the following series;

$$\sin A = A - \frac{A^3}{2.3} + \frac{A^5}{2.3.4.5} - \frac{A^7}{2.3.4.5.6.7} + \&c.$$

$$\text{or, } \sin A = A - \frac{A^3}{6} + \frac{A^5}{120} - \frac{A^7}{5040} + \&c.$$

And,

And, in the present enquiry, all the terms after the second may be neglected, because the 5th power of an arc of 4° divided by 120, gives a quotient not exceeding 0''·01. Consequently, we may assume $\sin b = b - \frac{1}{6}b^3$, $\sin a = a - \frac{1}{6}a^3$; and thus the preceding equation will become,

$$b - \tfrac{1}{6}b^3 = \sin B \, (a - \tfrac{1}{6}a^3)$$

$$\text{or, } b = a \sin B - \tfrac{1}{6}(a^3 \, . \sin B - b^3).$$

Now, if the triangle were considered as rectilinear, we should have $b = a \, . \sin B$; a theorem which manifestly gives the side b or AC too great by $\frac{1}{6}(a^3 \, . \sin B - b^3)$. But, neglecting quantities of the fifth order, for the reason already assigned, the last equation but one gives $b^3 = a^3 \, . \sin^3 B$. Therefore, by substitution, $e = -\frac{1}{6}a^3 \, . \sin B \, (1 - \sin^2 B)$: or, to have this error in seconds, take R'' = the radius expressed in seconds, so shall $e = -a \, . \sin B \, . \dfrac{a^2 \, . \cos^2 B}{6 R'' R''}$.

Cor. 1. If $a = 4°$, and $B = 35°16'$, in which case the value of $\sin B \, . \cos^2 B$ is a maximum, we shall find $e = -4\frac{1}{2}''$.

Cor. 2. If, with the same data, the correction be applied, to find the side c adjacent to the given angle, we should have

$$e' = a \, . \cos B \, \frac{a^2 \, . \sin^2 B}{3R''R''}.$$

So that this error exists in a contrary sense to the other; the one being subtractive, the other additive.

Cor. 3. The data being the same, if we have to find the angle C, the error to be corrected will be

$$e'' = a^2 \, . \frac{\sin 2B}{4R''}.$$

As to the excess of the arc over its chord, it is easy to find it correctly from the expressions in prob. 5: but for arcs that are very small, compared with the radius, a near approximation to that excess will be found in the same measures as the radius of the earth, by taking $\frac{1}{24}$ of the quotient of the cube of the length of the arc divided by the square of the radius.

PROBLEM VIII.

It is required to Investigate a Theorem, by means of which, Spherical Triangles, whose Sides are Small compared with the radius, may be solved by the rules for Plane Trigonometry, without considering the Chords of the respective Arcs or Sides.

Let a, b, c, be the sides, and A, B, C, the angles of a spherical triangle, on the surface of a sphere whose radius is r; then

then a similar triangle on the surface of a sphere whose radius $= 1$, will have for its sides $\frac{a}{r}, \frac{b}{r}, \frac{c}{r}$; which, for the sake of brevity, we represent by α, β, γ, respectively: then, by equa. I chap. iv, we have $\cos A = \frac{\cos \alpha - \cos \beta . \cos \gamma}{\sin \beta . \sin \gamma}$.

Now, r being very great with respect to the sides a, b, c, we may, as in the investigation of the last problem, omit all the terms containing higher than 4th powers, in the series for the sine and cosine of an arc, given at pa. 381 vol. i: so shall we have, without perceptible error,

$$\cos \alpha = 1 - \frac{\alpha^2}{2} + \frac{\alpha^4}{2.3.4} \ldots \sin \beta = \beta - \frac{\beta^3}{2.3}.$$

And similar expressions may be adopted for $\cos \beta$, $\cos \gamma$, $\sin \gamma$. Thus, the preceding equation will become

$$\cos A = \frac{\frac{1}{2}(\beta^2 + \gamma^2 - \alpha^2) + \frac{1}{24}(\alpha^4 - \beta^4 - \gamma^4) - \frac{1}{4}\beta^2\gamma^2}{\beta\gamma(1 - \frac{1}{6}\beta^2 - \frac{1}{6}\gamma^2)}.$$

Multiplying both terms of this fraction by $1 + \frac{1}{6}(\beta^2 + \gamma^2)$, to simplify the denominator, and reducing, there will result,

$$\cos A = \frac{\beta^2 + \gamma^2 - \alpha^2}{2\beta\gamma} + \frac{\alpha^4 + \beta^4 + \gamma^4 - 2\alpha^2\beta^2 - 2\alpha^2\gamma^2 - 2\beta^2\gamma^2}{24\beta\gamma}.$$

Here, restoring the values of α, β, γ, the second member of the equation will be entirely constituted of like combinations of the letters, and therefore the whole may be represented by

$$\cos A = \frac{M}{2bc} + \frac{N}{24bcr^2} \ldots (1.)$$

Let, now, A' represent the angle opposite to the side a, in the rectilinear triangle whose sides are equal in length to the arcs a, b, c; and we shall have

$$\cos A' = \frac{b^2 + c^2 - a^2}{2bc} = \frac{M}{2bc}.$$

Squaring this, and substituting for $\cos^2 A'$ its value $1 - \sin^2 A'$, there will result

$$-4b^2c^2 \sin^2 A' = a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 = N.$$

So that, equa. I, reduces to the form

$$\cos A = \cos A' - \frac{bc}{6r^2} \sin^2 A'.$$

Let $A = A' + x$, then, as x is necessarily very small, its second power may be rejected, and we may assume $\cos A = \cos A' - x . \sin A'$: whence, substituting for $\cos A$ this value of it, we shall have $x = \frac{bc}{6r^2} . \sin A'$.

It hence appears that x is of the second order, with respect to $\frac{b}{r}$ and $\frac{c}{r}$; and of course that the result is exact to quan-

ities within the fourth order. Therefore, because $A = A' + x$,

$$A = A' + \frac{bc}{6r^2} \cdot \sin A'.$$

But, by prob. 2 rule 2, Mensuration of Planes $\frac{1}{2}bc \sin A'$ is he area of the rectilinear triangle, whose sides are a, b, and c.

$$\text{Therefore } A = A' + \frac{\text{area}}{3r^2};$$

$$\text{or } A' = A - \frac{\text{area}}{3r^2}.$$

$$\text{In like manner} \left\{ \begin{array}{l} B' = B - \dfrac{\text{area}}{3r^2}. \\ C' = C - \dfrac{\text{area}}{3r^2}. \end{array} \right.$$

$$\text{And } A' + B' + C' = 180° = A + B + C - \frac{\text{area}}{r^2}.$$

$$\text{or, } \frac{\text{area}}{r^2} = A + B + C - 180°.$$

Whence, since the spherical excess is a measure of the area (th. 5 ch. iv), we have this theorem : viz.

A spherical triangle being proposed, of which the sides are very small, compared with the radius of the sphere; if from each of its angles one third of the excess of the sum of its three angles above two right angles be subtracted, the angles so diminished may be taken for the angles of a rectilinear triangle, whose sides are equal in length to those of the proposed spherical triangle.*

Scholium.

We have already given, at th. 5 chap. iv, expressions for finding the spherical excess, in the two cases, where two sides and the included angle of a triangle are known, and where the three sides are known. A few additional rules may with propriety be presented here.

1. The spherical excess E, may be found in seconds, by the expression $E = \frac{R''s}{r}$; where s is the surface of the triangle $= \frac{1}{2}bc \cdot \sin A = \frac{1}{2}ab \cdot \sin C = \frac{1}{2}ac \cdot \sin B = \frac{1}{2}a^2 \cdot \frac{\sin B \cdot \sin C}{\sin(B + C)}$, r is the radius of the earth, in the same measures as a, b, and c, and $R'' = 206264''.8$, the seconds in an arc equal in length to the radius.

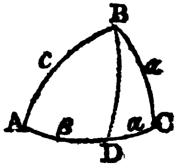
If this formula be applied logarithmically; then log $R'' =$ log $\frac{1}{\text{arc } 1''} = 5.3144251$.

* This curious theorem was first announced by M. Legendre, in the Memoirs of the Paris Academy, for 1787. Legendre's investigation is nearly the same as the above : a shorter investigation is given by Swanberg, at pa. 40, of his " Exposition des Opérations faites en Laponie ;" but it is defective in point of perspicuity.

2. From

2. From the logarithm of the area of the triangle, taken as a plane one, in feet. subtract the constant log 9·3267737 then the remainder is the logarithm of the excess above 180° in seconds nearly*.

3. Since $s = \frac{1}{2}bc \cdot \sin A$, we shall manifestly have $E = \frac{R''}{2r^2}bc \cdot \sin A$. Hence, if from the vertical angle B we demit the perpendicular BD upon the base AC, dividing it into the two segments α, β, we shall have $b = \alpha + \beta$, and thence $E = \frac{R''}{2r^2}c(\alpha + \beta)\sin A = \frac{R''}{2r^2}\alpha c \cdot \sin A + \frac{R''}{2r^2}\beta c \cdot \sin A$. But the two right-angled triangles ABD, CBD, being nearly rectilinear, give $\alpha = a \cdot \cos C$, and $\beta = c \cdot \cos A$; whence we have



$$E = \frac{R''}{2r^2}ac \cdot \sin A \cdot \cos C + \frac{R''}{2r^2}c^2 \cdot \sin A \cdot \cos. A.$$

In like manner, the triangle ABC, which itself is so small as to differ but little from a plane triangle, gives $c \cdot \sin A = a \cdot \sin C$. Also, $\sin A \cdot \cos A = \frac{1}{2}\sin 2A$, and $\sin C \cdot \cos C = \frac{1}{2}\sin 2C$ (equa. xv ch. iii). Therefore, finally,

$$E = \frac{R''}{4r^2}a^2 \cdot \sin 2C + \frac{R''}{4r^2}c^2 \cdot \sin 2A.$$

From this theorem a table may be formed, from which the spherical excess may be found; entering the table with each of the sides above the base and its adjacent angle, as arguments.

4. If the base b and height h, of the triangle are given, then we have evidently $E = \frac{1}{2}bh\frac{R''}{r^2}$. Hence results the following simple logarithmic rule: Add the logarithm of the base of the triangle, taken in feet, to the logarithm of the perpendicular, taken in the same measure; deduct from the sum the logarithm 9·6278037; the remainder will be the common logarithm of the spherical excess in seconds and decimals.

5. Lastly, when the three sides of the triangle are given in feet: add to the logarithm of half their sum, the logs. of the three differences of those sides and that half sum, divide the total of these 4 logs. by 2, and from the quotient subtract the log. 9·3267737; the remainder will be the logarithm of the spherical excess in seconds &c, as before.

One or other of these rules will apply to all cases in whic the spherical excess will be required.

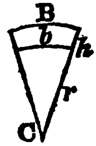
* This is General Roy's rule given in the Philosophical Transactions, for 1790, pa. 171.

PROBLE

PROBLEM IX.

Given the Measure of a Base on any Elevated Level; to find its Measure when Reduced to the Level of the Sea.

Let r represent the radius of the earth, or the distance from its centre to the surface of the sea, $r+h$ the radius referred to the level of the base measured, the altitude h being determined by the rule for the measurement of such altitudes by the barometer and thermometer, (in this volume); let B be the length of the base measured at the elevation h, and b that of the base referred to the level of the sea.



Then because the measured base is all along reduced to the horizontal plane, the two, B and b, will be concentric and similar arcs, to the respective radii $r+h$ and r. Therefore, since similar arcs, whether of spheres or spheroids, are as their radii of curvature, we have

$$r+h : r :: \text{B} : b = \frac{r\text{B}}{r+h}.$$

Hence, also $\text{B} - b = \text{B} - \frac{r\text{B}}{r+h} = \frac{\text{B}h}{r+h}$; or, by actually dividing Bh by $r+h$, we shall have

$$\text{B} - b = \text{B} \times \left(\frac{h}{r} - \frac{h^2}{r^2} + \frac{h^3}{r^3} - \frac{h^4}{r^4} + \&\text{c.}\right)$$

Which is an *accurate* expression for the excess of B above b.

But the mean radius of the earth being more than 21 million feet, if h the difference of level were 50 feet, the second and all succeeding terms of the series could never exceed the fraction $\frac{1}{1760000000000}$; and may therefore safely be neglected: so that for all practical purposes we may assume $\text{B} - b = \frac{\text{B}h}{r}$. Or, in logarithms, add the logarithm of the measured base in feet, to the logarithm of its height above the level of the sea, subtract from the sum the logarithm 7·3223947, the remainder will be the logarithm of a number, which taken from the measured base will leave the reduced base required.

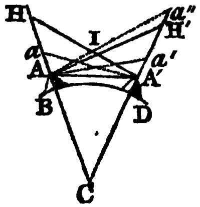
PROBLEM X.

To determine the Horizontal Refraction.

1. Particles of light, in passing from any object through the atmosphere, or part of it, to the eye, do not proceed in a right line; but the atmosphere being composed of an infinitude of strata (if we may so call them) whose density increases as they are posited nearer the earth, the luminous rays which pass

pass through it are acted on as if they passed successively through media of increasing density, and are therefore inflected more and more towards the earth as the density augments. In consequence of this it is, that rays from objects, whether celestial or terrestrial, proceed in curves which are *concave* towards the earth; and thus it happens, since the eye always refers the place of objects to the direction in which the rays reach the eye, that is, to the direction of the tangent to the curve at that point, that the apparent, or observed elevations of objects, are always *greater* than the true ones. The difference of these elevations, which is, in fact, the *effect* of refraction, is, for the sake of brevity, called *refraction:* and it is distinguished into two kinds, *horizontal* or *terrestrial* refraction, being that which affects the altitudes of hills, towers, and other objects on the earth's surface; and *astronomical* refraction, or that which is observed with regard to the altitudes of heavenly bodies. Refraction is found to vary with the state of the atmosphere, in regard to heat or cold, humidity or dryness, &c: so that, determinations obtained for one state of the atmosphere, will not answer correctly for another, without modification. Tables commonly exhibit the refraction at different altitudes, for some assumed mean state.

2. With regard to the *horizontal* refraction the following method of determining it has been successfully practised in the English Trigonometrical Survey.



Let A, A′, be two elevated stations on the surface of the earth, BD the intercepted arc of the earth's surface, C the earth's centre, AH′, A′H, the horizontal lines at A, A′, produced to meet the opposite vertical lines CH′, CH. Let a, a', represent the apparent places of the objects A, A′, then is a'AA′ the refraction observed at A, and aA′A the refraction observed at A′; and half the sum of those angles will be the horizontal refraction, if we assume it equal at each station.

Now, an instrument being placed at each of the stations A, A′, the reciprocal observations are made at the same instant of time, which is determined by means of signals or watches previously regulated for that purpose; that is, the observer at A takes the apparent depression of A′, at the same moment that the other observer takes the apparent depression of A.

In the quadrilateral ACA′I, the two angles A, A′, are right angles, and therefore the angles I and C are together equal to two right angles: but the three angles of the triangle IAA′ are

are together equal to two right angles; and consequently the angles A and A′ are together equal to the angle C, which is measured by the arc BD. If therefore the sum of the two depressions HA′a, H′Aa′, be taken from the sum of the angles HA′A H′AA′, or, which is equivalent, from the angle C (which is known, because its measure BD is known); the remainder is the sum of both refractions, or angles aA′A, a′AA′. Hence this rule, *take the sum of the two depressions from the measure of the intercepted terrestrial arc, half the remainder is the refraction.*

3. If by reason of the minuteness of the contained arc BD, one of the objects, instead of being depressed, appears elevated, as suppose A′ to a″: then the sum of the angles a″AA′ and aA′A will be greater than the sum IAA′+IA′A, or than C, by the angle of elevation a″AA′; but if from the former sum there be taken the depression HA′A, there will remain the sum of the two refractions. So that in this case the rule becomes as follows: *take the depression from the sum of the contained arc and elevation, half the remainder is the refraction.*

5. The quantity of this terrestrial refraction is estimated by Dr. Maskelyne at one-tenth of the distance of the object observed, expressed in degrees of a great circle. So, if the distance be 10000 fathoms, its 10th part 1000 fathoms, is the 60th part of a degree of a great circle on the earth, or 1′, which therefore is the refraction in the altitude of the object at that distance.

But M. Legendre is induced, he says, by several experiments, to allow only $\frac{1}{14}$th part of the distance for the refraction in altitude. So that, on the distance of 10000 fathoms, the 14th part of which is 714 fathoms, he allows only 44″ of terrestrial refraction, so many being contained in the 714 fathoms. See his Memoir concerning the Trigonometrical operations, &c.

Again, M. Delambre, an ingenious French astronomer, makes the quantity of the terrestrial refraction to be the 11th part of the arch of distance. But the English measurers, especially Col. Mudge, from a multitude of exact observations, determine the quantity of the medium refraction to be the 12th part of the said distance.

The quantity of this refraction, however, is found to vary considerably, with the different states of the weather and atmosphere, from the $\frac{1}{7}$th to the $\frac{1}{18}$th of the contained arc. See Trigonometrical Survey, vol. I pa. 160, 355.

Scholium

Scholium.

Having given the mean results of observations on the terrestrial refraction, it may not be amiss, though we cannot enter at large into the investigation, to present here a correct table of mean astronomical refractions. The table which has been most commonly given in books of astronomy is Dr Bradley's, computed from the rule $r = 57'' \times \cot(a + 3r)$, where a is the altitude, r the refraction, and $r = 2'35''$ when $a = 20°$. But it has been found by numerous observations, that the refractions thus computed are rather too *small*.—Laplace, in his Mecanique Celeste (tome iv pa. 27) deduces a formula which is strictly similar to Bradley's; for it is $r = m \times \tan(z - nr)$, where z is the zenith distance, and m and n are two constant quantities to be determined from observation. The only advantage of the formula given by the French philosopher, over that given by the English astronomer, is that Laplace and his colleagues have found more correct coefficients than Bradley had.

Now, if $R = 57°\cdot2957795$, the arc equal to the radius, if we make $m = \frac{kR}{n}$, (where k is a constant coefficient which, as well as n, is an abstract number,) the preceding equation will become $\frac{nr}{R} = k \times \tan(z - nr)$. Here, as the refraction r is always very small, as well as the correction nr, the trigonometrical tangent of the arc nr may be substituted for $\frac{nr}{R}$; thus we shall have $\tan nr = k \cdot \tan(z - nr)$.

But $nr = \frac{1}{2}z - (\frac{1}{2}z - nr) \ldots . z - nr = \frac{1}{2}z + (\frac{1}{2}z - nr)$;

$$\text{Conseq. } \frac{\tan nr}{\tan(z-nr)} = \frac{\tan\left(\frac{z}{2} - \frac{z-2nr}{2}\right)}{\tan\left(\frac{z}{2} + \frac{z-2nr}{2}\right)} = \frac{\sin z - \sin(z-2nr)}{\sin z + \sin(z-2nr)} = k$$

Hence, $\sin(z - 2nr) = \frac{1-k}{1+k} \cdot \sin z$.

This formula is easy to use, when the coefficients n and $\frac{1-k}{1+k}$ are known: and it has been ascertained, by a mean of many observations, that these are 4 and $\cdot99765175$ respectively. Thus Laplace's equation becomes

$$\sin(z - 8r) = \cdot99765175 \sin z:$$

and from this the following table has been computed. Besides the refractions, the differences of refraction, for every 10 minutes of altitude, are given; an addition which will render the table more extensively useful in all cases where great accuracy is required.

Table

Table of Refractions.

***Barom.* 29·92 *inc.Fah. Thermom.* 54°.**

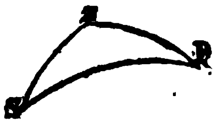
Alt. app.	Refrac.	Diff. on 10′	Alt. app	Refr	Diff. 10′.	Alt app.	Refr	Diff. 10′.	Alt app	Ref.	Diff. 10′.
D. M.	M. S.	S.	D. M.	M. S.	S.	D.	M. S.	S.	D.	S.	S.
0 0	33 46·3	112·0	7 0	7 24·8	9·5	14	3 49·8	2·58	56	39·3	0·25
10	31 54·3	105·0	10	7 15·3	9·0	15	3 34·3	2·28	57	37·8	0·24
20	30 9·3	97·3	20	7 6·3	8·6	16	3 20·6	2·02	58	36·4	0·24
30	28 32·1	89·8	30	6 57·7	8·1	17	3 8·5	1·82	59	35·0	0·23
40	27 2·2	83·6	40	6 49·6	7·7	18	2 57·6	1·66	60	33·6	0·22
50	25 38·6	77·4	50	6 41·9	7·5	19	2 47·7	1·48	61	32·3	0·22
1 0	24 21·2	71·6	8 0	6 34·4	7·3	20	2 38·8	1·37	62	31·0	0·21
10	23 9·6	66·2	10	6 27·1	7·1	21	2 30·6	1·24	63	29·7	0·21
20	22 3·4	61·5	20	6 20·0	6·9	22	2 23·2	1·11	64	28·4	0·20
30	21 1·9	57·1	30	6 13·1	6·7	23	2 16·5	1·05	65	27·2	0·20
40	20 4·8	53·3	40	6 6·4	6·5	24	2 10·2	0·98	66	25·9	0·20
50	19 11·5	49·3	50	5 59·9	6·3	25	2 4·3	0.90	67	24·7	0·20
2 0	18 22·2	45·9	9 0	5 53·6	6·2	26	1 58·9	0.83	68	23·5	0·20
10	17 36·3	43·1	10	5 47·4	5·9	27	1 53·9	0.78	69	22·4	0·20
20	16 53·2	39·8	20	5 41·5	5·7	28	1 49·2	0.73	70	21·2	0·20
30	16 13·4	37·4	30	5 35·8	5·5	29	1 44·8	0.70	71	20·0	0·19
40	15 36·0	35·1	40	5 30·3	5·3	30	1 40·6	0.65	72	18·9	0·18
50	15 0·9	32·8	50	5 25·0	5·2	31	1 36·7	0.60	73	17·8	0·18
3 0	14 28·1	30·8	10 0	5 19·8	5·1	32	1 33·1	0.58	74	16·7	0·18
10	13 57·3	28·8	10	5 14·7	5·0	33	1 29·6	0.56	75	15·6	0·18
20	13 28·5	27·2	20	5 9·7	4·8	34	1 26·2	0.53	76	14·5	0·17
30	13 1·3	25·7	30	5 4·9	4·6	35	1 23·1	0.50	77	13·5	0·17
40	12 35·6	24·3	40	5 0·3	4·4	36	1 20·1	0.48	78	12·4	0·17
50	12 11·3	23·0	50	4 55·9	4·2	37	1 17·2	0.47	79	11·3	0·17
4 0	11 48·3	21·7	11 0	4 51·7	4·1	38	1 14·4	0.43	80	10·3	0·17
10	11 26·6	20·5	10	4 47·6	4·0	39	1 11·8	0.42	81	9·2	0·17
20	11 6·1	19·4	20	4 43·6	4·0	40	1 9·3	0.40	82	8·2	0·17
30	10 46·7	18·4	30	4 39·6	3·9	41	1 6·9	0.38	83	7·2	0·17
40	10 28·3	17·4	40	4 35·7	3·9	42	1 4·6	0.37	84	6·1	0·17
50	10 10·9	16·6	50	4 31·8	3·8	43	1 2·4	0.35	85	5·1	0·17
5 0	9 54·3	15·9	12 0	4 28·0	3·7	44	1 0·3	0.34	86	4·1	0·17
10	9 38·4	15·0	10	4 24·3	3.6	45	0 58·2	0.33	87	3·1	0·17
20	9 23·4	14·4	20	4 20·7	3·5	46	0 56·2	0.32	88	2·0	0·17
30	9 9·0	13·7	30	4 17·2	3·4	47	0 54·3	0.31	89	1·0	0·17
40	8 55·3	13·0	40	4 13·8	3·2	48	0 52·4	0.30	90	0·0	
50	8 42·3	12·4	50	4 10·6	3·1	49	0 50·6	0·29			
6 0	8 29·9	11·8	13 0	4 7·5	3·1	50	0 48·9	0·28			
10	8 18·1	11·5	10	4 4·4	3·0	51	0 47·2	0·27			
20	8 6·6	11·0	20	4 1·4	3·0	52	0 45·5	0·26			
30	7 55·6	10 6	30	3 58·4	2·9	53	0 43·9	0·26			
40	7 45·0	10·3	40	3 55·5	2·9	54	0 42·3	0·25			
50	7 34·7	9·9	50	3 52·6	2·8	55	0 40·8	0·25			
7 0	7 24·8		14 0	3 49·8		56	0 39·3				

PROBLEM XI.

To find the Angle made by a Given Line with the Meridian.

1. The easiest method of finding the angular distance of a given line from the meridian, is to measure the greatest and the least angular distance of the vertical plane in which is the star marked *a* in Ursa minor (commonly called the *pole star*), from the said line: for half the sum of these two measures will manifestly be the angle required.

2. Another method is to observe when the sun is on the given line; to measure the altitude of his centre at that time, and correct it for refraction and parallax. Then, in the spherical triangle ZPS, where Z is the zenith of the place of observation, P the elevated pole, and S the centre of the sun, there are supposed given ZS the zenith distance, or co-altitude of the sun, PS the co-declination of that luminary, PZ the co-latitude of the place of observation, and ZPS the hour angle, measured at the rate of 15° to an hour, to find the angle SZP between the meridian PZ and the vertical ZS, on which the sun is at the given time. And here, as three sides and one angle are known, the required angle is readily found, by saying, as sine ZS : sine ZPS : : sine PS : sine PZS; that is, as the cosine of the sun's altitude, is to the sine of the hour angle from noon; so is the cosine of the sun's declination, to the sine of the angle made by the given vertical and the meridian.

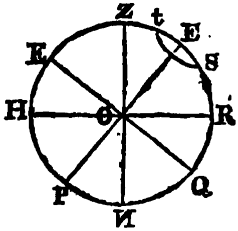


Note. Many other methods are given in books of Astronomy; but the above are sufficient for our present purpose. The first is independent of the latitude of the place; the second requires it.

PROBLEM XII.

To find the latitude of a Place.

The latitude of a place may be found by observing the greatest and least altitude of a circumpolar star, and then applying to each the correction for refraction; so shall half the sum of the altitudes, thus corrected, be the altitude of the pole, or the latitude.



For, if P be the elevated pole, *st* the circle described by the star, PR = EZ the latitude: then since P*s* = P*t*, PR must be = ½(R*t* + R*s*).

This method is obviously independent of the declination of the star: it is therefore most commonly adopted in trigonometrical surveys, in which the telescopes employed are of such power as to enable the observer to see stars in the day-time: the pole-star being here also made use of.

Numerous other methods of solving this problem likewise are given in books of Astronomy; but they need not be detailed here.

Corol. If the mean altitude of a circumpolar star be thus measured, at the two extremities of any arc of a meridian, the difference of the altitudes will be the measure of that arc: and if it be a small arc, one for example not exceeding a degree of the terrestrial meridian, since such small arcs differ extremely little from arcs of the circle of curvature at their middle points, we may, by a simple proportion, infer the length of a degree whose middle point is the middle of that arc.

Scholium.

Though it is not consistent with the purpose of this chapter to enter largely into the doctrine of astronomical spherical problems; yet it may be here added, for the sake of the young student, that if a = right ascension, d = declination, l = latitude, λ = longitude, p = angle of position (or, the angle at a heavenly body formed by two great circles, one passing through the pole of the equator and the other through the pole of the ecliptic), i = inclination or obliquity of the ecliptic, then the following equations, most of which are new, obtain generally, for all the stars and heavenly bodies.

1. $\tan a = \tan \lambda \cdot \cos i - \tan l \cdot \sec \lambda \cdot \sin i$.
2. $\sin d = \sin \lambda \cdot \cos l \cdot \sin i + \sin l \cdot \cos i$.
3. $\tan \lambda = \sin i \cdot \tan d \cdot \sec a + \tan a \cdot \cos i$.
4. $\sin l = \sin d \cdot \cos i - \sin a \cdot \cos d \cdot \sin i$.
5. $\cot p = \cos d \cdot \sec a \cdot \cot i + \sin d \cdot \tan a$.
6. $\cot p = \cos l \cdot \sec \lambda \cdot \cot i - \sin l \cdot \tan \lambda$.
7. $\cos a \cdot \cos d = \cos l \cdot \cos \lambda$.
8. $\sin p \cdot \cos d = \sin i \cdot \cos \lambda$.
9. $\sin p \cdot \cos \lambda = \sin i \cdot \cos a$.

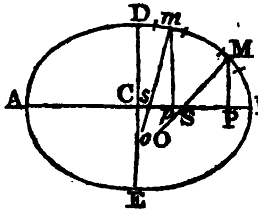
$$\left.\begin{array}{l} 10.\ \tan a = \tan \lambda \cdot \cos i. \\ 11.\ \cos \lambda = \cos a \cdot \cos d. \end{array}\right\} \text{when } l = 0 \text{, as is always the case with the sun.}$$

The

The investigation of these equations, which is omitted for the sake of brevity, depends on the resolution of the spherical triangle whose angles are at the poles of the ecliptic and equator, and the given star, or luminary.

PROBLEM XIII.

To determine the Ratio of the Earth's Axes, and their Actual Magnitude, from the Measure of a Degree or Smaller Portion of a Meridian in Two Given Latitudes; the earth being supposed a spheroid generated by the rotation of an ellipse upon its minor axis.



Let ADBE represent a meridian of the earth, DE its minor axis, AB a diameter of the equator, M, m, arcs of the same number of degrees, or the same parts of a degree, of which the lengths are measured, and which are so small, compared with the magnitude of the earth, that they may be considered as coinciding with arcs of the osculatory circles at their respective middle points; let MO, mo, the radii of curvature of those middle points, be $=$ R and r respectively; MP, mp, ordinates perpendicular to AB: suppose further CD $=c$, CB $=d$; $d^2 - c^2 = e^2$ CP $=x$; C$p=u$; the radius or sine total $=1$; the known angle BSM, or the latitude of the middle point M, $=$ L; the known angle Bsm, or the latitude of the the point m, $=l$; the measured lengths of the arcs M and m being denoted by those letters respectively.

Now the similar sectors whose arcs are M, m, and radii of curvature R, r, give R : r : : M : m; and consequently R$m =$ rM. The central equation to the ellipse investigated at p. 29 of this volume gives PM $= \frac{c}{d}\sqrt{(d^2 - x^2)}$; $pm = \frac{c}{d}\sqrt{(d^2 - u^2)}$; also SP $= \frac{c^2 x}{d^2}$ $sp = \frac{c^2 u}{d^2}$ (by th. 17 Ellipse). And the method of finding the radius of curvature (Flux. art. 74, 75), applied to the central equations above, gives

$\text{R} = \frac{(d^4 - e^2 x^2)^{\frac{3}{2}}}{c^4 d}$; and $r = \frac{(d^4 - e^2 u^2)^{\frac{3}{2}}}{c^4 d}$. On the other hand, the triangle SPM gives SP : PM : : cos L : sin L; that is, $\frac{c^2 x}{d^2} : \frac{c}{d}\sqrt{(d^2 - x^2)}$: : cos L : sin L; whence $x^2 = \frac{d^4 \cos^2 \text{L}}{d^2 - e^2 \sin^2 \text{L}}$.

And from a like process there results, $u^2 = \frac{d^4 \cos^2 l}{d^2 - e^2 \sin^2 l}$

Sub-

Substituting in the equation $\text{R}m = r\text{M}$, for R, and r their values, for x^2 and u^2 their values just found, and observing that $\sin^2 \text{L} + \cos^2 \text{L} = 1$, and $\sin^2 l + \cos^2 l = 1$, we shall find

$$\frac{m}{(d^2 - e^2 \sin^2 \text{L})^{\frac{3}{2}}} = \frac{\text{M}}{(d^2 - e^2 \sin^2 l)^{\frac{3}{2}}},$$

$$\text{or } m(d^2 - e^2 \sin^2 l)^{\frac{3}{2}} = \text{M}(d^2 - e^2 \sin^2 \text{L})^{\frac{2}{3}},$$

$$\text{or } m^{\frac{2}{3}}(d^2 - e^2 \sin^2 l) = \text{M}^{\frac{2}{3}}(d^2 - e^2 \sin^2 \text{L}).$$

From this there arises $e^2 = d^2 - c^2$ (by hyp.) $=$

$$\frac{d^2(\text{M}^{\frac{2}{3}} - m^{\frac{2}{3}})}{\text{M}^{\frac{2}{3}} \sin^2 \text{L} - m^{\frac{2}{3}} \sin^2 l}. \quad \text{But, } \frac{c^2}{d^2} = 1 - \frac{d^2 - c^2}{d^2};$$

and consequently the reciprocal of this fraction, or

$$\frac{d^2}{c^2} = \frac{\text{M}^{\frac{2}{3}} \sin^2 \text{L} - m^{\frac{2}{3}} \sin^2 l}{m^{\frac{2}{3}} \cos^2 l - \text{M}^{\frac{2}{3}} \cos^2 \text{L}} = \frac{(\text{M}^{\frac{1}{3}} \sin \text{L} + m^{\frac{1}{3}} \sin l).(\text{M}^{\frac{1}{3}} \sin \text{L} - \frac{1}{3} m \sin l)}{(m^{\frac{1}{3}} \cos l + \text{M}^{\frac{1}{3}} \cos \text{L}).(m^{\frac{1}{3}} \cos l - \text{M}^{\frac{1}{3}} \cos \text{L})}.$$

Whence, by extracting the root, there results finally

$$\frac{d}{c} = \sqrt{\frac{(\text{M}^{\frac{1}{3}} \sin \text{L} + m^{\frac{1}{3}} \sin l).(\text{M}^{\frac{1}{3}} \sin \text{L} - m^{\frac{1}{3}} \sin l)}{(m^{\frac{1}{3}} \cos l + \text{M}^{\frac{1}{3}} \cos \text{L}).(m^{\frac{1}{3}} \cos l - \text{M}^{\frac{1}{3}} \cos \text{L})}}.$$

This expression, which is simple and symmetrical, has been obtained without any developement into series, without any omission of terms on the supposition that they are indefinitely small, or any possible deviation from correctness, except what may arise from the want of coincidence of the circles of curvature at the middle points of the arcs measured, with the arcs themselves; and this source of error may be diminished at pleasure, by diminishing the magnitude of the arcs measured: though it must be acknowledged that such a procedure may give rise to errors in the practice, which may more than counterbalance the small one to which we have just adverted.

Cor. Knowing the number of degrees, or the parts of degrees, in the measured arcs M, m, and their lengths, which are here regarded as the lengths of arcs to the circles which have R, r, for radii, those radii evidently become known in magnitude. At the same time there are given the algebraic values of R and r: thus, taking R for example, and exterminating e^2 and x^2, there results $\text{R} = \frac{d^4}{c(d^2 - (d^2 - c^2)\sin^2 \text{L})^{\frac{3}{2}}}$. Therefore, by putting in this equation the known ratio of d to c, there will remain only one unknown quantity d or c, which may of course be easily determined by the reduction of the last equation; and thus all the dimensions of the terrestrial spheroid will become known.

General

General Scholium and Remarks.

1. The value $\frac{d}{c}-1, =\frac{d-c}{c}$, is called the *compression* of the terrestrial spheriod, and it manifestly becomes known when the ratio $\frac{d}{c}$ is determined. But the measurements of philosophers, however carefully conducted, furnish resulting compressions, in which the discrepancies are much greater than might be wished. General Roy has recorded several of these in the Phil. Trans. vol 77, and later measurers have deduced others. Thus, the degree measured at the equator by Bouguer, compared with that of France measured by Mechain and Delambre, gives for the compression $\frac{1}{334}$, also $d = 3271208$ toises, $c \doteq 3261443$ toises, $d-c = 9765$ toises. General Roy's sixth spheroid, from the degrees at the equator and in latitude 45°, gives $\frac{1}{309\cdot3}$. Mr. Dalby makes $d =$ 3489932 fathoms, $c = 3473656$. Col. Mudge $d = 3491420$, $c = 3468907$, or 7935 and 7882 miles. The Degree measured at Quito, compared with that measured in Lapland by Swanberg, gives compression $= \frac{1}{309\cdot4}$. Swanberg's observations, compared with Bouguer's give $\frac{1}{329\cdot25}$. Swanberg's compared with the degree of Delambre and Mechain $\frac{1}{307\cdot4}$. Compared with Major Lambton's degree $\frac{1}{307\cdot17}$. A minimum of errors in Lapland, France, and Peru gives $\frac{1}{323\cdot4}$. Laplace, from the lunar motions, finds compression $= \frac{1}{314}$. From the theory of gravity as applied to the latest observations of Burg, Maskelyne, &c, $\frac{1}{309\cdot05}$. From the variation of the pendulum in different latitudes $\frac{1}{335\cdot78}$. Dr. Robison, assuming the variation of gravity at $\frac{1}{180}$, makes the compression $\frac{1}{319}$. Others give results varying from $\frac{1}{178\cdot4}$ to $\frac{1}{577}$: but far the greater number of observations differ but little from $\frac{1}{304}$, which the computation from the phenomena of the precession of the equinoxes and the nutation of the earth's axis, gives for the maximum limit of the compression. 2. From

2. From the various results of careful admeasurements it happens, as Gen. Roy has remarked, "that philosophers are not yet agreed in opinion with regard to the exact figure of the earth; some contending that it has no regular figure, that is, not such as would be generated by the revolution of a curve around its axis. Others have supposed it to be an ellipsoid; regular, if both polar sides should have the same degree of flatness; but irregular if one should be flatter than the other. And lastly, some suppose it to be a spheroid differing from the ellipsoid, but yet such as would be formed by the revolution of a curve around its axis." According to the theory of gravity however, the earth must of necessity have its axes approaching nearly to either the ratio of 1 to 680 or 303 to 304; and as the former ratio obviously does not obtain, the figure of the earth *must* be such as to correspond nearly with the latter ratio.

3. Besides the method above described, others have been proposed for determining the figure of the earth by measurement. Thus that figure might be ascertained by the measurement of a degree in two parallels of latitude; but not so accurately as by meridional arcs, 1st. Because, when the distance of the two stations, in the same parallel is measured, the celestial arc is not that of a parallel circle, but is nearly the arc of a great circle, and always exceeds the arc that corresponds truly with the terrestrial arc. 2dly. The interval of the meridian's passing through the two stations must be determined by a time-keeper, a very small error in the going of which will produce a very considerable error in the computation. Other methods which have been proposed, are, by comparing a degree of the meridian in any latitude, with a degree of the curve perpendicular to the meridian in the same latitude; by comparing the measures of degrees of the curves perpendicular to the meridian in different latitudes; and by comparing an arc of a meridian with an arc of the parallel of latitude that crosses it. The theorems connected with these and some other methods are investigated by Professor Playfair in the Edinburgh Transactions, vol. v, to which, together with the books mentioned at the end of the 1st section of this chapter, the reader is referred for much useful information on this highly interesting subject.

Having thus solved the chief problems connected with Trigonometrical Surveying, the student is now presented with the following examples by way of exercise.

Ex. 1. The angle subtended by two distant objects at a third object is 66°30′39″; one of those objects appeared under an elevation of 25′47″, the other under a depression of 1″. Required the reduced horizontal angle. Ans. 66°30′37″.

Ex. 2.

Ex. 2. Going along a straight and horizontal road which passed by a tower, I wished to find its height, and for this purpose measured two equal distances each of 84 feet, and at the extremities of those distances took three angles of elevation of the top of the tower, viz 36° 50′, 21° 24′ and 14°. What is the height of the tower? Ans. 53·96 feet.

Ex. 3. Investigate General Roy's rule for the spherical excess, given in the scholium to prob. 8.

Ex. 4. The three sides of a triangle measured on the earth's surface (and reduced to the level of the sea) are 17, 18, and 10 miles: what is the spherical excess?

Ex. 5. The base and perpendicular of another triangle are 24 and 15 miles. Required the spherical excess.

Ex. 6. In a triangle two sides are 18 and 23 miles, and they include an angle of 58° 24′ 36″. What is the spherical excess.

Ex. 7. The length of a base measured at an elevation of 38 feet above the level of the sea is 34286 feet: required the length when reduced to that level.

Ex. 8. Given the latitude of a place 48° 51′ N, the sun's declination 18° 30′ N, and the sun's altitude at $10^h\ 11^m\ 26^s$ AM, 52°35′; to find the angle that the vertical on which the sun is, makes with the meridian,

Ex. 9. When the sun's longitude is 29° 13′ 43″, what is his right ascension? The obliquity of the elliptic being 23° 27′ 40″.

Ex. 10. Required the longitude of the sun, when his right ascension and declination are 32° 46′ 52″½. and 13° 13′ 27″. N respectively. See the theorems in the scholium to prob. 12.

Ex. 11. The right ascension of the star α Ursæ majoris is 162° 50′ 34″, and the declination 62° 50′ N: what are the longitude and latitude? The obliquity of the ecliptic being as above.

Ex. 12. Given the measure of a degree on the meridian in N. lat. 49°3′, 60833 fathoms, and of another in N. lat. 12°32′, 60494 fathoms: to find the ratio of the earth's axes.

Ex. 13. Demonstrate that, if the earth's figure be that of an oblate spheroid, a degree of the earth's equator is the first of two mean proportionals between the last and first degrees of latitude.

Ex. 14. Demonstrate that the degrees of the terrestrial meridian, in receeding from the equator towards the poles, are increased

increased very nearly in the duplicate ratio of the sine of the latitude.

Ex. 15. If p be the measure of a degree of a great circle perpendicular to a meridian at a certain point, m that of the corresponding degree on the meridian itself, and d the length of a degree on an oblique arc, that arc making an angle a with the meridian, then is $d = \frac{pm}{p+(m-p)\sin^2 a}$. Required a demonstration of this theorem.

CHAPTER VI.

PRINCIPLES OF POLYGONOMETRY.

The theorems and problems in Polygonometry bear an intimate connection and close analogy to those in plane trigonometry; and are in great measure deducible from the same common principles. Each comprises three general cases.

1. A triangle is determined by means of two sides and an angle; or, which amounts to the same, by its sides except one, and its angles except two. In like manner, any rectilinear polygon is determinable when all its sides except one, and all its angles except two, are known.

2. A triangle is determined by one side and two angles; that is, by its sides except two, and all its angles. So likewise, any rectilinear figure is determinable when all its sides except two, and all its angles, are known.

3. A triangle is determinable by its three sides; that is, when all its sides are known, and all its angles, but three. In like manner, any rectilinear figure is determinable by means of all its sides, and all its angles except three.

In each of these cases, the three unknown quantities may be determined by means of three independent equations; the manner of deducing which may be easily explained, after the following theorems are duly understood.

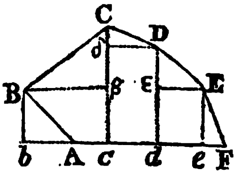
THEOREM I.

In Any Polygon, any One Side is Equal to the Sum of all The Rectangles of Each of the Other Sides drawn into the Cosine of the Angle made by that Side and the Proposed Side*.

*This theorem and the following one, were announced by Mr. Lexel of Petersburg, in Phil. Trans. vol. 65, p. 282: but they were first demonstrated by Dr. Hutton, in Phil. Trans. vol. 66, pa. 600.

Let

Let ABCDEF be a polygon: then will $AF = AB \cdot \cos A + BC \cdot \cos CB^{\wedge}FA + CD \cdot \cos CD^{\wedge}AF + DE \cdot \cos DE^{\wedge}AF + EF^{\wedge}AF$*.



For, drawing lines from the several angles, respectively parallel and perpendicular to AF; it will be

$$Ab = AB \cdot \cos BAF,$$
$$bc = B\beta = BC \cdot \cos CB\beta = BC \cdot \cos CB^{\wedge}AF,$$
$$cd = \delta D = CD \cdot \cos CD\delta = CD \cdot \cos CD^{\wedge}AF,$$
$$de = \varepsilon E = DE \cdot \cos DE\varepsilon = DE \cdot \cos DE^{\wedge}AF,$$
$$eF = \ldots\ldots EF \cdot \cos EFe = EF \cdot \cos EF^{\wedge}AF.$$

But $AF = bc + cd + de + eF - Ab$; and Ab, as expressed above, is in effect subtractive, because the cosine of the obtuse angle BAF is negative. Consequently,
$AF = Ac + cd + de + eF = AB \cdot \cos BAF + BC \cdot \cos CB^{\wedge}AF +$ &c, as in the proposition. A like demonstration will apply, *mutatis mutandis*, to any other polygon.

Cor. When the sides of the polygon are reduced to three, this theorem becomes the same as the fundamental theorem in chap. ii, from which the whole doctrine of Plane Trigonometry is made to flow.

THEOREM II.

The Perpendicular let fall from the Highest Point or Summit of a Polygon, upon the Opposite Side or Base, is Equal to the Sum of the Products of the Sides Comprised between that Summit and the Base, into the Sines of their Respective Inclinations to that Base.

Thus, in the preceding figure, $Cc = CB \cdot \sin CB^{\wedge}FA + BA \cdot \sin A$; or $Cc = CD \cdot \sin CD^{\wedge}AF + DE \cdot \sin DE^{\wedge}AF + EF \cdot \sin F$. This is evident from an inspection of the figure.

Cor. 1. In like manner $Dd = DE \cdot \sin DE^{\wedge}AF + EF \cdot \sin F$, or $Dd = CB \cdot \sin CB^{\wedge}FA + BA \cdot \sin A - CD \cdot \sin CD^{\wedge}AF$.

Cor. 2. Hence, the sum of the products of each side, into the sine of the sum of the *exterior* angles, (or into the sine of the sum of the supplements of the interior angles), comprised between those sides and a determinate side, is $= +$ perp. $-$ perp. or $= 0$. That is to say, in the preceding figure,

$$AB \cdot \sin A + BC \cdot \sin (A + B) + CD \cdot \sin (A + B + C) + DE \cdot \sin (A + B + C + D) + EF \cdot \sin (A + B + C + D + E) = 0.$$

---

* When a caret is put between two letters or pairs of letters denoting lines, the expression altogether denotes the angle which would be made by those two lines if they were produced till they met. thus $CB^{\wedge}FA$ denotes the inclination of the line CB to FA.

Here it is to be observed, that the sines of angles greater than 180° are negative (ch. ii equa. VII).

Cor. 3. Hence again, by putting for sin (A+B), sin (A+B+C), their values sin A . cos B + sin B . cos A, sin A . cos (B + C) + sin (B + C . cos A, &c (ch ii equa. v), and recollecting that $\text{tang} = \frac{\sin}{\cos}$ (ch. ii p. 55), we shall have,

$$\sin A . (AB + BC . \cos B + CD . \cos(B+C) + DE . \cos(B+C+D) + \&c)$$
$$+ \cos A . (BC . \sin B + CD . \sin(B+C) + DE . \cos(B+C+D) + \&c) = 0;$$

and thence finally, tan 180° − A, or tan BAF =

$$\frac{BC . \sin B + CD . \sin(B+C) + DE . \sin(B+C+D) + EF . \sin(B+C+D+E)}{AB + BC . \cos B + CD . \cos(B+C) + DE . \cos(B+C+D) + EF . \cos(B+C+D+E)}$$

A similar expression will manifestly apply to any polygon; and when the number of sides exceeds four, it is highly useful in practice.

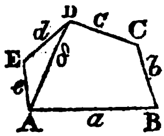
Cor. 4. In a triangle ABC, where the sides AB, BC, and the angle ABC, or its supplement B, are known, we have

$$\tan CAB = \frac{BC . \sin B}{AB + BC \cos B} \ldots . \tan BCA = \frac{AB . \sin B}{BC + AB . \cos B};$$

in both which expressions, the second term of the denominator will become subtractive whenever the angle ABC is acute, or B obtuse.

THEOREM III.

The Squre of Any Side of a Polygon, is Equal to the Sum of the Squares of All the Other Sides, Minus Twice the Sum of the Products of All the Other Sides Multiplied two and two, and by the Cosines of the Angles they Include.



For the sake of brevity, let the sides be represented by the small letters which stand against them in the annexed figure: then, from theor. 1, we shall have the subjoined equations, viz.

$$a = b . \cos a^{\wedge}b + c . \cos a^{\wedge}c + \delta . \cos a^{\wedge}\delta,$$
$$b = a . \cos a^{\wedge}b + c . \cos b^{\wedge}c + \delta . \cos b^{\wedge}\delta,$$
$$c = a . \cos a^{\wedge}c + b . \cos b^{\wedge}c + \delta . \cos c^{\wedge}\delta,$$
$$\delta = a . \cos a^{\wedge}d + b . \cos b^{\wedge}d + c . \cos c^{\wedge}\delta.$$

Multiplying the first of these equations by a, the second by b, the third by c, the fourth by δ; subtracting the three latter products from the first, and transposing b^2, c^2, δ^2, there will result

$$a^2 = b^2 + c^2 + \delta^2 - 2(bc . \cos b^{\wedge}c + b\delta . \cos b^{\wedge}\delta + c\delta . \cos c^{\wedge}\delta).$$

In like manner,

$$c^2 = a^2 + b^2 + \delta^2 - 2(ab . \cos a^{\wedge}b + a\delta . \cos a^{\wedge}\delta + b\delta . \cos b^{\wedge}\delta).$$

&c. &c.

Or,

Or, since $b_A c = \text{C}$, $b_A\partial = \text{C} + \text{D} - 180°$, $c^A\partial = \text{D}$, we have

$$a^2 = b^2 + c^2 + \partial^2 - 2(bc \,.\, \cos \text{C} - b\partial \,.\, \cos(\text{C}+\text{D}) + c\partial \,.\, \cos \text{D}),$$
$$c^2 = a^2 + b^2 + \partial^2 - 2(ab \,.\, \cos \text{B} - b\partial \,.\, \cos(\text{A}+\text{B}) + a\partial \,.\, \cos \text{A}).$$

&c. &c

The same method applied to the pentagon ABCDE, will give

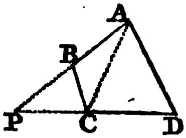
$$a^2 = b^2 + c^2 + a^2 + e^2 - 2 \left\{ \begin{array}{l} bc \,.\, \cos \text{C} - bd \,.\, \cos(\text{C}+\text{D}) + be \,.\, \cos(\text{C}+\text{D}+\text{E}) \\ + cd \,.\, \cos \text{D} - ce \,.\, \cos(\text{D}+\text{E}) + de \,.\, \cos \text{E}. \end{array} \right\}.$$

And a like process is obviously applicable to any number of sides; whence the truth of the theorem is manifest.

Cor. The property of a plane triangle expressed in equa. 1 ch. ii, is only a particular case of this general theorem.

THEOREM IV.

Twice the Surface of Any Polygon, is Equal to the Sum of the Rectangles of its Sides, except one, taken two and two, by the Sines of the Sums of the *Exterior** Angles Contained by those sides.



1. For a trapezium, or polygon of four sides. Let two of the sides AB, DC, be produced till they meet at P. Then the trapezium ABCD is manifestly equal to the difference between the triangles PAD and PBC. But twice the surface of the triangle PAD is (Mens. of Planes pr. 2 rule 2) AP . PD . sin P = (AB + BP) . (DC + CP) . sin P; and twice the surface of the triangle PBC is = BP . PC . sin P: therefore their difference, or twice the area of the trapezium is = (AB . DC + AB . CP + DC . BP) . sin P. Now, in $\triangle$ PBC,

$$\sin \text{P} : \sin \text{B} :: \text{BC} : \text{PC}, \text{ whence PC} = \frac{\text{BC} \,.\, \sin \text{B}}{\sin \text{P}},$$

$$\sin \text{P} : \sin \text{C} :: \text{BC} : \text{PB}, \text{ whence PB} = \frac{\text{BC} \,.\, \sin \text{C}}{\sin \text{P}}.$$

Substituting these values of PB, PC, for them in the above equation, and observing that sin P = sin (PBC + PCB) = sin sum of *exterior* angles B and C, there results at length,

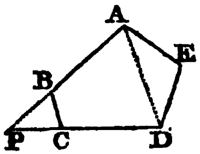
$$\left.\begin{array}{l} \text{Twice surface} \\ \text{of trapezium.} \end{array}\right\} = \left\{ \begin{array}{l} \text{AB . BC . sin B} \\ + \text{AB . DC . sin (B + C)} \\ + \text{BC . DC . sin C.} \end{array} \right.$$

Cor. Since AB . BC . sin B = twice triangle ABC, it follows that twice triangle ACD is equal to the remaining two terms, viz,

$$\text{twice area ACD} = \left\{ \begin{array}{l} \text{AB . DC . sin (B + C)} \\ + \text{BC . DC . sin C.} \end{array} \right.$$

* The *exterior* angles here meant, are those formed by producing the sides in the same manner as in th. 20 Geometry, and in cors. 1, 2, th. 2, of this chap.

2. For



2. For a pentagon, as ABCDE. Its area is obviously equal to the sum of the areas of the trapezium ABCD, and of the triangle ADE. Let the sides AB, DC, as before, meet when produced at P. Then, from the above, we have

$$\left.\begin{array}{c}\text{Twice area of}\\ \text{the trapezium}\\ \text{ABCD}\end{array}\right\}=\left\{\begin{array}{l}\text{AB . BC . sin B}\\ +\text{AB . DC . sin (B + C)}\\ +\text{BC . DC . sin C.}\end{array}\right.$$

And, by the preceding corollary,

$$\left.\begin{array}{c}\text{Twice triangle}\\ \text{DAE}\end{array}\right\}=\left\{\begin{array}{l}\text{AP . DE . sin (P+D) or sin (B+C+D)}\\ +\text{DP . DE . sin D.}\end{array}\right.$$

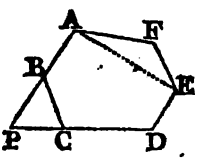
$$\left.\begin{array}{c}\text{That is twice}\\ \text{triangle DAE}\end{array}\right\}=\left\{\begin{array}{l}\text{AB . DE . sin (B+C+D)}\\ +\text{DC . DE . sin D}\\ +\text{BP . DE . sin (B+C+D)}\\ +\text{CP . DE . sin D.}\end{array}\right.$$

Now, $\text{BP} = \frac{\text{BC. sin C}}{\text{sin (B + C)}}$, and $\text{CP} = \frac{\text{BC. sin B}}{\text{sin (B + C)}}$: therefore the last two terms become $\frac{\text{BC . DE . sin C . sin(B+C+D)}}{\text{sin (B+C)}} + \frac{\text{BC.DE . sinB sinD}}{\text{sin (B+C)}}$ $= \text{BC . DE .} \frac{\text{sin B . sin D + sin C . sin (B+C+D)}}{\text{sin (B+C)}}$: and this expression by means of the formula for 4 arcs (art. 30 ch. iii,) becomes BC . DE . sin (C+D). Hence, collecting the terms, and arranging them in the order of the sides, they become

$$\left.\begin{array}{c}\text{Twice the area}\\ \text{of the penta-}\\ \text{gon ABCDE}\end{array}\right\}=\left\{\begin{array}{l}\text{AB . BC . sin B}\\ +\text{AB . DC . sin (B+C)}\\ +\text{AB . DE . sin (B+C+D)}\\ +\text{BC . DC . sin C}\\ +\text{BC . DE . sin (C+D)}\\ +\text{DC . DE . sin D.}\end{array}\right.$$

Cor. Taking away from this expression, the 1st, 2d, and 4th terms, which together make double the trapezium ABCD, there will remain

$$\left.\begin{array}{c}\text{Twice area of}\\ \text{the triangle}\\ \text{DAE.}\end{array}\right\}=\left\{\begin{array}{l}\text{AB . DE . sin (B+C+D)}\\ +\text{BC . DE . sin (C+D)}\\ +\text{DC . DE . sin D.}\end{array}\right.$$



3. For a hexagon, as ABCDEF. The double area will be found, by supposing it divided into the pentagon ABCDE, and the triangle AEF. For, by the last rule, and its corallary, we have,

Twice

$$\left.\begin{array}{r}\text{Twice area of}\\ \text{the pentagon}\\ \text{ABCDE}\end{array}\right\} = \left\{\begin{array}{l} \text{AB . BC . sin B}\\ +\text{AB . CD . sin (B+C)}\\ +\text{AB . DE . sin (B+C+D)}\\ +\text{BC . CD . sin C}\\ +\text{BC . DE . sin (C+D)}\\ +\text{CD . DE . sin D.}\end{array}\right.$$

$$\left.\begin{array}{r}\text{Twice area of}\\ \text{the triangle}\\ \text{AEF}\end{array}\right\} = \left\{\begin{array}{l} \text{AP . EF . sin (B+C+D+E)}\\ +\text{DP . EF . sin (D+E)}\\ +\text{DE . EF . sin E.}\end{array}\right.$$

$$\left.\begin{array}{r}\text{Or, twice area of}\\ \text{the triangle}\\ \text{AEF.}\end{array}\right\} = \left\{\begin{array}{l} \text{AB . EF . sin (B+C+D+E)}\\ +\text{DC . EF . sin (D+E)}\\ +\text{DE . EF . sin E}\\ +\text{BP . EF . sin (B+C+D+E)}\\ +\text{CP . EF . sin (D+E).}\end{array}\right.$$

Now, writing for BP, CP, their respective values, $\frac{\text{BC . sin C}}{\text{sin (B+C)}}$ and $\frac{\text{BC . sin B}}{\text{sin (B+C)}}$, the sum of the last two expressions, in the double areas of AEF, will become

$$\text{BC . EF .} \frac{\text{sin C . sin (B+C+D+E) + sin B . sin (D+E)}}{\text{sin (B+C)}}:$$

and this, by means of the formula for 5 arcs (art. 30 ch. iii) becomes BC . EF sin (C+D+E) Hence, collecting and properly arranging the several terms as before, we shall obtain

$$\left.\begin{array}{r}\text{Twice the area}\\ \text{of the hexa-}\\ \text{gon ABCDEF.}\end{array}\right\} = \left\{\begin{array}{l} \text{AB . BC . sin B}\\ +\text{AB . CD . sin (B+C)}\\ +\text{AB . DE . sin (B+C+D)}\\ +\text{AB . EF . sin (B+C+D+E)}\\ +\text{BC . CD . sin C}\\ +\text{BC . DE . sin (C+D)}\\ +\text{BC . EF . sin (C+D+E)}\\ +\text{CD . DE . sin D}\\ +\text{CD . EF . sin (D+E)}\\ +\text{DE . EF . sin E.}\end{array}\right.$$

4 In a similar manner may the area of a heptagon be determined, by finding the sum of the areas of the hexagon and the adjacent triangle; and thence the area of the octagon, nonagon, or of any other polygon, may be inferred; the law of continuation being sufficiently obvious from what is done above, and the number of terms $= \frac{n-1}{1} \cdot \frac{n-2}{2}$, when the number of sides of the polygon is n: for the number of terms is evidently the same as the number of ways in which $n-1$ quantities can be taken, two and two; that is, (by the nature of Permutations) $= \frac{n-1}{1} \cdot \frac{n-2}{2}$.

Scholium.

Scholium.

This curious theorem was first investigated by *Simon Lhuillier*, and published in 1789. Its principal advantage over the common method for finding the areas of irregular polygons is, that in this method there is no occasion to construct the figures, and of course the errors that may arise from such constructions are avoided.

In the application of the theorem to practical purposes, the expressions above become simplified by dividing any proposed polygon into two parts by a diagonal, and computing the surface of each part separately.

Thus, by dividing the trapezium ABCD into two triangles, by the diagonal AC, we shall have

$$\left.\begin{matrix}\text{Twice area}\\ \text{trapezium}\end{matrix}\right\} = \left\{\begin{matrix} AB . BC . \sin B \\ + CD . AD . \sin D.\end{matrix}\right.$$

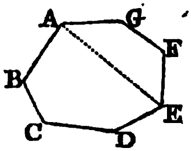
The pentagon ABCDE may be divided into the trapezium ABCD, and the triangle ADE, whence

$$\left.\begin{matrix}\text{Twice area of}\\ \text{pentagon}\end{matrix}\right\} = \left\{\begin{matrix} AB . BC . \sin B \\ + AB . DC . \sin (B+C) \\ + BC . DC . \sin C \\ + DE . AE . \sin E.\end{matrix}\right.$$

Thus again, the hexagon may be divided into two trapeziums, by a diagonal drawn from A to D, which is to be the line excepted in the theorem; then will

$$\left.\begin{matrix}\text{Twice area of}\\ \text{hexagon}\end{matrix}\right\} = \left\{\begin{matrix} AB . BC . \sin B \\ + AB . DC . \sin (B+C) \\ + BC . DC . \sin C \\ + DE . EF . \sin E \\ + DE . AF . \sin (E+F) \\ + EF . AF . \sin F.\end{matrix}\right.$$

And lastly, the heptagon may be divided into a pentagon and a trapezium, the diagonal, as before, being the excepted line: so will the double area be expressed by 9 instead of 15 products, thus :



$$\left.\begin{matrix}\text{Twice area of}\\ \text{heptagon}\end{matrix}\right\} = \left\{\begin{matrix} AB . BC . \sin B \\ + AB . CD . \sin (B+C) \\ + AB . DE . \sin (B+C+D) \\ + BC . CD . \sin C \\ + BC . DE . \sin (C+D) \\ + CD . DE . \sin D \\ + EF . FG . \sin F \\ + EF . GA . \sin (F+G) \\ + FG . GA . \sin G.\end{matrix}\right.$$

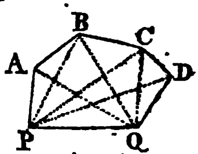
The same method may obviously be extended to other polygons, with great ease and simplicity.

It

It often happens, however, that only one side of a polygon can be measured, and the distant angles be determined by intersection; in this case the area may be found, independent of construction, by the following problem.

PROBLEM. I.

Given the Length of One of the Sides of a Polygon, and the Angles made at its two extremities by that Side and Lines drawn to all the Other Angles of the Polygon; to find an Expression for the Surface of that Polygon.



Here we suppose known PQ; also
$APQ = a'$, $BPQ = b'$, $CPQ = c'$, $DPQ = d'$;
$AQP = a''$, $BQP = b''$, $CQP = c''$, $DQP = d''$.
Now, $\sin PAQ = \sin(a'+a'')$; $\sin PBQ = \sin(b'+q'')$.

Therefore, $\sin(a'+a'') : PQ :: \sin a'' : PA = \dfrac{\sin a''}{\sin(a'+a'')} PC.$

And, . . . $\sin(b'+b'') : PQ :: \sin b'' : PB = \dfrac{\sin b''}{\sin(b'+b'')} PQ.$

But, triangle $APB = AP \cdot PB \cdot \frac{1}{2} \sin APB = \frac{1}{2} AP \cdot PB \cdot \sin(a'-b')$.

Hence, surface $\triangle APB = \frac{1}{2} PQ^2 \cdot \dfrac{\sin a'' \cdot \sin b'' \cdot \sin(a'-b')}{\sin(a'+a'') \cdot \sin(b'+b'')}.$

In like manner, $\triangle BPC = \frac{1}{2} PQ^2 \cdot \dfrac{\sin b'' \cdot \sin c'' \cdot \sin(b'-c')}{\sin(b'+b'') \cdot \sin(c'+c'')}.$

$$\triangle CPD = \tfrac{1}{2} PQ^2 \cdot \frac{\sin c'' \cdot \sin d'' \cdot \sin(c'-d')}{\sin(c'+c'') \cdot \sin(d'+d'')},$$

&c. &c. &c.

$\triangle DPQ = QP \cdot PD \cdot \frac{1}{2} \sin DPQ = PQ \cdot \dfrac{\sin d''}{\sin(d'+d'')} \cdot \frac{1}{2} PQ \cdot \sin d' =$

$\frac{1}{2} PQ^2 \cdot \dfrac{\sin d' \cdot \sin d''}{\sin(d'+d'')}.$ Consequently,

$$\text{Surface } PABCDQ = \tfrac{1}{2} PQ^2 \cdot \left\{ \begin{array}{l} \dfrac{\sin a'' \cdot \sin b'' \cdot \sin(a'-b')}{\sin(a'+a'') \cdot \sin(b'+b'')} \\ + \dfrac{\sin b'' \cdot \sin c'' \cdot \sin(b'-c')}{\sin(b'+b'') \cdot \sin(c'+c'')} \\ + \dfrac{\sin c'' \cdot \sin d'' \cdot \sin(c'-d')}{\sin(c'+c'') \cdot \sin(d'+d'')} \\ + \dfrac{\sin d' \cdot \sin d''}{\sin(d'+d'')} \end{array} \right.$$

The

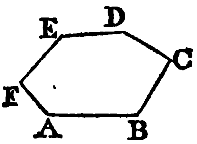
The same method manifestly applies to polygons of *any* number of sides: and all the terms except the last are so perfectly symmetrical, while that last term is of so obvious a form, that there cannot be the least difficulty in extending the formula to any polygon whatever.

PROBLEM II.

Given, in a Polygon, All the Sides and Angles, except three; to find the Unknown Parts.

This problem may be divided into three general cases, as shown at the beginning of this chapter: but the analytical solution of all of them depends on the same principles; and these are analogous to those pursued in the analytical investigations of plane trigonometry. In polygonometry, as well as trigonometry, when three unknown quantities are to be found, it must be by means of three independent equations, involving the known and unknown parts. These equations may be deduced from either theorem 1, or 3, as may be most suited to the case in hand; and then the unknown parts may each be found by the usual rules of extermination.

For an exsmple, let it be supposed that in an irregular hexagon ABCDEF, there are given all the sides except AB, BC, and all the angles except B; to determine those three quantities.



The angle B is evidently equal to $(2n-4)$ right angles $-$ $(A + C + D + E + F)$; n being the number of sides, and the angles being here supposed the interior ones.

Let $AB = x$, $BC = y$: then, by th. 1,

$$x = y \cdot \cos B + DC \cdot \cos AB^{\wedge}CD + DE \cdot \cos AB^{\wedge}ED$$
$$+ EF \cdot \cos AB^{\wedge}EF + AF \cdot \cos AB^{\wedge}AF;$$
$$y = x \cdot \cos B + AF \cdot \cos BC^{\wedge}AF + FE \cdot \cos BC^{\wedge}FE.$$
$$+ DE \cdot \cos BC^{\wedge}DE + DC \cdot \cos BC^{\wedge}CD.$$

In the first of the above equations, let the sum of all the terms after $y \cdot \cos B$, be denoted by c; and in the second the sum of all those which fall after $x \cdot \cos B$, by d; both sums being manifestly constituted of known terms: and let the known coefficients of x and y be m and n respectively. Then will the preceding equations become

$$x = ny + c \;\ldots\; y = mx + d.$$

Substituting for y, in the first of the two latter equations, its value in the second, we obtain $x = mnx + nd + c$. Whence there will readily be found

$$x = \frac{nd+c}{1-mn}, \text{ and } y = \frac{mc+d}{1-mn}.$$

Thus

Thus AB and BC are determined. Like expressions will serve for the determination of any other two sides, whether contiguous or not: the coefficients of x and y being designated by different letters for that express purpose; which would have been otherwise unnecessary in the solution of the individual case proposed.

Remark. Though the algebraic investigations commonly lead to results which are apparently simple, yet they are often, especially in polygons of many sides, inferior in practice to the methods suggested by subdividing the figures. The following examples are added for the purpose of explaining those methods: the operations however are merely indicated; the detail being omitted to save room.

EXAMPLES.

Ex. 1. In a hexagon ABCDEF, all the sides except AF, and all the angles except A and F, are known. Required the unknown parts. Suppose we have

	Ext. ang.	Whence	
AB = 1284	B = 32°	B + C	= 80°
BC = 1782	C = 48°	B + C + D	= 132°
CD = 2400	D = 52°	B + C + D + E	= 198°
DE = 2700	E = 66°	A + F	= 162°.
EF = 2860			

Then, by cor. 3 th. 2, tan BAF =

$$\frac{BC \cdot \sin B + CD \cdot \sin(B+C) + DE \cdot \sin(B+C+D) + EF \cdot \sin(B+C+D+E)}{AB + BC \cdot \cos B + CD \cdot \cos(B+C) + DE \cdot \cos(B+C+D) + EF \cdot \cos(B+C+D+E)}$$

$$= \frac{BC \cdot \sin 32° + CD \cdot \sin 80° + DE \cdot \sin 132° + EF \cdot \sin 198°}{AB + BC \cdot \cos 32° + CD \cdot \cos 80° + DE \cdot \cos 132° + EF \cdot \cos 198°}$$

$$= \frac{BC \cdot \sin 32° + CD \cdot \sin 80° + DE \cdot \sin 48° - EF \cdot \sin 18°}{AB + BC \cdot \cos 32° + CD \cdot \cos 80° - DE \cdot \cos 48° - EF \cdot \cos 18°}$$

Whence BAF is found 106°31′38″; and the other angle AFE = 91°28′22″. So that the exterior angles A and F are 73°28′22″, and 88°31′38″ respectively: all the exterior angles making 4 right angles, as they ought to do. Then, all the angles being known, the side AF is found by th. 1 = 4621·5.

If one of the angles had been a re-entering one, it would have made no other difference in the computation than what would arise from its being considered as subtractive.

Ex. 2. In a hexagon ABCDEF, all the sides except AF, and all the angles except C and B, are known: viz,

AB = 2400	Ex. Ang.	
BC = 2700	A = 54°	
CD = 3200	B = 62°	
DE = 3500	E = 64°	
EF = 3750	F = 72°	

We shall have, by th. 2 cor 1,

$$\left.\begin{array}{l} \text{AB} \ \sin \text{A} \\ +\text{BC} \cdot \sin(\text{A}+\text{B}) \\ +\text{CD} \cdot \sin(\text{A}+\text{B}+\text{C}) \end{array}\right\} = \left\{\begin{array}{l} \text{DE} \cdot \sin(\text{E}+\text{F}) \\ +\text{EF} \cdot \sin \text{F}. \end{array}\right.$$

$$\text{Therefore, CD} \cdot \sin(116^\circ + \text{C}) = \left\{\begin{array}{l} -\text{AB} \cdot \sin\ 54^\circ \\ -\text{BC} \cdot \sin\ 116^\circ \\ +\text{DE} \cdot \sin\ 136^\circ \\ +\text{EF} \cdot \sin\ \ 72^\circ. \end{array}\right.$$

$$\text{Or, } 116^\circ + \text{C} = \left\{\begin{array}{l} 149^\circ 23' 26'', \\ +\ 33^\circ 36' 34''. \end{array}\right.$$

The second of these will give for C, a re-entering angle; the first will give exterior angle C = 33° 23′ 26″, and then will D = 14°36′34″. Lastly,

$$\text{AF} = \left\{\begin{array}{l} -\text{AB} \cdot \cos 54^\circ \\ +\text{BC} \cdot \cos 64^\circ \\ +\text{CD} \cdot \cos 30^\circ 36' 34'' \\ +\text{DE} \cdot \cos 44^\circ \\ -\text{EF} \cdot \cos 72^\circ \end{array}\right\} = 3885 \cdot 905.$$

Ex. 3. In a hexagon ABCDEF, are known, all the sides except AF, and all the angles except B and E; to find the rest.

Given AB = 1200		Exterior angles A = 64°	
BC = 1500			
CD = 1600		C = 72°	
DE = 1800		D = 75°	
EF = 2000			
		F = 84°.	

Suppose the diagonal BE drawn, dividing the figure into two trapeziums. Then, in the trapezium BCDE the sides, except BE, and the angles except B and E, will be known; and these may be determined as in exam. 1. Again, in a trapezium ABEF, there will be known the sides except AF, and the angles except the adjacent ones B and E. Hence, first for BCDE: (cor. 3 th. 2),

$$\tan \text{CBE} = \frac{\text{CD} \cdot \sin \text{C} + \text{DE} \cdot \sin(\text{C}+\text{D})}{\text{BC} + \text{CD} \cdot \cos \text{C} + \text{DE} \cdot \cos(\text{C}+\text{D})} =$$

$$\frac{\text{CD} \cdot \sin 72^\circ + \text{DE} \cdot \sin 147^\circ}{\text{BC} + \text{CD} \cdot \cos 72^\circ + \text{DE} \cdot \cos 147^\circ} = \frac{\text{CD} \cdot \sin 72^\circ + \text{DE} \cdot \sin 33^\circ}{\text{BC} + \text{CD} \cdot \cos 72^\circ - \text{DE} \cdot \cos 33^\circ}.$$

Whence CBE = 79° 2′ 1″; and therefore DEB = 67°57′59″.

$$\text{Then EB} = \left\{\begin{array}{l} \text{BC} \cdot \cos 79^\circ\ 2'\ 1'' \\ +\text{CD} \cdot \cos\ \ 7^\circ\ 2'\ 1'' \\ +\text{DE} \cdot \cos 67^\circ 57' 59'' \end{array}\right\} = 2548 \cdot 581.$$

Secondly, in the trapezium ABEF,

AB . sin A + BE . sin (A + B) = EF . sin F: whence

$$\sin(\text{A} + \text{B}) = \frac{\text{EF} \cdot \sin \text{F} - \text{AB} \cdot \sin \text{B}}{\text{BE}} = \sin \left\{\begin{array}{l} 20^\circ 55' 54'', \\ 159^\circ\ 4'\ 6''. \end{array}\right.$$

Taking

Taking the lower of these, to avoid re-entering angles, we have B (exterior ang.)$=95°4'6''$; ABE$= 84°55'54''$; FEB $= 63°4'6''$: therefore ABC $= 163°57'55''$; and FED $= 131°2'5''$: and consequently the exterior angles at B and E are $16° 2'5''$ and $48° 57' 55''$ respectively.

Lastly, AF $= -$AB . cos A $-$ BE . cos (A + B) $-$ EF cos F $= -$ AB . cos $64°$ + BE . cos $20° 55' 54''$ $-$ EF . cos $84° = 1645{\cdot}292$.

Note. The preceding three examples comprehend all the varieties which canoccur in Polygonometry, when all the sides except one, and all the angles but two, are known. The unknown angles may be about the unknown side; or they may be adjacent to each other, though distant from the unknown side; and they may be remote from each other, as well as from the unknown side.

Ex. 4. In a hexagon ABCDEF, are known all the angles, and all the sides except AF and CD: to find those sides.

Given AB $= 2200$		Ext. Ang. A $=$	$96°$
BC $= 2400$		B $=$	$54°$
		C $=$	$20°$
DE $= 4800$		D $=$	$24°$
EF $= 5200$		E $=$	$18°$
		F $=$	$148°$.

Here, reasoning from the principle of cor. th. 2, we have

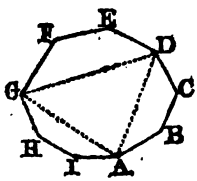
$$\left\{\begin{array}{l}\text{AB}.\sin 96°\\ +\text{BC}.\sin 150°\\ +\text{CD}.\sin 170°\end{array}\right\} = \left\{\begin{array}{l}\text{DE}.\sin 166°\\ +\text{EF}.\sin 148°.\end{array}\right. \text{ or } \left\{\begin{array}{l}\text{AB}.\sin 84°\\ +\text{BC}.\sin 30°\\ +\text{CD}.\sin 10°\end{array}\right\} = \left\{\begin{array}{l}\text{DE}.\sin 14°\\ +\text{EF}.\sin 32°.\end{array}\right.$$

$$\text{Whence CD} = \left\{\begin{array}{l}\text{DE}.\sin 14°.\text{cosec }10° - \text{AB}.\sin 84°\ \text{cosec }10°\\ +\text{EF}.\sin 32°.\text{cosec }10° - \text{BC}.\sin 30°.\text{cosec }10°\end{array}\right\} = 3045{\cdot}58.$$

$$\text{And AF} = \left\{\begin{array}{l}\text{DE}.\sin 24°.\text{cosec }10° - \text{CB}\ \sin 20°\\ +\text{EF}\ \sin 42°\ \text{cosec }10° - \text{BA}.\sin 74°\end{array}\right\} = 14874{\cdot}98.$$

Ex. 5. In the nonagon ABCDEFGHI, all the sides are known, and all the angles except A, D, G: it is required to find those angles.

Given AB $= 2400$	FG $= 3800$	Ext. Ang. B $=$	$40°$
BC $= 2700$	GH $= 4000$	C $=$	$32°$
CD $= 2800$	HI $= 4200$	E $=$	$36°$
DE $= 3200$	IA $= 4500$.	F $=$	$45°$
EF $= 3500$		H $=$	$48°$
		I $=$	$50°$.



Suppose diagonals drawn to join the unknown angles, and dividing the polygon into three trapeziums and a triangle; as in the marginal figure. Then,

1st. In the trapezium ABCD, where AD and the angles about it are unknown; we have (cor. 3 th. 2)

tan

$$\tan \text{BAD} = \frac{\text{BC} \cdot \sin \text{B} + \text{CD} \cdot \sin(\text{B}+\text{C})}{\text{AB} + \text{BC} \cdot \cos \text{B} + \text{CD} \cdot \cos(\text{B}+\text{C})} = \frac{\text{BC} \cdot \sin 40^\circ + \text{CD} \sin 70^\circ}{\text{AB} + \text{BC} \cdot \cos 40^\circ + \text{CD} \cdot \cos 72^\circ}.$$

Whence BAD $= 39^\circ 30' 42$, CDA $= 32^\circ 29' 18''$.

$$\text{And AD} = \left\{ \begin{array}{l} \text{AB} \cdot \cos 39^\circ 30' 42'' \\ + \text{BC} \cdot \cos \ \ 0 \ 29 \ 18 \\ + \text{CD} \cdot \cos 32 \ 29 \ 18 \end{array} \right\} = 6913 \cdot 292.$$

2dly. In the quadrilateral DEFG, where DG and the angles about it are unknown; we have

$$\tan \text{EDG} = \frac{\text{EF} \cdot \sin \text{E} + \text{FG} \cdot \sin(\text{E}+\text{F})}{\text{DE} + \text{EF} \cdot \cos \text{E} + \text{FG} \cdot \cos(\text{E}+\text{F})} = \frac{\text{EF} \cdot \sin 36^\circ + \text{FG} \cdot \sin 81^\circ}{\text{DE} + \text{EF} \cdot \cos 36^\circ + \text{FG} \cdot \cos 81^\circ}.$$

Whence EDG $= 41^\circ 14' 53''$, FGD $= 39^\circ 45' 7''$.

$$\text{And DG} = \left\{ \begin{array}{l} \text{DE} \cdot \cos 41^\circ 14' 53'' \\ + \text{EF} \cdot \cos \ \ 5^\circ 14' 53'' \\ + \text{FG} \cdot \cos 39^\circ 45' \ \ 7'' \end{array} \right\} = 8812 \cdot 803.$$

3dly. In the trapezium GHIA, an exactly similar process gives HGA $= 50^\circ 46' 53''$, IAG $= 47^\circ 13' 7''$, and AG $= 9780 \cdot 591$.

4thly. In the triangle ADG, the three sides are now known, to find the angles: viz, DAG $= 60^\circ 53' 26''$, AGD $= 43^\circ 15' 54''$, ADG $= 75^\circ 50' 40''$. Hence there results, lastly,

IAB $= 47^\circ 13' 7'' + 60^\circ 53' 26'' + 39^\circ 30' 42'' = 147^\circ 37' 15''$,
CDE $= 32^\circ 29' 18'' + 70^\circ 50' 40'' + 41^\circ 14' 53'' = 149^\circ 34' 51''$,
FGH $= 39^\circ 45' 7'' + 43^\circ 15' 54'' + 50^\circ 46' 53'' = 133^\circ 47' 54''$.

Consequently, the required exterior angles are A $= 32^\circ 22' 45''$, D $= 30^\circ 25' 9''$, G $= 46^\circ 12' 6''$.

Ex. 6. Required the area of the hexagon in ex. 1.

Ans. 16530191.

Ex. 7. In a quadrilateral ABCD, are given AB $= 24$, BC $= 30$, CD $= 34$; angle ABC $= 92^\circ 18'$, BCD $= 97^\circ 23'$. Required the side AD, and the area.

Ex. 8. In prob. 1, suppose PQ $= 2538$ links, and the angles as below; what is the area of the field ABCDQP?

APQ $= 89^\circ 14'$, BPQ $= 68^\circ 11'$, CPQ $= 36^\circ 24'$, DPQ $= 19^\circ 57'$;
AQP $= 25^\circ 18'$, BQP $= 69^\circ 24'$, CQP $= 94^\circ 6'$, DQP $= 121^\circ 18'$.

OF

OF MOTION, FORCES, &c.

DEFINITIONS.

Art. 1. BODY is the mass, or quantity of matter, in any material substance; and it is always proportional to its weight or gravity, whatever its figure may be.

2. Body is either Hard, Soft, or Elastic. A Hard Body is that whose parts do not yield to any stroke or percussion, but retains its figure unaltered. A Soft Body is that whose parts yield to any stroke or impression, without restoring themselves again; the figure of the body remaining altered. And an Elastic Body is that whose parts yield to any stroke, but which presently restore themselves again, and the body regains the same figure as before the stroke.

We know of no bodies that are absolutely, or perfectly, either hard, soft, or elastic; but all partaking these properties, more or less, in some intermediate degree.

3. Bodies are also either Solid or Fluid. A Solid Body, is that whose parts are not easily moved among one another, and which retains any figure given to it. But a Fluid Body is that whose parts yield to the slightest impression, being easily moved among one another; and its surface, when left to itself, is always observed to settle in a smooth plane at the top.

4. Density is the proportional weight or quantity of matter in any body. So, in two spheres, or cubes, &c, of equal size or magnitude; if the one weigh only one pound, but the other two pounds; then the density of the latter is double the density of the former; if it weigh 3 pounds, its density is triple; and so on.

5. Motion is a continual and successive change of place.—If the body move equally, or pass over equal spaces in equal times, it is called Equable or Uniform Motion. But if it increase or decrease, it is Variable Motion; and it is called Accelerated Motion in the former case, and Retarded Motion in the latter.—Also, when the moving body is considered with

with respect to some other body at rest, it is said to be Absolute Motion. But when compared with others in motion, it is called Relative Motion.

6. Velocity, or Celerity, is an affection of motion, by which a body passes over a certain space in a certain time. Thus, if a body in motion pass uniformly over 40 feet in 4 seconds of time, it is said to move with the velocity of 10 feet per second; and so on.

7 Momentum, or Quantity of Motion, is the power or force in moving bodies, by which they continually tend from their present places, or with which they strike any obstacle that opposes their motion.

8. Force is a power exerted on a body to move it, or to stop it. If the force act constantly, or incessantly, it is a Permanent Force: like pressure or the force of gravity. But if it act instantaneously, or but for an imperceptibly small time, it is called Impulse, or Percussion: like the smart blow of a hammer.

9. Forces are also distinguished into Motive, and Accelerative or Retarding. A Motive or Moving Force, is the power of an agent to produce motion; and it is equal or proportional to the momentum it will generate in any body, when acting, either by percussion, or for a certain time as a permanent force.

10. Accelerative, or Retardive Force, is commonly understood to be that which affects the velocity only; or it is that by which the velocity is accelerated or retarded; and it is equal or proportional to the motive force directly, and to the mass or body moved inversely.—So, if a body of 2 pounds weight, be acted on by a motive force of 40; then the accelerating force is 20. But if the same force of 40 act on another body of 4 pounds weight; then the accelerating force in this latter case is only 10; and so is but half the former, and will produce only half the velocity.

11. Gravity, or Weight, is that force by which a body endeavours to fall downwards. It is called Absolute Gravity, when the body is in empty space; and Relative Gravity, when emersed in a fluid.

12. Specific Gravity is the proportion of the weights of different bodies of equal magnitude; and so is proportional to the density of the body.

AXIOMS.

AXIOMS.

13. Every body naturally endeavours to continue in its present state, whether it be at rest, or moving uniformly in a right line.

14. The Change or Alteration of Motion, by any external force, is always proportional to that force, and in the direction of the right line in which it acts.

15. Action and Re-action, between any two bodies, are equal and contrary. That is, by Action and Re-action, equal changes of motion are produced in bodies acting on each other; and these changes are directed towards opposite or contrary parts.

GENERAL LAWS OF MOTION, &c.

PROPOSITION I.

16. *The Quantity of Matter, in all Bodies, is in the Compound Ratio of their Magnitudes and Densities.*

That is, b is as md; where b denotes the body or quantity of matter, m its magnitude, and d its density.

For, by art. 4, in bodies of equal magnitude, the mass or quantity of matter is as the density. But, the densities remaining, the mass is as the magnitude: that is, a double magnitude contains a double quantity of matter, a triple magnitude a triple quantity, and so on. Therefore the mass is in the compound ratio of the magnitude and density.

17. *Corol.* 1. In similar bodies, the masses are as the densities and cubes of the diameters, or of any like linear dimensions.—For the magnitudes of bodies are as the cubes of the diameters, &c.

18. *Corol.* 2. The masses are as the magnitudes and specific gravities.—For, by art. 4 and 12, the densities of bodies are as the specific gravities.

19. *Scholium.* Hence, if b denote any body, or the quantity of matter in it, m its magnitude, d its density, g its specific

specific gravity, and a its diameter or other dimension; then, $\propto$ (pronounced or named *as*) being the mark for general proportion, from this proposition and its corollaries we have these general proportions:

$$b \propto md \propto mg \propto a^3 d,$$

$$m \propto \frac{b}{d} \propto \frac{b}{g} \propto a^3,$$

$$d \propto \frac{b}{m} \propto g \propto \frac{mg}{a^3},$$

$$a^3 \propto \frac{b}{d} \propto m \propto \frac{mg}{d}.$$

PROPOSITION II.

20. *The Momentum, or Quantity of Motion, generated by a Single Impulse, or any Momentary Force, is as the Generating Force.*

That is, m is as f; where m denotes the momentum, and f the force.

For every effect is proportional to its adequate cause. So that a double force will impress a double quantity of motion; a triple force, a triple motion; and so on. That is, the motion impressed, is as the motive force which produces it.

PROPOSITION III.

21. *The Momenta, or Quantities of Motion, in moving Bodies, are in the Compound Ratio of the Masses and Velocities.*

That is, m is as bv.

For, the motion of any body being made up of the motions of all its parts, if the velocities be equal, the momenta will be as the masses; for a double mass will strike with a double force; a triple mass, with a triple force, and so on. Again, when the mass is the same, it will require a double force to move it with a double velocity, a triple force with a triple velocity, and so on; that is, the motive force is as the velocity; but the momentum impressed, is as the force which produces it, by prop. 2; and therefore the momentum is as the velocity when the mass is the same. But the momentum was found to be as the mass when the velocity is the same. Consequently,

Consequently, when neither are the same, the momentum is in the compound ratio of both the mass and velocity.

PROPOSITION IV.

22. *In Uniform Motions, the Spaces described are in the Compound Ratio of the Velocities and the Times of their Description.*

That is, s is as tv.

For, by the nature of uniform motion, the greater the velocity, the greater is the space described in any one and the same time; that is, the space is as the velocity, when the times are equal. And when the velocity is the same, the space will be as the time; that is, in a double time a double space will be described; in a triple time, a triple space; and so on. Therefore universally, the space is in the compound ratio of the velocity and the time of description.

23. *Corol.* 1. In uniform motions, the time is as the space directly, and velocity reciprocally; or as the space divided by the velocity. And when the velocity is the same, the time is as the space. But when the space is the same, the time is reciprocally as the velocity.

24. *Corol.* 2. The velocity is as the space directly and the time reciprocally; or as the space divided by the time. And when the time is the same, the velocity is as the space. But when the space is the same, the velocity is reciprocally as the time.

Scholium.

25. In uniform motions generated by momentary impulse, let b = any body or quantity of matter to be moved,
f = force of impulse acting on the body b,
v = the uniform velocity generated in b,
m = the momentum generated in b,
s = the space described by the body b,
t = the time of describing the space s with the veloc. v.

Then from the last three propositions and corollaries, we have these three general proportions, namely, $f \propto m$, $m \propto bv$, and $s \propto tv$; from which is derived the following table of the general relations of those six quantities, in uniform motions and impulsive or percussive forces:

 $f \propto m$

$$f \propto m \propto bv \propto \frac{bs}{t}.$$

$$m \propto f \propto bv \propto \frac{bs}{t}.$$

$$b \propto \frac{f}{v} \propto \frac{m}{v} \propto \frac{ft}{s} \propto \frac{mt}{s}.$$

$$s \propto tv \propto \frac{ft}{b} \propto \frac{tm}{b}.$$

$$v \propto \frac{s}{t} \propto \frac{f}{b} \propto \frac{m}{b}.$$

$$t \propto \frac{s}{v} \propto \frac{bs}{f} \propto \frac{bs}{m}.$$

By means of which, may be resolved all questions relating to uniform motions, and the effects of momentary or impulsive forces.

PROPOSITION V.

26. *The Momentum generated by a Constant and Uniform Force acting for any Time, is in the Compound Ratio of the Force and Time of Acting.*

That is, m is as ft.

For, supposing the time divided into very small parts, by prop. 2, the momentum in each particle of time is the same, and therefore the whole momentum will be as the whole time, or sum of all the small parts. But by the same prop. the momentum for each small time, is also as the motive force. Consequently the whole momentum generated, is in the compound ratio of the force and time of acting.

27. *Corol.* 1. The motion, or momentum, lost or destroyed in any time, is also in the compound ratio of the force and time. For whatever momentum any force generates in a given time; the same momentum will an equal force destroy in the same or equal time; acting in a contrary direction.

And the same is true of the increase or decrease of motion, by forces that conspire with, or oppose the motion of bodies.

28. *Corol.* 2. The velocity generated, or destroyed, in any time, is directly as the force and time, and reciprocally as the body or mass of matter.—For, by this and the 3d prop. the compound ratio of the body and velocity, is as that of the force and time; and therefore the velocity is as the force and time divided by the body. And if the body and force be given, or constant, the velocity will be as the time.

PROPOSITION

PROPOSITION VI.

29. *The Spaces passed over by Bodies, urged by any Constant and Uniform Forces, acting during any Times, are in the compound Ratio of the Forces and Squares of the Times directly, and the Body or Mass reciprocally.*
Or, the Spaces are as the Squares of the Times, when the Force and Body are given.

That is, s is as $\frac{ft^2}{b}$, or as t^2 when f and b are given. For, let v denote the velocity acquired at the end of any time t, by any given body b, when it has passed over the space s. Then, because the velocity is as the time, by the last corol. therefore $\frac{1}{2}v$ is the velocity at $\frac{1}{2}t$, or at the middle point of the time; and as the increase of velocity is uniform, the same space s will be described in the same time t, by the velocity $\frac{1}{2}v$, uniformly continued from beginning to end. But, in uniform motions, the space is in the compound ratio of the time and velocity; therefore s is as $\frac{1}{2}tv$, or indeed $s = \frac{1}{2}tv$. But, by the last corol. the velocity v is as $\frac{ft}{b}$, or as the force and time directly, and as the body reciprocally. Therefore s, or $\frac{1}{2}tv$, is as $\frac{ft^2}{b}$; that is, the space is as the force and square of the time directly, and as the body reciprocally. Or s is as t^2, the square of the time only, when b and f are given.

30. *Corol.* 1. The space s is also as tv, or in the compound ratio of the time and velocity; b and f being given. For, $s = \frac{1}{2}tv$ is the space actually described. But tv is the space which might be described in the same time t, with the last velocity v, if it were uniformly continued for the same or an equal time. Therefore the space s, or $\frac{1}{2}tv$, which is actually described, is just half the space tv, which would be described, with the last or greatest velocity, uniformly continued for an equal time t.

31. *Corol.* 2. The space s is also as v^2, the square of the velocity; because the velocity v is as the time t.

Scholium.

32. Propositions 3, 4, 5, 6, give theorems for resolving all questions relating to motions uniformly accelerated. Thus, put

put b = any body or quantity of matter,
f = the force constantly acting on it,
t = the time of its acting,
v = the velocity generated in the time t,
s = the space described in that time,
m = the momentum at the end of the time.

Then, from these fundamental relations, $m \propto bv$, $m \propto ft$, $s \propto tv$, and $v \propto \frac{ft}{b}$, we obtain the following table of the general relations of uniformly accelerated motions:

$$m \propto bv \propto ft \propto \frac{bs}{t} \propto \frac{fs}{v} \propto \frac{ft^2v}{s} \propto \sqrt{bfs} \propto \sqrt{bftv}.$$

$$b \propto \frac{m}{v} \propto \frac{ft}{v} \propto \frac{mt}{s} \propto \frac{ft^2}{s} \propto \frac{f^2t^2}{ms} \propto \frac{m^2}{fs} \propto \frac{m^2}{ftv} \propto \frac{fs}{v^2}.$$

$$f \propto \frac{m}{t} \propto \frac{bv}{t} \propto \frac{mv}{s} \propto \frac{ms}{t^2v} \propto \frac{m^2}{bs} \propto \frac{m^2}{btv} \propto \frac{bv^2}{s} \propto \frac{bs}{t^2}.$$

$$v \propto \frac{s}{t} \propto \frac{ft}{b} \propto \frac{m}{b} \propto \frac{ms}{ft^2} \propto \frac{fs}{m} \propto \frac{m^2}{bft} \propto \sqrt{\frac{fs}{b}} \propto \frac{f^2st}{m^2}.$$

$$s \propto tv \propto \frac{ft^2}{b} \propto \frac{mt}{b} \propto \frac{ft^2v}{m} \propto \frac{mv}{f} \propto \frac{m^2}{bf} \propto \frac{bv^2}{f} \propto \frac{m^2v}{f^2t}.$$

$$t \propto \frac{s}{v} \propto \frac{m}{f} \propto \frac{bv}{f} \propto \frac{bs}{m} \propto \sqrt{\frac{bs}{f}} \propto \sqrt{\frac{ms}{fv}} \propto \frac{m^2}{bfv}, \text{ \&c.}$$

33. And from these proportions those quantities are to be left out which are given, or which are proportional to each other. Thus, if the body or quantity of matter be always the same, then the space described is as the force and square of the time. And if the body be proportional to the force, as all bodies are in respect to their gravity; then the space described is as the square of the time, or square of the velocity; and in this case, if F be put $= \frac{f}{b}$, the accelerating force; then will

$$s \propto tv \propto Ft^2 \propto \frac{v^2}{F}.$$

$$v \propto \frac{s}{t} \propto Ft \propto \sqrt{Fs}.$$

$$t \propto \frac{s}{v} \propto \frac{v}{F} \propto \sqrt{\frac{s}{F}}.$$

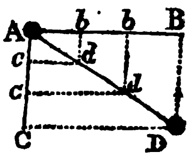
THE

The COMPOSITION and RESOLUTION of FORCES.

34. Composition of Forces, is the uniting of two or more forces into one, which shall have the same effect; or the finding of one force that shall be equal to several others taken together, in any different directions. And the resolution of Forces, is the finding of two or more forces which, acting in any different directions, shall have the same effect as any given single force.

PROPOSITION VII.

35. *If a Body at* A *be urged in the Directions* AB *and* AC, *by any two Similar Forces, such that they would separately cause the Body to pass over the Spaces* AB, AC, *in an equal Time; then if both Forces act together, they will cause the Body to move in the same Time, through* AD *the Diagonal of the Parallelogram* ABCD.



Draw *cd* parallel to AB, and *bd* parallel to AC. And while the body is carried over A*b* or *cd* by the force in that direction, let it be carried over *bd* by the force in that direction; by which means it will be found at *d*. Now, if the forces be impulsive or momentary, the motions will be uniform, and the spaces described will be as the times of description:

theref. A*b* or *cd* : AB or CD : : time in A*b* : time in AB,
and *bd* or A*c* : BD or AC : : time in A*c* : time in AC;
but the time in A*b* : = time in A*c*, and the time in AB = time in AC; therefore A*b* : *bd* : : AB : BD by equality: hence the point *d* is in the diagonal AD.

And as this is always the case in every point *d*, *d*, &c, therefore the path of the body is the straight line A*d*D, or the diagonal of the parallelogram.

But if the similar forces, by means of which the body is moved in the directions AB, AC, be uniformly accelerating ones, then the spaces will be as the squares of the times; in which case, call the time in *bd* or *cd*, *t*, and the time in AB or AC, T; then

it will be A*b* or *cd* : AB or CD : : t^2 : T^2,
and - *bd* or A*c* : BD or AC : : t^2 : T^2,
theref. by equality, A*b* : *bd* : : AB : BD;

and so the body is always found in the diagonal, as before.

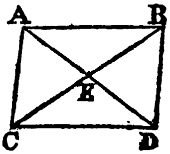
36. *Corol.*

36. *Corol.* 1. If the forces be not similar, by which the body is urged in the directions AB, AC, it will move in some curved line, depending on the nature of the forces.

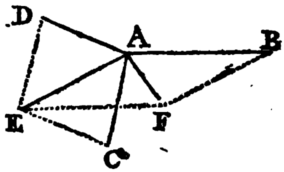
37. *Corol.* 2. Hence it appears, that the body moves over the diagonal AD, by the compound motion, in the very same time that it would move over the side AB, by the single force impressed in that direction, or that it would move over the side AC by the force impressed in that direction.

38. *Corol.* 3. The forces in the directions AB, AC, AD, are respectively proportional to the lines AB, AC, AD, and in these directions.

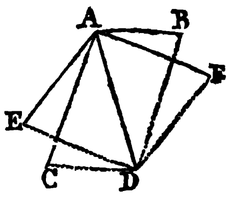
39. *Corol.* 4. The two oblique forces AB, AC, are equivalent to the single direct force AD, which may be compounded of these two, by drawing the diagonal of the parallelogram. Or they are equivalent to the double of AE, drawn to the middle of the line BC. And thus any force may be compounded of two or more other forces; which is the meaning of the expression *composition of forces.*



40. *Exam.* Suppose it were required to compound the three forces AB, AC, AD; or to find the direction and quantity of one single force, which shall be equivalent to, and have the same effect, as if a body A were acted on by three forces in the directions AB, AC, AD, and proportional to these three lines. First reduce the two AC, AD, to one AE, by completing the parallelogram ADEC. Then reduce the two AE, AB to one AF by the parallelogram AEFB. So shall the single force AF be the direction, and as the quantity, which shall of itself produce the same effect, as if all the three AB, AC, AD acted together.

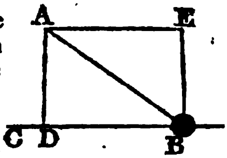


41. *Corol.* 5. Hence also any single direct force AD, may be resolved into two oblique forces, whose quantities and directions are AB, AC, having the same effect, by describing any parallelogram whose diagonal may be AD: and this is called the resolution of forces. So the force AD may be resolved into the two AB, AC, by the parallelogram



ABDC

ABDC; or into the two AE, AF, by the parallelogram AEDF; and so on, for any other two. And each of these may be resolved again into as many others as we please.



42. *Corol.* 6. Hence too may be found the effect of any given force, in any other direction, besides that of the line in which it acts; as, of the force AB in any other given direction CB. For draw AD perpendicular to CB; then shall DB be the effect of the force AB in the direction CB. For the given force AB is equivalent to the two AD, DB, or AE; of which the former AD, or EB, being perpendicular, does not alter the velocity in the direction CB; and therefore DB is the whole effect of AB in the direction CB. That is, a direct force expressed by the line DB acting in the direction DB, will produce the same effect or motion in a body B, in that direction, as the oblique force expressed by, and acting in, the direction AB, produces in the same direction CB. And hence any given force AB, is to its effect in DB, as AB to DB, or as radius to the cosine of the angle ABD of inclination of those directions. For the same reason, the force or effect in the direction AB, is to the force or effect in the direction AD or EB, as AB to AD; or as radius to sine of the same angle ABD, or cosine of the angle DAB of those directions.

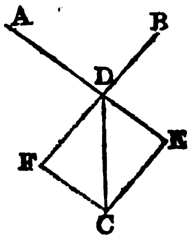
43. *Corol.* 7. Hence also, if the two given forces, to be compounded, act in the same line, either both the same way, or the one directly opposite to the other; then their joint or compounded force will act in the same line also, and will be equal to the sum of the two when they act the same way, or to the difference of them when they act in opposite directions; and the compound force, whether it be the sum or difference, will always act in the direction of the greater of the two.

PROPOSITION VIII.

44. *If Three Forces* A, B, C, *acting all together in the same Plane, keep one another in Equilibrio; they will be proportional to the Three Sides* DE, EC, CD, *of a Triangle, which are drawn Parallel to the Directions of the Forces* AD, DB, CD.

PRODUCE AD, BD, and draw CF, CE parallel to them. Then

Then the force in CD is equivalent to the two AD, BD, by the supposition; but the force CD is also equivalent to the two ED and CE or FD; therefore, if CD represent the force C, then ED will represent its opposite force A, and CE, or FD, its opposite force B. Consequently the three forces, A, B, C, are proportional to DE, CE, CD, the three lines parallel to the directions in which they act.

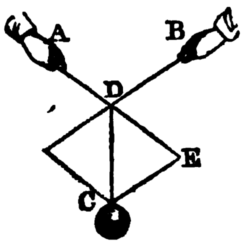


45. *Corol.* 1. Because the three sides CD, CE, DE, are proportional to the sines of their opposite angles E, D, C; therefore the three forces, when in equilibrio, are proportional to the sines of the angles of the triangle made of their lines of direction; namely, each force proportional to the sine of the angle made by the directions of the other two.

46. *Corol.* 2. The three forces, acting against, and keeping one another in equilibrio, are also proportional to the sides of any other triangle made by drawing lines either perpendicular to the directions of the forces, or forming any given angle with those directions. For such a triangle is always similar to the former, which is made by drawing lines parallel to the directions; and therefore their sides are in the same proportion to one another.

47. *Corol.* 3. If any number of forces be kept in equilibrio by their actions against one another; they may be all reduced to two equal and opposite ones.—For, by cor. 4, prop. 7, any two of the forces may be reduced to one force acting in the same plane; then this last force and another may likewise be reduced to another force acting in their plane; and so on, till at last they be all reduced to the action of only two opposite forces; which will be equal, as well as opposite, because the whole are in equilibrio by the supposition.

48. *Corol.* 4. If one of the forces, as C, be a weight, which is sustained by two strings drawing in the directions DA, BD: then the force or tension of the string AD, is to the weight C, or tension of the string DC, as DE to DC; and the force or tension of the other string BD, is to the weight C, or tension of CD, as CE to CD.



49. *Corol.*

49. *Corol.* 5. If three forces be in equilibrio by their mutual actions; the line of direction of each force, as DC, passes through the opposite angle C of the parallelogram formed by the directions of the other two forces.

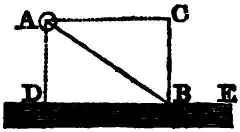
50. *Remark.* These properties, in this proposition and its corollaries, hold true of all similar forces whatever, whether they be instantaneous or continual, or whether they act by percussion, drawing, pushing, pressing, or weighing; and are of the utmost importance in mechanics and the doctrine of forces.

ON THE COLLISION OF BODIES.

PROPOSITION IX.

51. *If a Body strike or act Obliquely on a Plain Surface, the Force or Energy of the Stroke, or Action, is as the Sine of the Angle of Incidence.*

Or, the Force on the Surface is to the same if it had acted Perpendicularly, as the Sine of Incidence is to Radius.



Let AB express the direction and the absolute quantity of the oblique force on the plane DE; or let a given body A, moving with a certain velocity, impinge on the plane at B; then its force will be to the action on the plane, as radius to the sine of the angle ABD, or as AB, to AD or BC, drawing AD and BC perpendicular, and AC parallel to DE.

For, by prob. 7, the force AB is equivalent to the two forces AC, CB; of which the former AC does not act on the plane, because it is parallel to it. The plane is therefore only acted on by the direct force CB, which is to AB, as the sine of the angle BAC, or ABD, to radius.

52. *Corol.* 1. If a body act on another, in any direction, and by any kind of force, the action of that force on the second body, is made only in a direction perpendicular to the surface on which it acts. For the force in AB acts on DE only by the force CB, and in that direction.

53. *Corol.* 2. If the plane DE be not absolutely fixed, it will move, after the stroke, in the direction perpendicular to its surface. For it is in that direction that the force is exerted.

PROPOSITION X.

54. *If one Body* A, *strike another Body* B, *which is either at Rest or moving towards the Body* A, *or moving from it, but with a less Velocity than that of* A; *then the Momenta, or Quantities of Motion, of the two Bodies, estimated in any one Direction, will be the very same after the Stroke that they were before it.*

For, because action and re-action are always equal, and in contrary directions, whatever momentum the one body gains one way by the stroke, the other must just lose as much in the same direction; and therefore the quantity of motion in that direction, resulting from the motions of both the bodies remains still the same as it was before the stroke.

55. Thus, if A with a momentum of 10, strike B at rest, and communicate to it a momentum of 4, in the direction AB. Then A will have only a momentum of 6 in that direction; which, together with the momentum of B, viz. 4, make up still the same momentum between them as before, namely, 10.

56. If B were in motion before the stroke with a momentum of 5, in the same direction, and receive from A an additional momentum of 2. Then the motion of A after the stroke will be 8, and that of B, 7; which between them make 15, the same as 10 and 5, the motions before the stroke.

57 Lastly, if the bodies move in opposite directions, and meet one another, namely, A with a motion of 10, and B, of 5; and A communicate to B a motion of 6 in the direction AB of its motion. Then, before the stroke, the whole motion from both, in the direction of AB, is 10—5 or 5. But, after the stroke, the motion of A is 4 in the direction AB, and the motion of B is 6—5 or 1 in the same direction AB; therefore the sum 4 + 1, or 5, is still the same motion from both, as it was before.

PROPOSITION XI.

58. *The Motion of Bodies included in a Given Space, is the same with regard to each other, whether that Space be at Rest, or move uniformly in a Right Line.*

For, if any force be equally impressed both on the body and the line in which it moves, this will cause no change in the

the motion of the body along the right line. For the same reason, the motions of all the other bodies, in their several directions, will still remain the same. Consequently their motions among themselves will continue the same, whether the including space be at rest, or be moved uniformly forward. And therefore their mutual actions on one another, must also remain the same in both cases.

PROPOSITION XII.

59. *If a Hard and Fixed Plane be struck by either a Soft or a Hard Unelastic Body, the Body will adhere to it. But if the Plane be struck by a Perfectly Elastic Body, it will rebound from it again with the same Velocity with which it struck the Plane.*

For, since the parts which are struck, of the elastic body, suddenly yield and give way by the force of the blow, and as suddenly restore themselves again with a force equal to the force which impressed them, by the definition of elastic bodies; the intensity of the action of that restoring force on the plane, will be equal to the force or momentum with which the body struck the plane. And, as action and reaction are equal and contrary, the plane will act with the same force on the body, and so cause it to rebound or move back again with the same velocity as it had before the stroke.

But hard or soft bodies, being devoid of elasticity, by the definition, having no restoring force to throw them off again, they must necessarily adhere to the plane struck.

60. *Corol.* 1. The effect of the blow of the elastic body, on the plane, is double to that of the unelastic one, the velocity and mass being equal in each.

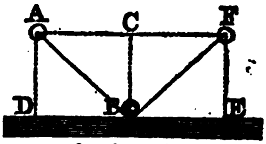
For the force of the blow from the unelastic body, is as its mass and velocity, which is only destroyed by the resistance of the plane. But in the elastic body, that force is not only destroyed and sustained by the plane; but another also equal to it is sustained by the plane, in consequence of the restoring force, and by virtue of which the body is thrown back again with an equal velocity. And therefore the intensity of the blow is doubled.

61. *Cor.* 2. Hence unelastic bodies lose, by their collision, only half the motion lost by elastic bodies; their mass and velocities being equal.—For the latter communicate double the motion of the former.

PROPOSITION

PROPOSITION XIII.

62. *If an Elastic Body* A *impinge on a Firm Plane* DE *at the Point* B, *it will rebound from it in an Angle equal to that in which it struck it ; or the Angle of Incidence will be equal to the Angle of Reflexion ; namely, the Angle* ABD *equal to the Angle* FBE.



LET AB express the force of the body A in the direction AB ; which let be resolved into the two AC, CB, parallel and perpendicular to the plane.—Take BE and CF equal to AC, and draw BF. Now action and reaction being equal, the plane will resist the direct force CB by another BC equal to it, and in a contrary direction; whereas the other AC, being parallel to the plane, is not acted on or diminished by it, but still continues as before. The body is therefore reflected from the plane by two forces BC, BE, perpendicular and parallel to the plane, and therefore moves in the diagonal BF by composition. But, because AC is equal to BE or CF, and that BC is common, the two triangles BCA, BCF are mutually similar and equal ; and consequently the angles at A and F are equal, as also their equal alternate angles ABD, FBE, which are the angles of incidence and reflexion.

PROPOSITION XIV.

63. *To determine the Motion of Non-elastic Bodies when they strike each other Directly, or in the same Line of Direction.*

LET the non-elastic body B, moving with the velocity V in the direction Bb, and the body b with the velocity v, strike each other. Then, because the momentum of any moving body is as the mass into the velocity, BV = M is the momentum of the body B, and $bv = m$ the momentum of the body b, which let be the less powerful of the two motions. Then, by prop. 10, the bodies will both move together as one mass in the direction BC after the stroke, whether before the stroke the body b moved towards C or towards B. Now, according as that motion of b was from or towards B, that is, whether the motions were in the same or contrary ways, the momentum after the stroke, in direction BC, will be

be the sum or difference of the momentums before the stroke; namely, the momentum in direction BC will be

$BV + bv$, if the bodies moved the same way, or
$BV - bv$, if they moved contrary ways, and
BV only, if the body b were at rest.

Then divide each momentum by the common mass of matter $B + b$, and the quotient will be the common velocity after the stroke in the direction BC; namely, the common velocity will be, in the first case,

$$\frac{BV + bv}{B + b}, \text{ in the 2d} \frac{BV - bv}{B + b}, \text{ and in the 3d} \frac{BV}{B + b}.$$

64. For example, if the bodies, or weights, B and b, be as 5 to 3, and their velocities V and v, as 6 to 4, or as 3 to 2, before the stroke; then 15 and 6 will be as their momentums, and 8 the sum of their weights; consequently, after the stroke, the common velocity will be as

$$\frac{15 + 6}{8} = \frac{21}{8} \text{ or } 2\tfrac{5}{8} \text{ in the first case,}$$
$$\frac{15 - 6}{8} = \frac{9}{8} \text{ or } 1\tfrac{1}{8} \text{ in the second, and}$$
$$\frac{15}{8} \text{ - - - - - or } 1\tfrac{7}{8} \text{ in the third.}$$

PROPOSITION XV.

65. *If two Perfectly Elastic Bodies impinge on one another: their Relative Velocity will be the same both Before and After the Impulse: that is, they will recede from each other with the Same Velocity with which they approached and met.*

For the compressing force is as the intensity of the stroke; which, in given bodies, is as the relative velocity with which they meet or strike. But perfectly elastic bodies restore themselves to their former figure, by the same force by which they were compressed; that is, the restoring force is equal to the compressing force, or to the force with which the bodies approach each other before the impulse. But the bodies are impelled from each other by this restoring force; and therefore this force, acting on the same bodies, will produce a relative velocity equal to that which they had before: or it will make the bodies recede from each other with the same

same velocity with which they before approached, or so as to be equally distant from one another at equal times before and after the impact.

66. *Remark.* It is not meant by this proposition, that each body will have the same velocity after the impulse as it had before; for that will be varied according to the relation of the masses of the two bodies; but that the velocity of the one will be, after the stroke, as much increased as that of the other is decreased, in one and the same direction. So, if the elastic body B move with a velocity V, and overtake the elastic body b moving the same way with the velocity v; then their relative velocity, or that with which they strike, is $V - v$, and it is with this same velocity that they separate from each other after the stroke. But if they meet each other, or the body b move contrary to the body B; then they meet and strike with the velocity $V + v$, and it is with the same velocity that they separate and recede from each other after the stroke. But whether they move forward or backward after the impulse, and with what particular velocities, are circumstances that depend on the various masses and velocities of the bodies before the stroke, and which make the subject of the next proposition.

PROPOSITION XVI.

67. *To determine the Motions of Elastic Bodies after Striking each other directly.*

LET the elastic body B move in the direction BC, with the velocity V; and let the velocity of the other body b be v in the same line; which latter velocity v will be positive if b move the same way as B, but negative if b move in the opposite direction to B. Then their relative velocity in the direction BC is $V - v$; also the momenta before the stroke are BV and bv, the sum of which is $BV + bv$ in the direction BC.

Again, put x for the velocity of B, and y for that of b, in the same direction BC, after the stroke; then their relative velocity is $y - x$, and the sum of their momenta $Bx + by$ in the same direction.

But the momenta before and after the collision, estimated in the same direction, are equal, by prop. 10, as also the relative velocities, by the last prop. Whence arise these two equations:

viz.

$$\text{viz. } BV + bv = Bx + by,$$
$$\text{and } V - v = y - x;$$

the resolution of which equations gives

$$x = \frac{(B - b)V + 2bv}{B + b}, \text{ the velocity of } B,$$

$$y = \frac{-(B - b)v + 2BV}{B + b}, \text{ the velocity of } b,$$

both in the direction BC, when V and v are both positive, or the bodies both moved towards C before the collision. But if v be negative, or the body b moved in the contrary direction before collision, or towards B; then, changing the sign of v, the same theorems become

$$x = \frac{(B - b)V - 2bv}{B + b}, \text{ the velocity of } B,$$

$$y = \frac{(B - b)v + 2BV}{B + b}, \text{ the veloc. of } b, \text{ in the direction BC.}$$

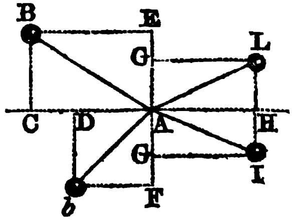
And if b were at rest before the impact, making its velocity $v = 0$, the same theorems give

$$x = \frac{B - b}{B + b}V, \text{ and } y = \frac{2B}{B + b}V, \text{ the velocities in this case.}$$

And in this case, if the two bodies B and b be equal to each other; then $B - b = 0$, and $\frac{2B}{B + b} = \frac{2B}{2B} = 1$; which give $x = 0$, and $y = V$; that is the body B will stand still, and the other body b will move on with the whole velocity of the former; a thing which we sometimes see happen in playing at billiards; and which would happen much oftener if the balls were perfectly elastic.

PROPOSITION XVII.

68 *If Bodies strike one another Obliquely, it is proposed to determine their Motions after the Stroke.*



LET the two bodies B, b, move in the oblique directions BA, bA, and strike each other at A, with velocities which are in proportion to the lines BA, bA; to find their motions after the impact. Let CAH represent the plane in which the bodies touch in the point of concourse; to which draw the perpendiculars BC, bD, and complete the rectangles CE, DF. Then the motion in BA is resolved

solved into the two BC, CA; and the motion in *b*A is resolved into the two *b*D, DA; of which the antecedents BC, *b*D, are the velocities with which they directly meet, and the consequents CA, DA, are parallel; therefore by these the bodies do not impinge on each other, and consequently the motions, according to these directions, will not be changed by the impulse; so that the velocities with which the bodies meet, are as BC and *b*D, or their equals EA and FA. The motions therefore of the bodies B, *b*, directly striking each other with the velocities EA, FA, will be determined by prop. 16 or 14, according as the bodies are elastic or non-elastic; which being done, let AG be the velocity, so determined, of one of them, as A; and since there remains also in the body a force of moving in the direction parallel to BE, with a velocity as BE, make AH equal to BE, and complete the rectangle GH: then the two motions in AH and AG, or HI, are compounded into the diagonal AI, which therefore will be the path and velocity of the body B after the stroke. And after the same manner is the motion of the other body *b* determined after the impact.

If the elasticity of the bodies be imperfect in any given degree, then the quantity of the corresponding lines must be diminished in the same proportion.

THE LAWS OF GRAVITY; THE DESCENT OF HEAVY BODIES; AND THE MOTION OF PROJECTILES IN FREE SPACE.

PROPOSITION XVIII.

69. *All the properties of Motion delivered in Proposition VI, its Corollaries and Scholium, for Constant Forces, are true in the Motions of Bodies freely descending by their own Gravity; namely, that the velocities are as the Times, and the Spaces as the Squares of the Times, or as the Squares of the Velocities.*

FOR, since the force of gravity is uniform, and constantly the same, at all places near the earth's surface, or at nearly the same distance from the centre of the earth; and since this is the force by which bodies descend to the surface; they therefore descend by a force which acts constantly and equally; consequently all the motions freely produced by gravity, are as above specified, by that proposition, &c.

SCHOLIUM.

70. Now it has been found, by numberless experiments, that

that gravity is a force of such a nature, that all bodies, whether light or heavy, fall perpendicularly through equal spaces in the same time, abstracting from the resistance of the air; as lead or gold and a feather, which in an exhausted receiver fall from the top to the bottom in the same time. It is also found that the velocities acquired by descending, are in the exact proportion of the times of descent: and further, that the spaces descended are proportional to the squares of the times, and therefore to the squares of the velocities. Hence then it follows, that the weights or gravities, of bodies near the surface of the earth, are proportional to the quantities of matter contained in them; and that the spaces, times, and velocities, generated by gravity, have the relations contained in the three general proportions before laid down. Further, as it is found, by accurate experiments, that a body in the latitude of London, falls nearly $16\frac{1}{12}$ feet in the first second of time, and consequently that at the end of that time it has acquired a velocity double, or of $32\frac{1}{6}$ feet by corol. 1, prop. 6; therefore if g denote $16\frac{1}{12}$ feet, the space fallen through in one second of time, or $2g$ the velocity generated in that time; then, because the velocities are directly proportional to the times, and the spaces to the squares of the times; therefore it will be,

as $1'' : t'' :: 2g : 2gt = v$ the velocity,
and $1^2 : t^2 :: g : gt^2 = s$ the space.

So that, for the descents of gravity, we have these general equations, namely,

$$s = gt^2 = \frac{v^2}{4g} = \tfrac{1}{2}tv.$$

$$v = 2gt = \frac{2s}{t} = 2\sqrt{gs}.$$

$$t = \frac{v}{2g} = \frac{2s}{v} = \sqrt{\frac{s}{g}}.$$

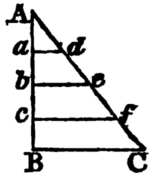
$$g = \frac{v}{2t} = \frac{s}{t^2} = \frac{v^2}{4s}.$$

Hence, because the times are as the velocities, and the spaces as the squares of either, therefore,

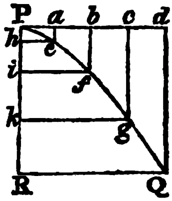
if the times be as the numbs.	1,	2,	3,	4,	5,	&c,
the velocities will also be as	1,	2,	3,	4,	5,	&c,
and the spaces as their squares	1,	4,	9,	16,	25,	&c,
and the space for each time as	1,	3,	5,	7,	9,	&c,

namely, as the series of the odd numbers, which are the differences of the squares denoting the whole spaces. So that if the first series of natural numbers be seconds of time,

namely, the times in seconds	1″,	2″,	3″,	4″,	&c,
the velocities in feet will be	$32\frac{1}{6}$,	$64\frac{1}{3}$,	$96\frac{1}{2}$,	$128\frac{2}{3}$,	&c,
the spaces in the whole times	$16\frac{1}{12}$,	$64\frac{1}{3}$,	$144\frac{3}{4}$,	$257\frac{1}{3}$,	&c,
and the space for each second	$16\frac{1}{12}$,	$48\frac{1}{4}$,	$80\frac{5}{12}$,	$112\frac{7}{12}$,	&c.



71. These relations, of the times, velocities, and spaces, may be aptly represented by certain lines and geometrical figures. Thus, if the line AB denote the time of any body's descent, and BC, at right angles to it, the velocity gained at the end of that time; by joining AC, and dividing the time AB into any number of parts at the points a, b, c; then shall ad, be, cf, parallel to BC, be the velocities at the points of time a, b, c, or at the ends of the times, Aa, Ab, AC; because these latter lines, by similar triangles are proportional to the former ad, be, cf, and the times are proportional to the velocities. Also, the area of the triangle ABC will represent the space descended by the force of gravity in the time AB, in which it generates the velocity BC; because that area is equal to $\frac{1}{2}$AB × BC, and the space descended is $s = \frac{1}{2}tv$, or half the product of the time and the last velocity. And, for the same reason, the less triangles Aad, Abe, Acf, will represent the several spaces described in the corresponding times Aa, Ab, AC, and velocities ad, be, cf; those triangles or spaces being also as the squares of their like sides Aa, Ab, AC, which represent the times, or of ad, be, cf, which represent the velocities.



72. But as areas are rather unnatural representations of the spaces passed over by a body in motion, which are lines, the relations may better be represented by the abscisses and ordinates of a parabola. Thus, if PQ be a parabola, PR its axis, and RQ its ordinate; and Pa, Pb, PC, &c, parallel to RQ, represent the times from the beginning, or the velocities, then ae, bf, cg, &c, parallel to the axis PR, will represent the spaces described by a falling body in those times; for, in a parabola, the abscisses Ph, Pi, Pk, &c, or ae, bf, cg, &c, which are the spaces described, are as the squares of the ordinates he, if, kg, &c, or Pa, PB, PC, &c, which represent the times or velocities.

73. And because the laws for the destruction of motion, are

are the same as those for the generation of it, by equal forces, but acting in a contrary direction; therefore,

1st, A body thrown directly upward, with any volocity, will lose equal velocities in equal times.

2d, If a body be projected upward, with the velocity it acquired in any time by descending freely, it will lose all its velocity in an equal time, and will ascend just to the same height from which it fell, and will describe equal spaces in equal times, in rising and falling, but in an inverse order; and it will have equal velocities at any one and the same point of the line described, both in ascending and descending.

3d, If bodies be projected upward, with any volocities, the height ascended to, will be as the squares of those velocities, or as the squares of the times of ascending, till they lose all their velocities.

74. To illustrate now the rules for the natural descent of bodies by a few examples, let it be required,

1st, To find the space descended by a body in 7 seconds of time, and the velocity acquired.

Ans. $788\frac{1}{12}$ space; and $225\frac{1}{6}$ velocity.

2d, To find the time of generating a velocity of 100 feet per second, and the whole space descended.

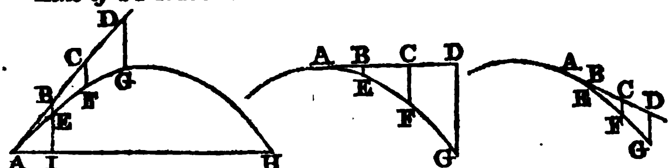
Ans. $3''\frac{21}{193}$ time; $155\frac{85}{193}$ space.

3d, To find the time of descending 400 feet, and the velocity at the end of that time.

Ans. $4''\frac{76}{77}$ time; and $160\frac{32}{77}$ velocity.

PROPOSITION XIX.

75. *If a Body be projected in Free Space either Parallel to the Horizon, or in an Oblique Direction, by the Force of Gun-Powder, or any other Impulse; it will by this Motion, in Conjunction with the Action of Gravity, describe the Curve Line of a Parabola.*



Let the body be projected from the point A, in the direction AD, with any uniform velocity; then, in any equal portions

portions of time, it would, by prop. 4, describe the equal spaces AB, BC, CD, &c, in the line AD, if it were not drawn continually down below that line by the action of gravity. Draw BE, CF, DG &c, in the direction of gravity, or perpendicular to the horizon, and equal to the spaces through which the body would descend by its gravity in the same time in which it would uniformly pass over the corresponding spaces AB, AC, AD, &c, by the projectile motion. Then, since by these two motions the body is carried over the space AB, in the same time as over the space BE, and the space AC in the same time as the space CF, and the space AD in the same time as the space DG, &c; therefore, by the composition of motions, at the end of those times, the body will be found respectively in the points E, F, G, &c; and consequently the real path of the projectile will be the curve line AEFG &c. But the spaces AB, AC, AD, &c, described by uniform motion, are as the times of description; and the spaces BE, CF, DG, &c, described in the same times by the accelerating force of gravity, are as the squares of the times; consequently the perpendicular descents are as the squares of the spaces in AD, that is BE, CF, DG, &c, are respectively proportional to AB^2, AC^2, AD^2, &c; which is the property of the parabola by theor. 8, Con. Sect. Therefore the path of the projectile is the parabolic line AEFG &c, to which AD is a tangent at the point A.

76. *Corol.* 1. The horizontal velocity of a projectile, is always the same constant quantity, in every point of the curve; because the horizontal motion is in a constant ratio to the motion in AD, which is the uniform projectile motion. And the projectile velocity is in proportion to the constant horizontal velocity, as radius to the cosine of the angle DAH, or angle of elevation or depression of the piece above or below the horizontal line AH.

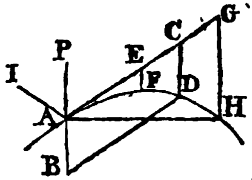
77. *Corol.* 2. The velocity of the projectile in the direction of the curve, or of its tangent at any point A is as the secant of its angle BAI of direction above the horizon. For the motion in the horizontal direction AI is constant, and AI is to AB, as radius to the secant of the angle A; therefore the motion at A, in AB, is everywhere as the secant of the angle A.

78. *Corol.* 3. The velocity in the direction DG of gravity, or perpendicular to the horizon, at any point G of the curve, is to the first uniform projectile velocity at A, or point of contact of a tangent, as 2GD is to AD. For, the times in AD and DG being equal, and the velocity acquired by freely descending

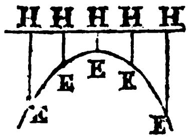
scending through DG, being such as would carry the body uniformly over twice DG in an equal time, and the spaces described with uniform motions being as the velocities, therefore the space AD is to the space 2DG, as the projectile velocity at A, to the perpendicular velocity at G.

PROPOSITION XX.

79. *The Velocity in the Direction of the Curve, at any Point of it, as* A, *is equal to that which is generated by Gravity in freely descending through a Space which is equal to One-Fourth of the Parameter of the Diameter of the Parabola at that Point.*



LET PA or AB be the height due to the velocity of the projectile at any point A, in the direction of the curve or tangent AC, or the velocity acquired by falling through that height; and complete the parallelogram ACDB. Then is CD = AB or AP, the height due to the velocity in the curve at A; and CD is also the height due to the perpendicular velocity at D, which must be equal to the former; but by the last corol. the velocity at A is to the perpendicular velocity at D, as AC to 2CD; and as these velocities are equal, therefore AC or BD is equal to 2CD, or 2AB; and hence AB or AP is equal to $\frac{1}{4}$BD, or $\frac{1}{4}$ of the parameter of the diameter AB, by corol. to theor. 13 of the Parabola.



80. *Corol.* 1. Hence, and from cor. 2, theor. 13 of the Parabola, it appears that, if from the directrix of the parabola which is the path of the projectile, several lines HE be drawn perpendicular to the directrix, or parallel to the axis; then the velocity of the projectile in the direction of the curve, at any point E, is always equal to the velocity acquired by a body falling freely through the perpendicular line HE.

81. *Corol.* 2. If a body, after falling through the height PA (last fig. but one), which is equal to AB, and when it arrives at A, have its course changed, by reflection from an elastic plane AI, or otherwise, into any direction AC, without altering the velocity; and if AC be taken = 2AP or 2AB, and

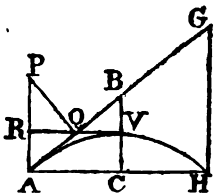
and the parallelogram be completed; then the body will describe the parabola passing through the point D.

82. *Corol.* 3. Because $AC = 2AB$ or $2CD$ or $2AP$, therefore $AC^2 = 2AP \times 2CD$ or $AP \cdot 4CD$; and, because all the perpendiculars EF, CD, GH, are as AE^2, AC^2, AG^2; therefore also $AP \cdot 4EF = AE^2$, and $AP \cdot 4GH = AG^2$, &c; and because the rectangle of the extremes is equal to the rectangle of the means of four proportionals, therefore always

it is $AP : AE :: AE : 4EF$,
and $AP : AC :: AC : 4CD$,
and $AP : AG :: AG : 4GH$,
and so on.

PROPOSITION XXI.

83. *Having given the Direction, and the Impetus, or Altitude due to the First Velocity of a Projectile; to determine the Greatest Height to which it will rise, and the Random or Horizontal Range.*

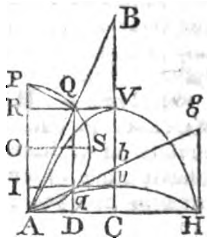


LET AP be the height due to the projectile velocity at A, AG the direction, and AH the horizon. On AG let fall the perpendicular PQ, and on AP the perpendicular QR; so shall AR be equal to the greatest altitude CV, and 4QR eqal to the horizontal range AH. Or, having drawn PQ perp. to AG, take $AG = 4AQ$, and draw GH perp. to AH; then AH is the range.

For, by the last corollary, $AP : AG :: AG : 4GH$;
and, by similar triangles, $AP : AG :: AQ : GH$,
or - - - - - $AP : AG :: 4AQ : 4GH$;

therefore $AG = 4AQ$; and, by similar triangles, $AH = 4\ QR$.

Also, if V be the vertex of the parabola, then AB or $\frac{1}{2}AG = 2AQ$, or $AQ = QB$; consequently $AR = BV$, which is $= CV$ by the property of the parabola.



84. *Corol.* 1. Because the angle Q is a right angle, which is the angle in a semicircle, therefore if, on AP as a diameter, a semicircle be described, it will pass through the point Q.

85. *Corol.* 2. If the horizontal range and the projectile velocity be given, the

the direction of the piece so as to hit the object H, will be thus easily found: Take AD = $\frac{1}{4}$ AH, draw DQ perpendicular to AH, meeting the semicircle, described on the diameter AP, in Q and q; then AQ or Aq will be the direction of the piece. And hence it appears, that there are two directions AB, Ab, which, with the same projectile velocity, give the very same horizontal range AH. And these two directions make equal angles qAD, QAP with AH and AP, because the arc PQ = the arc Aq.

86. *Corol.* 3. Or, if the range AH, and direction AB, be given; to find the altitude and velocity or impetus. Take AD = $\frac{1}{4}$AH, and erect the perpendicular DQ, meeting AB in Q; so shall DQ be equal to the greatest altitude CV. Also, erect AP perpendicular to AH, and QP to AQ; so shall AP be the height due to the velocity.

87. *Corol.* 4. When the body is projected with the same velocity, but in different directions: the horizontal ranges AH will be as the sines of double the angles of elevation.— Or, which is the same, as the rectangle of the sine and cosine of elevation. For AD or RQ, which is $\frac{1}{4}$AH, is the sine of the arc AQ, which measures double the angle QAD of elevation.

And when the direction is the same, but the velocities different; the horizontal ranges are as the square of the velocities, or as the height AP, which is as the square of the velocity; for the sine AD or RQ or $\frac{1}{4}$AH is as the radius or as the diameter AP.

Therefore, when both are different, the ranges are in the compound ratio of the squares of the velocities, and the sines of double the angles of elevation.

88. *Corol.* 5. The greatest range is when the angle of elevation is 45°, or half a right angle; for the double of 45 is 90, which has the greatest sine. Or the radius OS, which is $\frac{1}{4}$ of the range, is the greatest sine.

And hence the greatest range, or that at an elevation of 45°, is just double the altitude AP which is due to the velocity, or equal to 4VC. Consequently, in that case, C is the focus of the parabola, and AH its parameter. Also, the ranges are equal, at angles equally above and below 45°.

89. *Corol.* 6. When the elevation is 15°, the double of which, or 30°, has its sine equal to half the radius; consequently then its range will be equal to AP, or half the greatest range at the elevation of 45°; that is, the range at 15°, is equal to the impetus or height due to the projectile velocity.

90. *Corol.* 7.

90. *Corol.* 7. The greatest altitude CV, being equal to AE, is as the versed sine of double the angle of elevation, and also as AP or the square of the velocity. Or as the square of the sine of elevation, and the square of the velocity; for the square of the sine is as the versed sine of the double angle.

91. *Corol.* 8. The time of flight of the projectile, which is equal to the time of a body falling freely through EH or 4CV, four times the altitude, is therefore as the square root of the altitude, or as the projectile velocity and sine of the elevation.

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92. From the last proposition, and its corollaries, may be deduced the following set of theorems, for finding all the circumstances of projectiles on horizontal planes, having any two of them given. Thus, let s, c, t denote the sine, cosine, and tangent of elevation; S, v the sine and versed sine of the double elevation; R the horizontal range; T the time of flight; V the projectile velocity; H the greatest height of the projectile $g = 16\frac{1}{12}$ feet, and a the impetus, or the altitude due to the velocity V. Then,

$$\mathrm{R} = 2a\mathrm{S} = 4asc = \frac{\mathrm{S}\mathrm{V}^2}{2g} = \frac{sc\mathrm{V}^2}{g} = \frac{gc\mathrm{T}^2}{s} = \frac{g\mathrm{T}^2}{t} = \frac{4\mathrm{H}}{t}.$$

$$\mathrm{V} = \sqrt{4ag} = \sqrt{\frac{2g\mathrm{R}}{\mathrm{S}}} = \sqrt{\frac{g\mathrm{R}}{sc}} = \frac{g\mathrm{T}}{s} = \frac{2}{s}\sqrt{g\mathrm{H}}.$$

$$\mathrm{T} = \frac{s\mathrm{V}}{g} = 2s\sqrt{\frac{a}{g}} = \sqrt{\frac{t\mathrm{R}}{g}} = \sqrt{\frac{s\mathrm{R}}{gc}} = 2\sqrt{\frac{\mathrm{H}}{g}}.$$

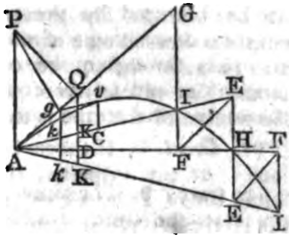
$$\mathrm{H} = as^2 = \tfrac{1}{2}av = \tfrac{1}{4}t\mathrm{R} = \frac{s\mathrm{R}}{4c} = \frac{s^2\mathrm{V}^2}{4g} = \frac{v\mathrm{V}^2}{8g} = \frac{g}{4}\mathrm{T}^2.$$

And from any of these, the angle of direction may be found. Also, in these theorems, g may, in many cases, be taken $= 16$, without the small fraction $\frac{1}{12}$, which will be near enough for common use.

PROPOSITION XXII.

93. *To determine the Range on an Oblique Plane; having given the Impetus or Velocity, and the Angle of Direction.*

LET AE be the oblique plane, at a given angle, either above or below the horizontal plane AH; AG the direction of



of the piece, and AP the altitude due to the projectile velocity at A.

By the last proposition, find the horizontal range AH to the given velocity and direction; draw HE perpendicular to AH, meeting the oblique plane in E; draw EF parallel to AG, and FI parallel to HE; so shall the projectile pass through I, and the range on the oblique plane will be AI. As is evident by theor. 15 of the Parabola, where it is proved, that if AH, AI be any two lines terminated at the curve, and IF, HE parallel to the axis; then is EF parallel to the tangent AG.

94. *Otherwise*, without the Horizontal Range.

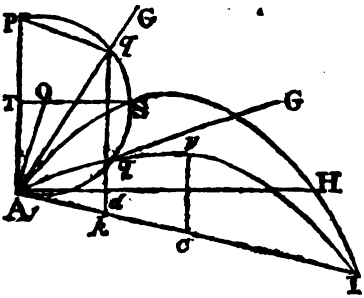
Draw PQ perp. to AG, and QD perp. to the horizontal plane AF, meeting the inclined plane in K; take AE = 4AK, draw EF parallel to AG, and FI parallel to AP or DQ; so shall AI be the range on the oblique plane. For AH = 4AD, therefore EH is parallel to FI, and so on, as above.

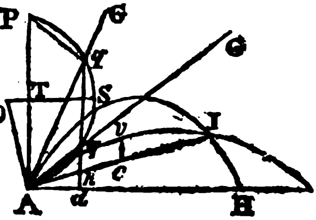
Otherwise.

95. Draw Pq making the angle APq = the angle GAI; then take AG = 4Aq, and draw GI perp. to AH. Or, draw qk perp. to AH, and take AI = 4Ak. Also kq will be equal to cv the greatest height above the plane.

For, by cor. 2, prop. 20, AP : AG : : AG : 4GI
and by sim. triangles, AP : AG : : Aq : GI,
or - - - AP : AG : : 4Aq : 4GI;

therefore AG = 4Aq; and by sim. triangles, AI = 4Ak.
Also qk, or $\frac{1}{4}$GI, is = to cv by theor. 13 of the Parabola.





96. *Corol.* 1. If AO be drawn perp. to the plane AI, and AP be

AP be bisected by the perpendicular STO; then with the centre O describing a circle through A and P, the same will also pass through q, because the angle GAI, formed by the tangent AI and AG, is equal to the angle APq, which will therefore stand on the same arc Aq.

97. *Corol.* 2. If there be given the range AI and the velocity, or the impetus, the direction will hence be easily found thus: Take Ak = $\frac{1}{4}$AI, draw kq perp. to AH, meeting the circle described with the radius AO in two points q and q; then Aq or Aq will be the direction of the piece. And hence it appears that there are two directions, which, with the same impetus, give the very same range AI. And these two directions make equal angles with AI and AP, because the arc Pq is equal the arc Aq. They also make equal angles with a line drawn from A through S, because the arc Sq is equal the arc Sq.

98. *Corol.* 3. Or, if there be given the range AI, and the direction Aq; to find the velocity or impetus. Take Ak = $\frac{1}{4}$ AI, and erect kq perp. to AH, meeting the line of direction in q; then draw qP making the ∠ AqP = ∠Akq; so shall AP be the impetus, or the altitude due to the projectile velocity.

99. *Corol.* 4. The range on an oblique plane, with a given elevation, is directly proportional to the rectangle of the cosine of the direction of the piece above the horizon, and the sine of the direction above the oblique plane, and reciprocally to the square of the cosine of the angle of the plane above or below the horizon.

For, put s = sin. ∠qAI or APq,
c = cos. ∠qAH or sin. PAq,
C = cos. ∠IAH or sin. Akd or Akq or AqP.

Then, in the triangle APq, C : s : : AP : Aq;
and in the triangle Akq, C : c : : Aq : Ak;
theref. by composition, $C^2 : cs$: : AP : AK = $\frac{1}{4}$AI.

So that the oblique range AI = $\frac{cs}{C^2} \times$ 4AP.

100. The range is the greatest when Ak is the greatest; that is, when kq touches the circle in the middle point S; and then the line of direction passes through S, and bisects the angle formed by the oblique plane and the vertex. Also, the ranges are equal at equal angles above and below this direction for the maximum.

101. *Corol.* 5. The greatest height *cv* or kq of the projectile,

tile, above the plane, is equal to $\frac{s^2}{c^2} \times$ AP. And therefore it is as the impetus and square of the sine of direction above the plane directly, and square of the cosine of the plane's inclination reciprocally.

For - c (sin. AQP) : s (sin. APQ) : : AP : AQ,
and c (sin, Akq) : s (sin kAq) : : Aq : kq,
theref. by comp. $c^2 : s^2 :: $ AP : kq.

102. *Corol.* 6. The time of flight in the curve AVI is $= \frac{2s}{c}\sqrt{\frac{AP}{g}}$, where $g = 16\frac{1}{12}$ feet. And therefore it is as the velocity and sine of direction above the plane directly, and cosine of the plane's inclination reciprocally. For the time of describing the curve, is equal to the time of falling freely through GI or 4kq or $\frac{4s^2}{c^2} \times$ AP. Therefore, the time being as the square root of the distance,

$\sqrt{g} : \frac{2s}{c}\sqrt{AP} :: 1'' : \frac{2s}{c}\sqrt{\frac{AP}{g}}$, the time of flight.

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103. From the foregoing corollaries may be collected the following set of theorems, relating to projects made on any given inclined planes, either above or below the horizontal plane. In which the letters denote as before, namely,

c = cos. of direction above the horizon,
C = cos. of inclination of the plane,
s = sin. of direction above the plane,
R the range on the oblique plane,
T the time of flight,
V the projectile velocity,
H the greatest height above the plane,
a the impetus, or alt. due to the velocity V,
$g = 16\frac{1}{12}$ feet. Then,

$$R = \frac{cs}{C^2} \times 4a = \frac{cs}{C^2 g} V^2 = \frac{gC}{s} T^2 = \frac{4c}{s} H.$$

$$H = \frac{s^2}{C^2} a = \frac{s^2 V^2}{4gC^2} = \frac{sR}{4c} = \frac{g}{4} T^2.$$

$$V = \sqrt{4ag} = C\sqrt{\frac{gR}{cs}} = \frac{gC}{s} T = \frac{2C}{s}\sqrt{gH}.$$

$$T = \frac{2s}{C}\sqrt{\frac{a}{g}} = \frac{sV}{gC} = \sqrt{\frac{sR}{gC}} = 2\sqrt{\frac{H}{g}}.$$

And from any of these, the angle of direction may be found.

PRAC-

PRACTICAL GUNNERY.

104. THE two foregoing propositions contain the whole theory of projectiles, with theorems for all the cases, regularly arranged for use, both for oblique and horizontal planes. But, before they can be applied to use in resolving the several cases in the practice of gunnery, it is necessary that some more data be laid down, as derived from good experiments made with balls or shells discharged from cannon or mortars, by gunpowder, under different circumstances. For, without such experiments and data, those theorems can be of very little use in real practice, on account of the imperfections and irregularities in the firing of gunpowder, and the expulsion of balls from guns, but more especially on account of the enormous resistance of the air to all projectiles made with any velocities that are considerable. As to the cases in which projectiles are made with small velocities, or such as do not exceed 200, or 300, or 400 feet per second of time, they may be resolved tolerably near the truth, especially for the larger shells, by the parabolic theory, laid down above. But, in cases of great projectile velocities, that theory is quite inadequate, without the aid of several data drawn from many and good experiments. For so great is the effect of the resistance of the air to projectiles of considerable velocity, that some of those which in the air range only between 2 and 3 miles at the most, would in vacuo range about ten times as far, or between 20 and 30 miles.

The effects of this resistance are also various, according to the velocity, the diameter, and the weight of the projectile. So that the experiments made with one size of ball or shell, will not serve for another size, though the velocity should be the same; neither will the experiments made with one velocity, serve for other velocities, though the ball be the same. And therefore it is plain that, to form proper rules for practical gunnery, we ought to have good experiments made with each size of mortar, and with every variety of charge, from the least to the greatest. And not only so, but these ought also to be repeated at many different angles of elevation, namely, for every single degree between 30° and 60° elevation, and at intervals of 5° above 60° and below 30°, from the vertical direction to point blank. By such a course of experiments it will be found, that the greatest range, instead of being constantly that at an elevation of 45°, as in the parabolic theory, will be at all intermediate degrees between 45 and 30; being

being more or less, both according to the velocity and the weight of the projectile; the smaller velocities and larger shells ranging farthest when projected almost at an elevation of 45°; while the greatest velocities, especially with the smaller shells, range farthest with an elevation of about 30°.

105. There have, at different times, been made certain small parts of such a course of experiments as is hinted at above. Such as the experiments or practice carried on in the year 1773, on Woolwich Common; in which all the sizes of mortars were used, and a variety of small charges of powder. But they were all at the elevation of 45°; consequently these are defective in the higher charges, and in all the other angles of elevation.

Other experiments were also carried on in the same place in the years 1784 and 1786, with various angles of elevation indeed, but with only one size of mortar, and only one charge of powder, and that but a small one too: so that all those nearly agree with the parabolic theory. Other experiments have also been carried on with the ballistic pendulum, at different times; from which have been obtained some of the laws for the quantity of powder, the weight and velocity of the ball, the length of the gun, &c. Namely, that the velocity of the ball varies as the square root of the charge directly, and as the square root of the weight of ball reciprocally; and that, some rounds being fired with a medium length of one-pounder gun, at 15° and 45° elevation, and with 2, 4, 8, and 12 ounces of powder, gave nearly the velocities, ranges, and times of flight, as they are here set down in the following Table.

Powder.	Elevation of gun.	Velocity of ball.	Range.	Time of flight.
oz.		feet.	feet.	
2	15°	860	4100	9″
4	15	1230	5100	12
8	15	1640	6000	$14\frac{1}{2}$
12	15	1680	6700	$15\frac{1}{2}$
2	45	860	5100	21

106. But as we are not yet provided with a sufficient number and variety of experiments, on which to establish true rules for practical gunnery, independent of the parabolic theory, we must at present content ourselves with the data of some

some one certain experimented range and time of flight, at a given angle of elevation; and then by help of these, and the rules in the parabolic theory, determine the like circumstances for other elevations that are not greatly different from the former, assisted by the following practical rules.—

SOME PRACTICAL RULES IN GUNNERY.

I. *To find the Velocity of any Shot or Shell.*

Rule. Divide double the weight of the charge of powder by the weight of the shot, both in lbs. Extract the square root of the quotient. Multiply that root by 1600, and the product will be the velocity in feet, or the number of feet the shot passes over per second.

Or say—As the root of the weight of the shot, is to the root of double the weight of the powder, so is 1600 feet, to the velocity.

II. *Given the range at One Elevation; to find the Range at Another Elevation.*

Rule. As the sine of double the first elevation, is to its range; so is the sine of double another elevation, to its range.

III. *Given the Range for One Charge; to find the Range for Another Charge, or the Charge for Another Range.*

Rule. The ranges have the same proportion as the charges; that is, as one range is to its charge, so is any other range to its charge: the elevation of the piece being the same in both cases.

107. *Example* 1. If a ball of 1 lb. acquire a velocity of 1600 feet per second, when fired with 8 ounces of powder; it is required to find with what velocity each of the several kinds of shells will be discharged by the full charges of powder, viz.

Nature of the shells in inches	13	10	8	$5\frac{1}{2}$	$4\frac{2}{5}$
Their weight in lbs. - -	196	90	48	16	8
Charge of powder in lbs. -	9	4	2	1	$\frac{1}{2}$
Ans. The velocities are -	485	477	462	566	566

108. *Exam.* 2. If a shell be found to range 1000 yards, when discharged at an elevation of 45°; how far will it range

range when the elevation is 30° 16′, the charge of powder being the same? Ans. 2612 feet, or 871 yards.

109. *Exam.* 3. The range of a shell, at 45° elevation, being found to be 3750 feet; at what elevation must the piece be set, to strike an object at the distance of 2810 feet, with the same charge of powder?

Ans. at 24° 16′ or at 65° 44′.

110. *Exam.* 4. With what impetus, velocity, and charge of powder, must a 13-inch shell be fired, at an elevation of 32° 12′, to strike an object at the distance of 3250 feet?

Ans. impetus 1802, veloc. 340, charge 4lb. $7\frac{1}{2}$oz.

111. *Exam.* 5. A shell being found to range 3500 feet, when discharged at an elevation of 25° 12′; how far then will it range at an elevation of 36° 15′ with the same charge of powder? Ans. 4332 feet.

112. *Exam.* 6. If, with a charge of 9lb. of powder, a shell range 4000 feet; what charge will suffice to throw it 3000 feet, the elevation being 45° in both cases?

Ans. $6\frac{3}{4}$lb. of powder.

113. *Exam.* 7. What will be the time of flight for any given range, at the elevation of 45°?

Ans. the time in secs. is $\frac{1}{4}$ the sq. root of the range in feet.

114. *Exam.* 8. In what time will a shell range 3250 feet, at an elevation of 32°? Ans. $11\frac{1}{4}$sec. nearly.

115. *Exam.* 9. How far will a shot range on a plane which ascends 8° 15′; and another which descends 8° 15′; the impetus being 3000 feet, and the elevation of the piece 32° 30′?

Ans. 4244 feet on the ascent,
and 6745 feet on the descent.

116. *Exam.* 10. How much powder will throw a 13-inch shell 4244 feet on an inclined plane, which ascends 8° 15′, the elevation of the mortar being 32° 30′?

Ans. 7·3765lb. or 7lb. 6oz.

117. *Exam.* 11. At what elevation must a 13-inch mortar be pointed, to range 6745 feet, on a plane which descends 8° 15′; the charge $7\frac{3}{8}$lb. of powder? Ans. 32° 28′.

118. *Exam.* 12. In what time will a 13-inch shell strike a plane which rises 8° 30′, when elevated 45°, and discharged with an impetus of 2304 feet? Ans. $14\frac{2}{3}$ seconds.

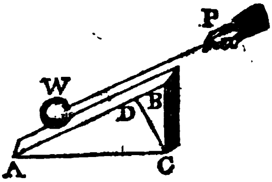
THE

THE DESCENT OF BODIES ON INCLINED PLANES AND CURVE SURFACES.—THE MOTION OF PENDULUMS.

PROPOSITION XXIII.

119. *If a weight* W *be Sustained on an Inclined Plane* AB, *by a Power* P, *acting in a Direction* WP, *Parallel to the Plane. Then*

The Weight of the Body, W	*The Length* AB,
The Sustaining Power P, *and*	*The Height* BC, *and*
The Pressure on the Plane, p,	*The Base* AC,
are respectively as	*of the Plane.*



For, draw CD perpendicular to the plane. Now here are three forces, keeping one another in equilibrio; namely, the weight, or force of gravity, acting perpendicular to AC, or parallel to BC; the power acting parallel to DB; and the pressure perpendicular to AB, or parallel to DC: but when three forces keep one another in equilibrio, they are proportional to the sides of the triangle CBD, made by lines in the direction of those forces, by prop. 8; therefore those forces are to one another as BC, BD, CD. But the two triangles ABC, CBD, are equiangular, and have their like sides proportional; therefore the three BC, BD, CD, are to one another respectively as the three AB, BC, AC; which therefore are as the three forces W, P, p.

120. *Corol.* 1. Hence the weight W, power P, and pressure p, are respectively as radius, sine, and cosine, of the plane's elevation BAC above the horizon.

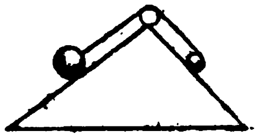
For, since the sides of triangles are as the sines of their opposite angles, therefore the three AB, BC, AC,
are respectively as - - - sin. C, sin. A, sin. B,
or as - - - - - radius, sine, cosine,
of the angle A of elevation.

Or, the three forces are as AC, CD, AD; perpendicular to their directions.

121. *Corol.* 2. The power or relative weight that urges a body W down the inclined plane, is $= \frac{BC}{AB} \times W$; or the force with

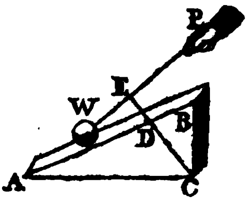
with which it descends, or endeavours to descend, is as the sine of the angle A of inclination.

122. *Corol.* 3. Hence, if there be two planes of the same height, and two bodies be laid on them which are proportional to the lengths of the planes; they will have an equal tendency to descend down the planes.



And consequently they will mutually sustain each other if they be connected by a string acting parallel to the planes.

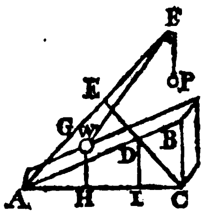
123. *Corol.* 4. In like manner, when the power P acts in any other direction whatever, WP; by drawing CDE perpendicular to the direction WP, the three forces in equilibrio, namely, the weight W, the power P, and the pressure on the plane, will still be respectively as AC, CD, AD, drawn perpendicular to the direction of those forces.



PROPOSITION XXIV.

124. *If a Weight* W *on an Inclined Plane* AB, *be in Equilibrio with another Weight* P *hanging freely; then if they be set a-moving, their Perpendicular Velocities, in that Place, will be Reciprocally as those Weights.*

LET the weight W descend a very small space, from W to A, along the plane, by which the string PFW will come into the position PFA. Draw WH perpendicular to the horizon AC, and WG perpendicular to AF: then WH will be the space perpendicularly descended by the weight W; and AG, or the difference between FA and FW, will be the space perpendicularly ascended by the weight P; and their perpendicular velocities are as those spaces WH and AG passed over in those directions, in the same time. Draw CDE perpendicular to AF, and DI perpendicular to AC.



Then,

in the sim. figs. AGWH and AEDI, AG : WH :: AE : DI;
and in the sim. tri. AEC, DIC, AC : CD :: AE : DI;
but, by cor. 4, prop. 23, AC : CD :: W : P;
therefore, by equality, AG : WH :: W : P;

 That

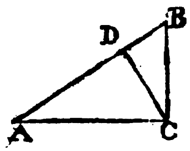
That is, their perpendicular spaces, or velocities, are reciprocally as their weights or masses.

125. *Corol.* 1. Hence it follows, that if any two bodies be in equilibrio on two inclined planes, and if they be set a-moving, their perpendicular velocity will be reciprocally as their weights. Because the perpendicular weight which sustains the one, would also sustain the other.

126. *Corol.* 2. And hence also, if two bodies sustain each other in equilibrio, on any planes, and they be put in motion; then each body multiplied by its perpendicular velocity, will give equal products.

PROPOSITION XXV.

127. *The Velocity acquired by a Body descending freely down an Inclined Plane* AB, *is to the Velocity acquired by a Body falling Perpendicularly, in the same Time; as the Height of the Plane* BC, *is to its Length* AB.



For the force of gravity, both perpendicularly and on the plane, is constant; and these two, by corol. 2, prop. 23, are to each other as AB to BC. But, by art. 28, the velocities generated by any constant forces, in the same time, are as those forces. Therefore the velocity down BA is to the velocity down BC, in the same time, as the force on BA to the force on BC: that is, as BC to BA.

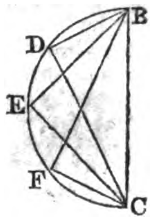
128. *Corol.* 1. Hence, as the motion down an inclined plane is produced by a constant force, it will be a motion uniformly accelerated; and therefore the laws before laid down for accelerated motions in general, hold good for motions on inclined planes; such, for instance, as the following: That the velocities are as the times of descending from rest; that the spaces descended are as the squares of the velocities, or squares of the times; and that if a body be thrown up an inclined plane, with the velocity it acquired in descending, it will lose all its motion, and ascend to the same height, in the same time, and will repass any point of the plane with the same velocity as it passed it in descending.

129. *Corol.* 2. Hence also, the space descended down an inclined plane, is to the space descended perpendicularly, in the same time, as the height of the plane CB, to its length AB, or as the sine of inclination to radius. For the spaces described

described by any forces, in the same time, are as the forces, or as the velocities.

130. *Corol.* 3. Consequently the velocities and spaces descended by bodies down different inclined planes, are as the sines of elevation of the planes.

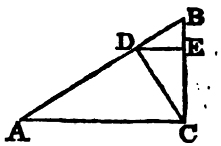
131. *Corol.* 4. If CD be drawn perpendicular to AB; then while a body falls freely through the perpendicular space BC, another body will, in the same time, descend down the part of the plane BD. For by similar triangles, - - - -
BC : BD : : BA : BC, that is, as the space descended, by corol. 2.



Or, in any right-angled triangle BDC, having its hypothenuse BC perpendicular to the horizon, a body will descend down any of its three sides BD, BC, DC, in the same time. And therefore, if on the diameter BC a circle be described, the time of descending down any chords BD, BE, BF, DC, EC, FC, &c, will be all equal, and each equal to the time of falling freely through the perpendicular diameter BC.

PROPOSITION XXVI.

132. *The Time of descending down the Inclined Plane* BA, *is to the Time of falling through the Height of the Plane* BC, *as the Length* BA *is to the Height* BC.



DRAW CD perpendicular to AB. Then the times of describing BD and BC are equal, by the last corol. Call that time t, and the time of describing BA call T.

Now, because the space described by constant forces, are as the squares of the times; therefore $t^2 : T^2 :: BD : BA$.

But the three BD, BC, BA, are in continual proportion; therefore $BD : BA :: BC^2 :: BA^2$;
hence, by equality, $t^2 : T^2 :: BC^2 : BA^2$,
or - - $t : T :: BC : BA$.

133. *Corol.* Hence the times of descending down different planes, of the same height, are to one another as the lengths of the planes.

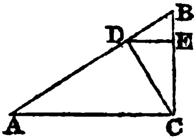
PROPOSITION

PROPOSITION XXVII.

134. *A Body acquires the Same Velocity in descending down any Inclined Plane* BA, *as by falling perpendicular through the Height of the Plane* BC.

For, the velocities generated by any constant forces, are in the compound ratio of the forces and times of acting.
But if we put

F to denote the whole force of gravity in BC,
f the force on the plane AB,
t the time of describing BC, and
T the time of descending down AB;
then by art. 119, F : f : : BA : BC;
and by art. 132, t : T : : BC : BA;
theref. by comp. Ft : fT : : 1 : 1.



That is, the compound ratio of the forces and times, or the ratio of the velocities, is a ratio of equality.

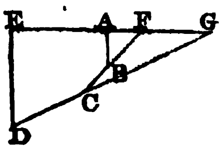
135. *Corol.* 1. Hence the velocities acquired, by bodies descending down any planes, from the same height, to the same horizontal line, are equal.

136. *Corol.* 2. If the velocities be equal, at any two equal altitudes, D, E; they will be equal at all other equal altitudes A, C.

137. *Corol.* 3. Hence also, the velocities acquired by descending down any planes, are as the square roots of the heights.

PROPOSITION XXVIII.

138. *If a Body descend down any Number of Contiguous Planes,* AB, BC, CD; *it will at last acquire the Same Velocity, as a Body falling perpendicularly through the Same Height* ED, *supposing the Velocity not altered by changing from one Plane to another.*



Produce the planes DC, CB, to meet the horizontal line EA produced in F and G. Then, by art. 135, the velocity at B is the same whether the body descend through AB or FB. And therefore the velocity at C will be the same, whether the body descend through ABC or through FC, which

which is also again, by art. 135, the same as by descending through GC. Consequently it will have the same velocity at D, by descending through the planes AB, BC, CD, as by descending through the plane GD; supposing no obstruction to the motion by the body impinging on the planes at B and C: and this again, is the same velocity as by descending through the same perpendicular height ED.

139. *Corol.* 1. If the lines ABCD, &c, be supposed indefinitely small, they will form a curve line, which will be the path of the body; from which it appears that a body acquires also the same velocity in descending along any curve, as in falling perpendicularly through the same height.

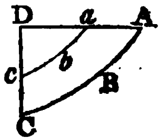
140. *Corol.* 2. Hence also, bodies acquire the same velocity by descending from the same height, whether they descend perpendicularly, or down any planes, or down any curve or curves. And if their velocities be equal, at any one height, they will be equal at all other equal heights. Therefore the velocity acquired by descending down any lines or curves, are as the square roots of the perpendicular heights.

141. *Corol.* 3. And a body, after its descent through any curve, will acquire a velocity which will carry it to the same height through an equal curve, or through any other curve, either by running up the smooth concave side, or by being retained in the curve by a string, and vibrating like a pendulum: Also, the velocities will be equal, at all equal altitudes; and the ascent and descent will be performed in the same time, if the curves be the same.

PROPOSITION XXIX.

142. *The Times in which Bodies descend through Similar Parts of Similar Curves,* ABC, abc, *placed alike, are as the Square Roots of their Lengths.*

THAT is, the time in AC is to the time in ac, as $\sqrt{}$ AC to $\sqrt{}$ ac.



For, as the curves are similar, they may be considered as made up of an equal number of corresponding parts, which are every where, each to each, proportional to the whole. And as they are placed alike, the corresponding small similar parts will also be parallel to each other. But the time of describing each of these pairs of corresponding parallel parts, by art. 128, are as the square roots of their lengths,

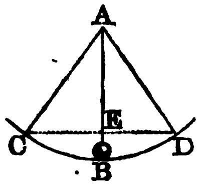
lengths, which, by the supposition, are as $\sqrt{AC}$ to $\sqrt{ac}$, the roots of the whole curves. Therefore, the whole times are in the same ratio of $\sqrt{AC}$ to $\sqrt{ac}$.

143. *Corol.* 1. Because the axes DC, DC, of similar curves, are as the lengths of the similar parts AC, ac; therefore the times of descent in the curves AC, ac, are as $\sqrt{DC}$ to $\sqrt{DC}$, or the square roots of their axes.

144. *Corol.* 2. As it is the same thing, whether the bodies run down the smooth concave side of the curves, or be made to describe those curves by vibrating like a pendulum, the lengths being DC, DC; therefore the times of the vibration of pendulums, in similar arcs of any curves, are as the square roots of the lengths of the pendulums.

SCHOLIUM.

145. Having, in the last corollary, mentioned the pendulum, it may not be improper here to add some remarks concerning it.

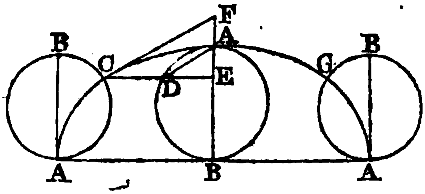


A pendulum consists of a ball, or any other heavy body B, hung by a fine string or thread, moveable about a centre A, and describing the arc CBD; by which vibration the same motions happen to this heavy body, as would happen to any body descending by its gravity along the spherical superficies CBD, if that superficies were perfectly hard and smooth. If the pendulum be carried to the situation AC, and then let fall, the ball in descending will describe the arc CB; and in the point B it will have that velocity which is acquired by decending through CB, or by a body falling freely through EB. This velocity will be sufficient to cause the ball to ascend through an equal arc BD, to the same height D from whence it fell at C; having there lost all its motion, it will again begin to descend by its own gravity; and in the lowest point B it will acquire the same velocity as before; which will cause it to re-ascend to C: and thus, by ascending and descending, it will perform continual vibrations in the circumference CBD. And if the motions of pendulums met with no resistance from the air, and if there were no friction at the centre of motion A, the vibrations of pendulums would never cease. But from these obstructions, though small, it happens, that the velocity of the ball in the point B is a little diminished in every vibration; and consequently it does not return precisely to the same points C or D, but the arcs described continually

tinually become shorter and shorter, till at length they are insensible; unless the motion be assisted by a mechanical contrivance, as in clocks, called a maintaining power.

DEFINITION.

146. If the circumference of a circle be rolled on a right line, beginning at any point A, and continued till the same point A arrive at the line



again, making just one revolution, and thereby measuring out a straight line ABA equal to the circumference of the circle, while the point A in the circumference traces out a curve line ACAGA; then this curve is called a cycloid; and some of its properties are contained in the following lemma.

LEMMA.

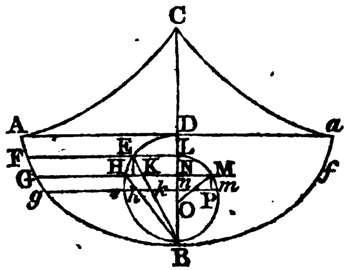
147. If the generating or revolving circle be placed in the middle of the cycloid, its diameter coinciding with the axis AB, and from any point there be drawn the tangent CF, the ordinate CDE perp. to the axis, and the chord of the circle AD: Then the chief properties are these:

The right line CD = the circular arc AD;
The cycloidal arc AC = double the chord AD;
The semi-cycloid ACA = double the diameter AB, and
The tangent CF is parallel to the chord AD.

PROPOSITION XXX.

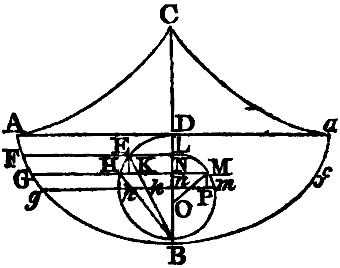
148. *When a Pendulum vibrates in a Cycloid; the Time of one Vibration, is to the Time in which a Body falls through Half the Length of the Pendulum, as the Circumference of a Circle is to its Diameter.*

Let ABa be the cycloid; DB its axis, or the diameter of the generating semicircle DEB; CB = 2DB the length of the pendulum, or radius of curvature at B. Let the ball descend from F, and, in vibrating, describe the arc FBf. Divide FB into innumerable small parts, one of which is Gg; draw FEL, GM, gm, perpendicular to



DB.

DB. On LB describe the semicircle LMB, whose centre is O; draw Mp parallel to DB; also draw the chords BF, BH, EH, and the radius OM.



Now the triangles BEH, BHK, are equiangular; therefore BK : BH : : BH : BE, or $BH^2 = BK \cdot BE$, or $BH = \sqrt{(BK \cdot BE)}$.

And the equiangular triangles Mmp, MON, give Mp : Mm : : MN : MO. Also, by the nature of the cycloid, Hh is equal to Gg.

If another body descend down the chord EB, it will have the same velocity as the ball in the cycloid has at the same height. So that Kk and Gg are passed over with the same velocity, and consequently the time in passing them will be as their lengths Gg, Kk, or as Hh to Kk, or BH, to BK by similar triangles, or $\sqrt{(BK \cdot BE)}$ to BK, or $\sqrt{BE}$ to $\sqrt{BK}$, or as $\sqrt{BL}$ to $\sqrt{BN}$ by similar triangles.

That is, the time in Gg : time in Kk : : $\sqrt{BL}$: $\sqrt{BN}$.

Again, the time of describing any space with a uniform motion, is directly as the space, and reciprocally as the velocity; also, the velocity in K or Kk, is to the velocity at B, as $\sqrt{EK}$ to $\sqrt{EB}$, or as $\sqrt{LN}$ to $\sqrt{LB}$; and the uniform velocity for EB is equal to half that at the point B, therefore the time in Kk : time in EB : : $\frac{Kk}{\sqrt{LN}} : \frac{EB}{\frac{1}{2}\sqrt{LB}} : : \frac{Nn}{\sqrt{LN}} : \frac{LB}{\frac{1}{2}\sqrt{LB}}$ (by sim. tri.) : : Nn or Mp : $2\sqrt{(BL \cdot LN)}$

That is, the time in Kk : time in EB : : Mp : $2\sqrt{(BL \cdot LN)}$

But it was, time in Gg : time in Kk : : $\sqrt{BL}$: $\sqrt{BN}$; theref.

by comp. time in Gg : time in EB : : Mp : $2\sqrt{(BN \cdot NL)}$ or 2NM.

But, by sim. tri Mm : 2OM or BL : : Mp : 2NM.

Theref. time in Gg : time in EB : : Mm : BL.

Consequently the sum of all the times in all the Gg's, is to the time in EB, or the time in DB, which is the same thing, as the sum of all the Mm's, is to LB;

that is, the time in Fg : time in DB : : Lm : LB,

and the time in FB : time in DB : : LMB : LB,

or the time in FBf : time in DB : : 2LMB : LB.

That is, the time of one whole vibration,
is to the time of falling through half CB,
as the circumference of any circle,
is to its diameter.

149. *Corol.*

149. *Corol.* 1. Hence all the vibrations of a pendulum in a cycloid, whether great or small, are performed in the same time, which time is to the time of falling through the axis, or half the length of the pendulum, as 3·1416, to 1, the ratio of the circumference to its diameter; and hence that time is easily found thus. Put $p = 3·1416$, and l the length of the pendulum, also g the space fallen by a heavy body in 1″ of time.

then $\sqrt{g} : \sqrt{\frac{1}{2}l} : : 1'' : \sqrt{\frac{l}{2g}}$ the time of falling through $\frac{1}{2}l$,

theref. $1 : p : : \sqrt{\frac{l}{2g}} : p\sqrt{\frac{l}{2g}}$, which therefore is the time of one vibration of the pendulum.

150. And if the pendulum vibrate in a small arc of a circle; because that small arc nearly coincides with the small cycloidal arc at the vertex B; therefore the time of vibration in the small arc of a circle, is nearly equal to the time of vibration in the cycloidal arc; consequently the time of vibration in a small circular arc, is equal to $p\sqrt{\frac{l}{2g}}$, where l is the radius of the circle.

151. So that, if one of these, g or l, be found by experiment, this theorem will give the other. Thus, if g, or the space fallen through by a heavy body in 1″ of time, be found, then this theorem will give the length of the second pendulum. Or, if the length of the second pendulum be observed by experiment, which is the easier way, this theorem will give g the descent of gravity in 1″. Now, in the latitude of London, the length of a pendulum which vibrates seconds, has been found to be $39\frac{1}{8}$ inches; and this being written for l in the theorem, it gives $p\sqrt{\frac{39\frac{1}{8}}{2g}} = 1''$: hence is found $g = \frac{1}{2}p^2 l = \frac{1}{2}p^2 \times 39\frac{1}{8} = 193·07$ inches $= 16\frac{1}{12}$ feet, for the descent of gravity in 1″; which it has also been found to be, very nearly, by many accurate experiments.

SCHOLIUM.

152. Hence is found the length of a pendulum that shall make any number of vibrations in a given time. Or, the number of vibrations that shall be made by a pendulum of a given length. Thus, suppose it were required to find the length of a half-seconds pendulum, or a quarter-seconds pendulum; that is, a pendulum to vibrate twice in a second, or 4 times in a second. Then, since the time of vibration is as the square root of the length,

 therefore

therefore $1 : \frac{1}{2} :: \sqrt{39\frac{1}{8}} : \sqrt{l}$,

or - - $1 : \frac{1}{4} :: 39\frac{1}{8} : \dfrac{39\frac{1}{8}}{4} = 9\frac{3}{4}$ inches nearly, the length of the half-seconds pendulum. Again $1 : \frac{1}{16} :: 39\frac{1}{8} : 2\frac{4}{9}$ inches, the length of the quarter-seconds pendulum.

Again, if it were required to find how many vibrations a pendulum of 80 inches long will make in a minute. Here

$$\sqrt{80} : \sqrt{39\tfrac{1}{8}} :: 60'' \text{ or } 1' : 60\sqrt{\frac{39\frac{1}{8}}{80}} = 7\tfrac{1}{2}\sqrt{31{\cdot}3} = -\ -$$

41·95987, or almost 42 vibrations in a minute.

153. In these propositions, the thread is supposed to be very fine, or of no sensible weight, and the ball very small, or all the matter united in one point; also, the length of the pendulum, is the distance from the point of suspension, or centre of motion, to this point, or centre of the small ball. But if the ball be large, or the string very thick, or the vibrating body be of any other figure; then the length of the pendulum is different, and is measured, from the centre of motion, not to the centre of magnitude of the body, but to such a point, as that if all the matter of the pendulum were collected into it, it would then vibrate in the same time as the compound pendulum; and this point is called the Centre of Oscillation; a point which will be treated of in what follows.

THE MECHANICAL POWERS, &c.

154. WEIGHT and Power, when opposed to each other, signify the body to be moved, and the body that moves it; or the patient and agent. The power is the agent, which moves, or endeavours to move, the patient or weight.

155. Equilibrium, is an equality of action or force, between two or more powers or weights, acting against each other, by which they destroy each other's effects, and remain at rest.

156. Machine, or Engine, is any mechanical instrument contrived to move bodies. And it is composed of the mechanical powers.

157. Mechanical Powers, are certain simple instruments, commonly employed for raising greater weights, or overcoming greater resistances, than could be effected by the natural strength without them. These are usually accounted six in number,

number, viz. the Lever, the Wheel and Axle, the Pulley, the Inclined Plane, the Wedge, and the Screw.

158. Mechanics, is the science of forces, and the effects they produce, when applied to machines, in the motion of bodies.

159. Statics, is the science of weights, especially when considered in a state of equilibrium.

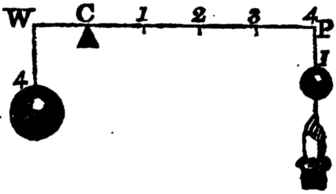
160. Centre of Motion, is the fixed point about which a body moves. And the Axis of Motion, is the fixed line about which it moves.

161. Centre of Gravity, is a certain point, on which a body being freely suspended, it will rest in any position.

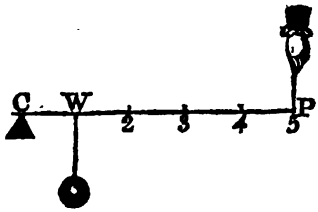
OF THE LEVER.

162. A LEVER is any inflexible rod, bar, or beam, which serves to raise weights, while it is supported at a point by a fulcrum or prop, which is the centre of motion. The lever is supposed to be void of gravity or weight, to render the demonstrations easier and simpler. There are three kinds of levers.

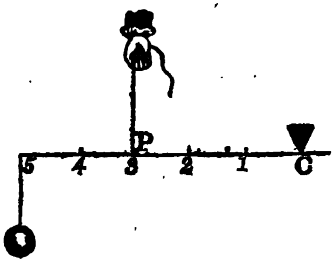
163. A Lever of the First kind has the prop C between the weight W and the power P. And of this kind are balances, scales, crows, hand-spikes, scissors, pinchers, &c.



164. A Lever, of the Second kind has the weight between the power and the prop. Such as oars, rudders, cutting knives that are fixed at one end, &c.

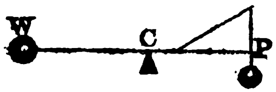


165. A Lever of the Third kind has the power between the weight and the prop. Such as tongs, the bones and muscles of animals, a man rearing a ladder, &c.



166. A

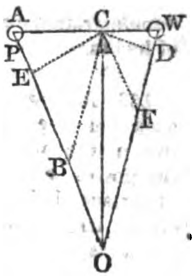
166. A Fourth kind is sometimes added, called the Bended Lever. As a hammer drawing a nail.



167. In all these instruments the power may be represented by a weight, which is its most natural measure, acting downward: but having its direction changed, when necessary, by means of a fixed pulley.

PROPOSITION XXXI.

168. *When the Weight and Power keep the Lever in Equilibrio, they are to each other Reciprocally as the Distances of their Lines of Direction from the Prop. That is,* P : W :: CD : CE; *where* CD *and* CE *are perpendicular to* WO *and* AO, *the Directions of the two Weights, or the Weight and Power* W *and* A.



For, draw CF parallel to AO, and CB parallel to WO: Also join CO, which will be the direction of the pressure on the prop C; for there cannot be an equilibrium unless the directions of the three forces all meet in, or tend to, the same point, as O. Then, because these three forces keep each other in equilibrio, they are proportional to the sides of the triangle CBO or CFO, drawn in the direction of those forces; therefore - - - - P : W :: CF : FO or CB.

But, because of the parallels, the two triangles CDF, CEB are equiangular, therefore - - - CD : CE :: CF : CB.

Hence, by equality, - - P : W :: CD : CE.

That is, each force is reciprocally proportional to the distance of its direction from the fulcrum.

And it will be found that this demonstration will serve for all the other kinds of levers, by drawing the lines as directed.

169. *Corol.* 1. When the angle A is = the angle W, then is CD : CE :: CW : CA :: P : W. Or when the two forces act perpendicularly on the lever, as two weights, &c; then, in case of an equilibrium, D coincides with W, and E with P; consequently then the above proportion becomes also P : W :: CW : CA, or the distances of the two forces from the fulcrum, taken on the lever, are reciprocally proportional to those forces.

170. *Corol.*

170. *Corol.* 2. If any force P be applied to a lever at A; its effect on the lever, to turn it about the centre of motion C, is as the length of the lever CA, and the sine of the angle of direction CAE. For the perp. CE is as $CA \times s. \angle A$.

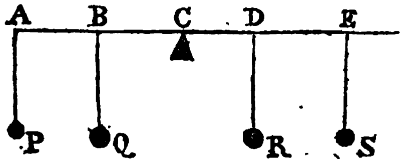
171. *Corol.* 3. Because the product of the extremes is equal to the product of the means, therefore the product of the power by the distance of its direction, is equal to the product of the weight by the distance of its direction.

That is, $P \times CE = W \times CD$.

172. *Corol.* 4. If the lever, with the weight and power fixed to it, be made to move about the centre C; the momentum of the power will be equal to the momentum of the weight; and their velocities will be in reciprocal proportion to each other. For the weight and power will describe circles whose radii are the distances CD, CE; and since the circumferences or spaces described, are as the radii, and also as the velocities, therefore the velocities are as the radii CD, CE; and the momenta, which are as the masses and velocities, are as the masses and radii; that is, as $P \times CE$ and $W \times CD$, which are equal by cor. 3.

173. *Corol.* 5. In a straight lever, kept in equilibrio by a weight and power acting perpendicularly; then, of these three, the power, weight, and pressure on the prop, any one is as the distance of the other two.

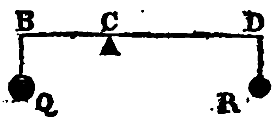
174. *Corol.* 6. If several weights P, Q, R, S, act on a straight lever, and keep it in equilibrio; then the sum of the products on one side of the prop, will be equal to the sum on the other side, made by multiplying each weight by its distance; namely,



$$P \times AC + Q \times BC = R \times DC + S \times EC.$$

For, the effect of each weight to turn the lever, is as the weight multiplied by its distance; and in the case of an equilibrium, the sums of the effects, or of the products on both sides, are equal.

175. *Corol.* 7. Because, when two weights Q and R are in equilibrio, $Q : R :: CD : CB$;

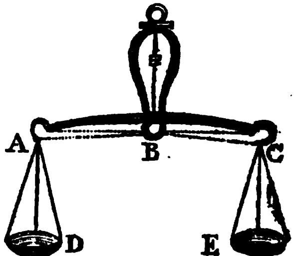


therefore, by composition, $Q + R : Q :: BD : CD$,
and, $Q + R : R :: BD : CB$.

That

That is, the sum of the weights is to either of them, as the sum of their distances is to the distance of the other.

SCHOLIUM.



176. On the foregoing principles depends the nature of scales and beams, for weighing all sorts of goods. For, if the weights be equal, then will the distances be equal also, which gives the construction of the common scales, which ought to have these properties:

1st, That the points of suspension of the scales and the centre of motion of the beam, A, B, C, should be in a straight line: *2d*, That the arms AB, BC, be of an equal length: *3d*, That the centre of gravity be in the centre of motion B, or a little below it: *4th*, That they be in equilibrio when empty: *5th*, That there be as little friction as possible at the centre B. A defect in any of these properties, makes the scales either imperfect or false. But it oftens happens that the one side of the beam is made shorter than the other, and the defect covered by making that scale the heavier, by which means the scales hang in equilibrio when empty; but when they are charged with any weights, so as to be still in equilibrio, those weights are not equal; but the deceit will be detected by changing the weights to the contrary sides, for then the equilibrium will be immediately destroyed.

177. To find the true weight of any body by such a false balance:—First weigh the body in one scale, and afterwards weigh it in the other; then the mean proportional between these two weights, will be the true weight required. For, if any body b weigh W pounds or ounces in the scale D, and only w pounds or punces in the scale E: then we have these two equations, namely, $AB \cdot b = BC \cdot W$.

and $BC \cdot b = AB \cdot w$;

the product of the two is $AB \cdot BC \cdot b^2 = AB \cdot BC \cdot Ww$;

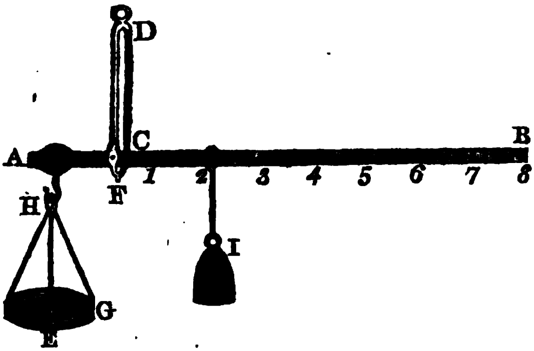
hence then - - - $b^2 = Ww$,

and - - - $b = \sqrt{Ww}$,

the mean proportional, which is the true weight of the body b.

178. The Roman Statera, or Steelyard, is also a lever, but of unequal brachia or arms, so contrived, that one weight only may serve to weigh a great many, by sliding it backward

ward and forward, to different distances, on the longer arm of the lever; and it is thus constructed:



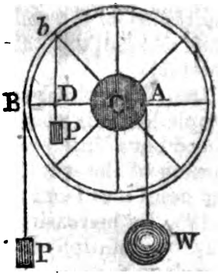
Let AB be the steelyard, and C its centre of motion, whence the divisions must commence if the two arms just balance each other: if not, slide the constant moveable weight I along from B towards C, till it just balance the other end without a weight, and there make a notch in the beam, marking it with a cipher 0. Then hang on at A a weight W equal to I, and slide I back towards B till they balance each other; there notch the beam, and mark it with 1. Then make the weight W double of I, and sliding I back to balance it, there mark it with 2. Do the same at 3, 4, 5, &c, by making W equal to 3, 4, 5, &c, times I; and the beam is finished. Then, to find the weight of any body *b* by the steelyard; take off the weight W, and hang on the body *b* at A; then slide the weight I backward and forward till it just balance the body *b*, which suppose to be at the number 5; then is *b* equal to 5 times the weight of I. So, if I be one pound, then *b* is 5 pounds; but if I be 2 pounds, then *b* is 10 pounds; and so on.

OF THE WHEEL AND AXLE.

PROPOSITION XXXII.

179. *In the Wheel-and-Axle; the Weight and Power will be in Equilibrio, when the Power* P *is to the Weight* W, *Reciprocally as the Radii of the Circles where they act; that is, as the Radius of the Axle* CA, *where the Weight hangs, to the Radius of the Wheel* CB, *where the Power acts. That is,* P : W :: CA : CB.

HERE the cord, by which the power P acts, goes about the

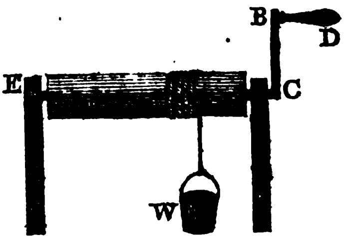


the circumference of the wheel, while that of the weight W goes round its axle, or another smaller wheel, attached to the larger, and having the same axis or centre C. So that BA is a lever moveable about the point C, the power P acting always at the distance BC, and the weight W at the distance CA; therefore P : W :: CA : CB.

180. *Corol.* 1. If the wheel be put in motion; then, the spaces moved being as the circumferences, or as the radii, the velocity of W will be to the velocity of P, as CA to CB; that is, the weight is moved as much slower, as it is heavier than the power; so that what is gained in power, is lost in time. And this is the universal property of all machines and engines.

181. *Corol.* 2. If the power do not act at right angles to the radius CB, but obliquely; draw CD perpendicular to the direction of the power; then, by the nature of the lever, P : W :: CA : CD.

SCHOLIUM.



182. To this power belong all turning or wheel machines, of different radii. Thus, in the roller turning on the axis or spindle CE, by the handle CBD; the power applied at B is to the weight W on the roller as the radius of the roller is to the radius CB of the handle.

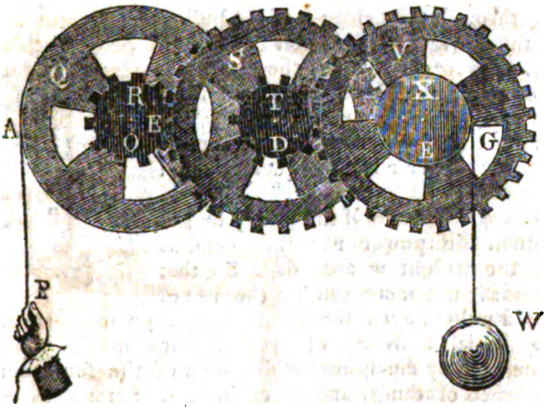
183. And the same for all cranes, capstans, windlasses, and such like; the power being to the weight, always as the radius or lever at which the weight acts, to that at which the power acts; so that they are always in the reciprocal ratio of their velocities. And to the same principle may be referred the gimblet and augur for boring holes.

184. But all this, however, is on supposition that the ropes or chords, sustaining the weights, are of no sensible thickness. For, if the thickness be considerable, or if there be several folds of them, over one another, on the roller or barrel; then we must measure to the middle of the outermost rope, for the

the radius of the roller; or, to the radius of the roller we must add half the thickness of the cord, when there is but one fold.

185. The wheel-and-axle has a great advantage over the simple lever, in point of convenience. For a weight can be raised but a little way by the lever; whereas, by the continual turning of the wheel and roller, the weight may be raised to any height, or from any depth.

186. By increasing the number of wheels too, the power may be multiplied to any extent, making always the less wheels to turn greater ones, as far as we please; and this is commonly called Tooth and Pinion Work, the teeth of one circumference working in the rounds or pinions of another, to turn the wheel. And then, in case of an equilibrium, the power is to the weight, as the continual product of the radii of all the axles, to that of all the wheels. So, if the power P



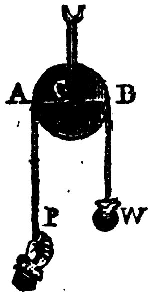
turn the wheel Q, and this turn the small wheel or axle R, and this turn the wheel S, and this turn the axle T, and this turn the wheel V; and this turn the axle X, which raises the weight W; then $P : W :: CB \cdot DE \cdot FG : AC \cdot BD \cdot EF$. And in the same proportion is the velocity of W slower than that of P. Thus, if each wheel be to its axle, as 10 to 1; then $P : W :: 1^3 : 10^3$ or as 1 to 1000. So that a power of one pound will balance a weight of 1000 pounds; but then, when put in motion, the power will move 1000 times faster than the weight.

OF THE PULLEY.

187. A PULLEY is a small wheel, commonly made of wood or brass, which turns about an iron axis passing through the centre, and fixed in a block, by means of a cord passed round its circumference, which serves to draw up any weight. The pulley is either single, or combined together, to increase the power. It is also either fixed or moveable, according as it is fixed to one place, or moves up and down with the weight and power.

PROPOSITION XXXIII.

188. *If a Power sustain a Weight by means of a Fixed Pulley: the Power and Weight are Equal.*



FOR through the centre C of the pulley draw the horizontal diameter AB: then will AB represent a lever of the first kind, its prop being the fixed centre C; from which the points A and B, where the power and weight act, being equally distant, the power P is consequently equal to the weight W.

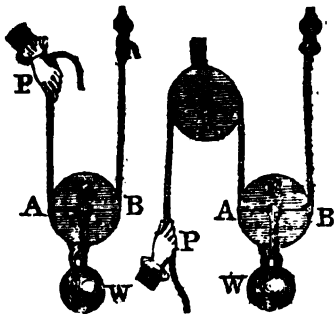
189. *Corol.* Hence, if the pulley be put in motion, the power P will descend as fast as the weight W ascends. So that the power is not increased by the use of the fixed pulley, even though the rope go over several of them. It is, however, of great service in the raising of weights, both by changing the direction of the force, for the convenience of acting, and by enabling a person to raise a weight to any height without moving from his place, and also by permitting a great many persons at once to exert their force on the rope at P, which they could not do to the weight itself; as is evident in raising the hammer or weight of a pile-driver, as well as on many other occasions.

PROPOSITION XXXIV.

190. *If a Power sustain a Weight by means of One Moveable Pulley; the Power is but Half the Weight.*

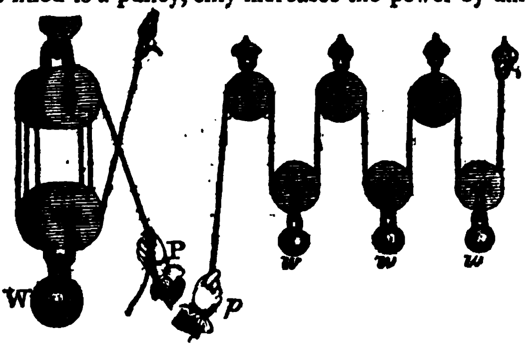
FOR, here AB may be considered as a lever of the second kind,

kind, the power acting at A, the weight at C, and the prop or fixed point at B; and because P : W : : CB : AB, and CB = ½AB, therefore P = ½W, or W = 2P.



191. *Corol.* 1. Hence it is evident, that when the pulley is put in motion, the velocity of the power will be double the velocity of the weight, as the point P moves twice as fast as the point C and weight W rises. It is also evident, that the fixed pulley F makes no difference in the power P, but is only used to change the direction of it, from upwards to downwards.

192. *Corol.* 2. Hence we may estimate the effect of a combination of any number of fixed and moveable pulleys; by which we shall find that every cord going over a moveable pulley always adds 2 to the powers; since each moveable pulley's rope bears an equal share of the weight; while each rope that is fixed to a pulley, only increases the power by unity.



Here $P = \frac{1}{6}W$. Here $p = \frac{1}{2}w = \frac{w + W + w}{6}$

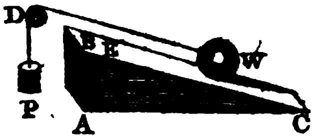
OF THE INCLINED PLANE.

193. THE INCLINED PLANE, is a plane inclined to the horizon, or making an angle with it. It is often reckoned one of the simple mechanic powers; and the double inclined plane makes the wedge. It is employed to advantage in raising heavy bodies in certain situations, diminishing their weights by laying them on the inclined planes.

PROPOSITION

PROPOSITION XXXV.

194. *The Power gained by the Inclined Plane, is in Proportion as the Length of the Plane is to its Height. That is, when a Weight* W *is sustained on an Inclined Plane;* BC, *by a Power* P *acting in the Direction* DW, *parallel to the Plane; then the Weight* W, *is in proportion to the Power* P, *as the Length of the Plane is to its Height; that is,* W : P : : BC : AB.



FOR, draw AE perp. to the plane BC, or to DW. Then we are to consider that the body W is sustained by three forces, viz. 1st, its own weight or the force of gravity, acting perp. to AC, or parallel to BA; 2d, by the power P, acting in the direction WD, parallel to BC, or BE; and 3dly, by the re-action of the plane, perp. to its face, or parallel to the line EA. But when a body is kept in equilibrio by the action of three forces, it has been proved, that the intensities of these forces are proportional to the sides of the triangle ABE, made by lines drawn in the directions of their actions; therefore those forces are to one another as the three lines - - - - AB, BE, AE; that is,
the weight of the body W is as the line AB,
the power P is as the line - - BE,
and the pressure on the plane as the line AE.
But the two triangles ABE, ABC are equiangular, and have therefore their like sides proportional; that is,
the three lines - - - - AB, BE, AE,
are to each other respectively as the three BC, AB, AC,
or also as the three - - - BC, AE, CE,
which therefore are as the three forces W, P, *p*,
where *p* denotes the pressure on the plane. That is, W : P : : BC : AB, or the weight is to the power, as the length of the plane is to its height.

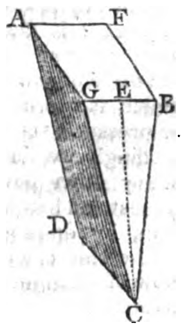
See more on the Inclined Plane, at p. 144, &c.

195. *Scholium.* The Inclined plane comes into use in some situations in which the other mechanical powers cannot be conveniently applied, or in combination with them. As, in sliding heavy weights either up or down a plank or other plane laid sloping: or letting large casks down into a cellar, or drawing them out of it. Also, in removing earth from a lower situation to a higher by means of wheel-barrows, or otherwise, as in making fortifications, &c; inclined planes, made of boards, laid aslope, serve for the barrows to run upon.

Of all the various directions of drawing bodies up an inclined plane, or sustaining them on it, the most favourable is where it is parallel to the plane BC, and passing through the centre of the weight; a direction which is easily given to it, by fixing a pulley at D, so that a chord passing over it, and fixed to the weight, may act or draw parallel to the plane. In every other position, it would require a greater power to support the body on the plane, or to draw it up. For if one end of the line be fixed at W, and the other end inclined down towards B, below the direction WD, the body would be drawn down against the plane, and the power must be increased in proportion to the greater difficulty of the traction. And, on the other hand, if the line were carried above the direction of the plane, the power must be also increased; but here only in proportion as it endeavours to lift the body off the plane.

If the length BC of the plane be equal to any number of times its perp. height AB, as suppose 3 times; then a power P of 1 pound hanging freely, will balance a weight W of 3 pounds, laid on the plane; and a power P of 2 pounds, will balance a weight W of 6 pounds; and so on, always 3 times as much. But then if they be set a-moving, the perp. descent of the power P, will be equal to 3 times as much as the perp. ascent of the weight W. For, though the weight W ascends up the direction of the oblique plane, BC, just as fast as the power P descends perpendicularly, yet the weight rises only the perp. height AB, while it ascends up the whole length of the plane BC, which is 3 times as much; that is, for every foot of the perp. rise, of the weight, it ascends 3 feet up in the direction of the plane, and the power P descends just as much, or 3 feet.

OF THE WEDGE.

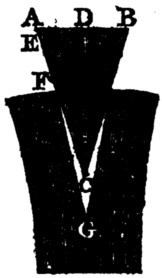


196. THE WEDGE is a piece of wood or metal, in form of half a rectangular prism. AF or BG is the breadth of its back; CE its height; GC, BC its sides; and its end GBC is composed of two equal inclined planes GCE, BCE.

PROPOSITION

PROPOSITION XXXVI.

197. *When a Wedge is in Equilibrio; the Power acting against the Back, is to the Force acting Perpendicularly against either Side, as the Breadth of the Back* AB, *is to the Length of the Side* AC *or* BC.



For, any three forces, which sustain one another in equilibrio, are as the corresponding sides of a triangle drawn perpendicular to the directions in which they act. But AB is perp. to the force acting on the back, to urge the wedge forward; and the sides AC, BC are perp. to the forces acting on them; therefore the three forces are as AB, AC, BC.

198. *Corol.* The force on the back, | AB,
Its effect in direct. perp. to AC, | AC,
And its effect parallel to AB; | DC,
are as the three lines | which are per. to them.

And therefore the thinner a wedge is, the greater is its effect, in splitting any body, or in overcoming any resistance against the sides of the wedge.

SCHOLIUM.

199. But it must be observed, that the resistance, or the forces above-mentioned, respect one side of the wedge only. For if those against both sides be taken in, then, in the foregoing proportions, we must take only half the back AD, or else we must take double the line AC or DC.

In the wedge, the friction against the sides is very great, at least equal to the force to be overcome, because the wedge retains any position to which it is driven; and therefore the resistance is double by the friction. But then the wedge has a great advantage over all the other powers, arising from the force of percussion or blow with which the back is struck, which is a force incomparably greater than any dead weight or pressure, such as is employed in other machines. And accordingly we find it produces effects vastly superior to those of any other power; such as the splitting and raising the largest and hardest rocks, the raising and lifting the largest ship, by driving a wedge below it, which a man can do by the blow of a mallet: and thus it appears that the small blow of a hammer, on the back of a wedge, is incomparably greater than any mere pressure, and will overcome it.

OF

OF THE SCREW.

200. THE SCREW is one of the six mechanical powers; chiefly used in pressing or squeezing bodies close, though sometimes also in raising weights.

The screw is a spiral thread or groove cut round a cylinder, and every where making the same angle with the length of it. So that if the surface of the cylinder, with this spiral thread on it, were unfolded and stretched into a plane, the spiral thread would form a straight inclined plane, whose length would be to its height, as the circumference of the cylinder, is to the distance between two threads of the screw: as is evident by considering that, in making one round, the spiral rises along the cylinder the distance between the two threads.

PROPOSITION XXXVII.

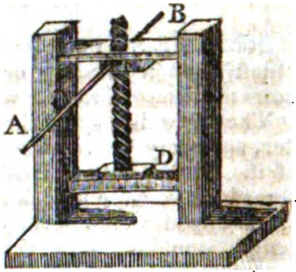
201. *The Force of a Power applied to turn a Screw round, is to the Force with which it presses upward or downward, setting aside the Friction, as the Distance between two Threads, is to the Circumference where the Power is applied.*

THE screw being an inclined plane, or half wedge, whose height is the distance between two threads, and its base the circumference of the screw; and the force in the horizontal direction, being to that in the vertical one, as the lines perpendicular to them, namely, as the height of the plane, or distance of the two threads, is to the base of the plane, or circumference of the screw; therefore the power is to the pressure, as the distance of two threads is to that circumference. But, by means of a handle or lever, the gain in power is increased in the proportion of the radius of the screw to the radius of the power, or length of the handle, or as their circumferences. Therefore, finally, the power is to the pressure, as the distance of the threads, is to the circumference described by the power.

202. *Corol.* When the screw is put in motion; then the power is to the weight which would keep it in equilibrio, as the velocity of the latter is to that of the former; and hence their two momenta are equal, which are produced by multiplying each weight or power by its own velocity. So that this is a general property in all the mechanical powers, namely, that the momentum of a power is equal to that of the weight which would balance it in equilibrio; or that each of them is reciprocally proportional to its velocity.

SCHOLIUM.

SCHOLIUM.



203. Hence we can easily compute the force of any machine turned by a screw. Let the annexed figure represent a press driven by a screw, whose threads are each a quarter of an inch asunder; and let the screw be turned by a handle of 4 feet long, from A to B; then, if the natural force of a man, by which he can lift, pull, or draw, be 150 pounds; and it be required to determine with what force the screw will press on the board at D, when the man turns the handle at A and B, with his whole force. Then the diameter AB of the power being 4 feet, or 48 inches, its circumference is 48 × 3·1416 or $150\frac{4}{5}$ nearly; and the distance of the threads being $\frac{1}{4}$ of an inch; therefore the power is to the pressure as 1 to $603\frac{1}{5}$; but the power is equal to 150lb; theref. as 1 : $603\frac{1}{5}$: : 150 : 90480; and consequently the pressure at D is equal to a weight of 90480 pounds, independent of friction.

204. Again, if the endless screw AB be turned by a handle AC of 20 inches, the threads of the screw being distant half an inch each; and the screw turns a toothed wheel E, whose pinion L turns another wheel F, and the pinion M of this another wheel G, to the pinion or barrel of which is hung a weight W; it is required to determine what weight the man will be able to raise, working at the handle C; supposing the diameters of the wheels to be 18 inches, and those of the pinions and barrel 2 inches; the teeth and pinions being all of a size.

Here

Here $20 \times 3{\cdot}1416 \times 2 = 125{\cdot}664$, is the circumference of the power.

And $125{\cdot}664$ to $\frac{1}{2}$, or $251{\cdot}328$ to 1, is the force of the screw alone.

Also, 18 to 2, or 9 to 1, being the proportion of the wheels to the pinions; and as there are three of them, therefore 9^3 to 1^3, or 729 to 1, is the power gained by the wheels.

Consequently $251{\cdot}328 \times 729$ to 1, or $183218\frac{1}{4}$ to 1 nearly, is the ratio of the power to the weight, arising from the advantage both of the screw and the wheels.

But the power is 150lb; therefore $150 \times 183218\frac{1}{4}$, or 27482716 pounds, is the weight the man can sustain, which is equal to 12269 tons weight.

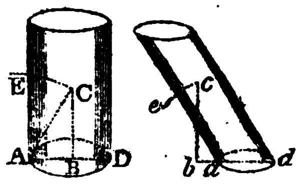
But the power has to overcome, not only the weight, but also the friction of the screw, which is very great, in some cases equal to the weight itself, since it is sometimes sufficient to sustain the weight, when the power is taken off.

ON THE CENTRE OF GRAVITY.

205. THE CENTRE of GRAVITY of a body, is a certain point within it, on which the body being freely suspended, it will rest in any position; and it will always descend to the lowest place to which it can get, in other positions.

PROPOSITION XXXVIII.

206. *If a Perpendicular to the Horizon, from the centre of Gravity of any body, fall within the Base of the Body, it will rest in that Position; but if the Perpendicular fall Without the Base, the Body will not rest in that Position, but will tumble down.*



For, if CB, be the perp. from the centre of gravity C, within the base: then the body cannot fall over towards A; because, in turning on the point A, the centre of gravity C would describe an arc which would rise from C to E; contrary to the nature of that centre, which only rests when in the lowest place. For the same reason, the body will not fall towards D. And therefore it will stand in that position.

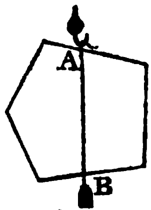
 But

But if the perpendicular fall without the base, as cb; then the body will tumble over on that side: because in turning on the point a, the centre c descends by describing the descending arc ce.

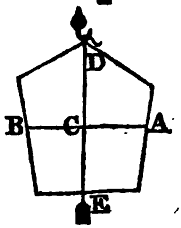
207. *Corol.* 1. If a perpendicular, drawn from the centre of gravity, fall just on the extremity of the base; the body may stand; but any the least force will cause it to fall that way. And the nearer the perpendicular is to any side, or the narrower the base is, the easier it will be made to fall, or be pushed over that way; because the centre of gravity has the less height to rise: which is the reason that a globe is made to roll on a smooth plane by any the least force. But the nearer the perpendicular is to the middle of the base, or the broader the base is, the firmer the body stands.

208. *Corol.* 2. Hence if the centre of gravity of a body be supported, the whole body is supported. And the place of the centre of gravity must be accounted the place of the body; for into that point the whole matter of the body may be supposed to be collected, and therefore all the force also with which it endeavours to descend.

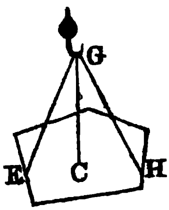
209. *Corol.* 3. From the property which the centre of gravity has, of always descending to the lowest point, is derived an easy mechanical method of finding that centre.



Thus, if the body be hung up by any point A, and a plumb line AB be hung by the same point, it will pass through the centre of gravity; because that centre is not in the lowest point till it fall in the plumb line. Mark the line AB on it. Then hang the body up by any other point D, with a plumb line DE, which will also pass through the centre of gravity, for the same reason as before; and therefore that centre must be at C where the two plumb lines cross each other.



210. Or, if the body be suspended by two or more cords GF, GH, &c. then a plumb line from the point G will cut the body in its centre of gravity C.



211. Like-

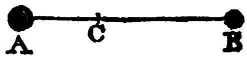
211. Likewise, because a body rests when its centre of gravity is supported, but not else; we hence derive another easy method of finding that centre mechanically. For, if the body be laid on the edge of a prism, or over one side of a table, and moved backward and forward till it rest, or balance itself; then is the centre of gravity just over the line of the edge. And if the body be then shifted into another position, and balanced on the edge again, this line will also pass by the centre of gravity; and consequently the intersection of the two will give the centre itself.

PROPOSITION XXXIX.

212. *The common Centre of Gravity* C *of any two Bodies* A, B, *divides the Line joining their Centres, into two Parts, which are Reciprocally as the Bodies.*

That is, AC : BC :: B : A.

For, if the centre of gravity C be supported, the two bodies A and B will be supported, and will rest in equilibrio. But by the nature of the lever, when



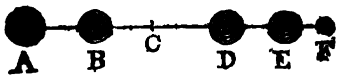
two bodies are in equilibrio about a fixed point C, they are reciprocally as their distances from that point; therefore A : B :: CB : CA.

213. *Corol.* 1. Hence AB : AC :: A + B : B; or, the whole distance between the two bodies, is to the distance of either of them from the common centre, as the sum of the bodies is to the other body.

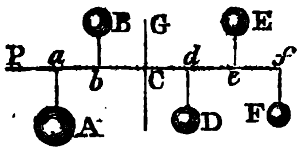
214. *Corol.* 2. Hence also, $CA \cdot A = CB \cdot B$; or the two products are equal, which are made by multiplying each body by its distance from the centre of gravity.

215. *Corol.* 3. As the centre C is pressed with a force equal to both the weights A and B, while the points A and B are each pressed with the respective weights A and B. Therefore, if the two bodies be both united in their common centre C, and only the ends A and B of the line AB be supported, each will still bear, or be pressed by the same weights A and B as before. So that, if a weight of 100lb. be laid on a bar at C, supported by two men at A and B, distant from C, the one 4 feet, and the other 6 feet; then the nearer will bear the weight of 60lb, and the farther only 40lb. weight.

216. Corol.



216. *Corol.* 4. Since the effect of any body to turn a lever about the fixed point C is as that body and as its distance from that point; therefore, if C be the common centre of gravity of all the bodies A, B, D, E, F, placed in the straight line AF; then is CA . A + CB . B = CD . D + CE . E + CF . F; or, the sum of the products on one side, equal to the sum of the products on the other, made by multiplying each body by its distance from that centre. And if several bodies be in equilibrium on any straight lever, then the prop is in the centre of gravity.

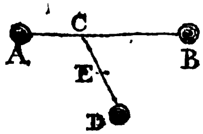


217 *Corol.* 5. And though the bodies be not situated in a straight line, but scattered about in any promiscuous manner, the same property as in the last corollary still holds true, if perpendiculars to any line whatever af be drawn through the several bodies, and their common centre of gravity, namely, that ca : A + cb = cd . D + ce . E + cf . F. For the bodies have the same effect on the line af, to turn it about the point C, whether they are placed at the points a, b, d, e, f, or in any part of the perpendiculars Aa, Bb, Dd, Ee, Ff.

PROPOSITION XL.

218. *If there be three or more Bodies, and if a line be drawn from any one Body* D *to the Centre of Gravity of the rest* C; *then the Common Centre of Gravity* E *of all the Bodies, divides the line* CD *into two Parts in* E, *which are Reciprocally Proportional as the Body* D *to the Sum of all the other Bodies.*

That is, CE : ED :: D : A + B &c.



For, suppose the bodies A and B to be collected into the common centre of gravity C, and let their sum be called S. Then, by the last prop. CE : ED :: D : S or A + B &c.

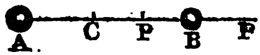
217. *Corol.* Hence we have a method of finding the common centre of gravity of any number of bodies; namely, by first finding the centre of any two of them, then the centre of that centre and a third, and so on for a fourth, or fifth, &c.

PROPOSITION XLI.

220. *If there be taken any Point* P, *in the Line passing through the Centres of two Bodies; then the Sum of the two Products, of each Body multiplied by its Distance from that Point, is equal to the Product of the Sum of the Bodies multiplied by the Distance of their Common Centre of Gravity* C *from the same Point* P.

That is, $PA \cdot A + PB \cdot B = PC \cdot \overline{A + B}$.

For, by the 38th, $CA \cdot A = CB \cdot B$, that is, $\overline{PA - PC} \cdot A = \overline{PC - PB} \cdot B$; therefore, by adding, $PA \cdot A + PB \cdot B = PC \cdot \overline{A + B}$.



221. *Corol.* 1. Hence, the two bodies A and B have the same force to turn the lever about the point P, as if they were both placed in C their common centre of gravity.

Or, if the line, with the bodies, move about the point P; the sum of the momenta of A and B, is equal to the momentum of the sum S or $A + B$ placed at the centre C.

222. *Corol.* 2. The same is also true of any number of bodies whatever, as will appear by cor. 4. prop. 39, namely, $PA \cdot A + PB \cdot B + PD \cdot D$ &c. $= PC \cdot \overline{A + B + D}$ &c, where P is in any point whatever in the line AC.

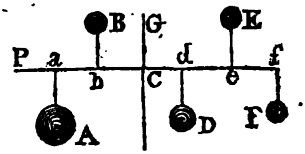
And, by cor. 5, prop. 39, the same thing is true when the bodies are not placed in that line, but any where in the perpendiculars passing through the points A, B, D, &c; namely, $Pa \cdot A + Pb \cdot B + Pd \cdot D$ &c $= Pc \cdot \overline{A + B + D}$ &c.

223. *Corol.* 3. And if a plane pass through the point P perpendicular to the line CP; then the distance of the common centre of gravity from that plane, is

$Pc = \frac{Pa \cdot A + Pb \cdot B + Pd \cdot D \,\&c}{A + B + D \,\&c}$, that is, equal to the sum of all the forces divided by the sum of all the bodies. Or, if A, B, D, &c, be the several particles of one mass or compound body; then the distance of the centre of gravity of the body, below any given point P, is equal to the forces of all the particles divided by the whole mass or body, that is, equal to all the $Pa \cdot A$, $Pb \cdot B$, $Pd \cdot D$, &c, divided by the body or sum of particles A, B, D, &c.

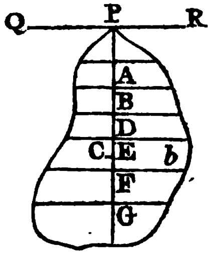
PROPOSITION XLII.

224. *To find the Centre of Gravity of any Body, or of any System of Bodies.*

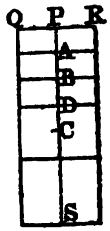


Through any point P draw a plane, and let Pa, Pb, Pd, &c, be the distance of the bodies A, B, D, &c, from the plane; then, by the last cor. the distance of the common centre of gravity from the plane, will be $PC = \frac{Pa \cdot A + Pb \cdot B + Pd \cdot D \ \&c}{A + B + D \ \&c}$.

225. Or, if b be any body, and QPR any plane; draw PAB &c, perpendicular to QR, and through A, B, &c, draw innumerable sections of the body b parallel to the plane QR. Let s denote any of these sections, and $d = PA$, or PB, &c, its distance from the plane QR. Then will the distance of the centre of gravity of the body from the plane be $PC = \frac{\text{sum of all the } d\text{'s}}{b}$. And if the distance be thus found for two intersecting planes, they will give the point in which the centre is placed.



226. But the distance from one plane is sufficient for any regular body, because it is evident that, in such a figure, the centre of gravity is in the axis, or line passing through the centres of all the parallel sections.



Thus, if the figure be a parallelogram, or a cylinder, or any prism whatever; then the axis or line, or plane PS, which bisects all the sections parallel to QR, will pass through the centre of gravity of all those sections, and consequently through that of the whole figure C. Then, all the sections s being equal, and the body $b = PS \cdot s$, the distance of the centre will be $PC =$

$$\frac{PA \cdot s + PB \cdot s + \&c.}{b} = \frac{PA + PB + PD \ \&c.}{b} \times s = \frac{PA + PB + \&c.}{PS}.$$

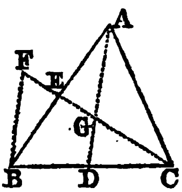
But

But PA + PB + &c, is the sum of an arithmetical progression, beginning at 0, and increasing to the greatest term PS, the number of the terms being also equal to PS; therefore the sum PA + PB + &c $= \frac{1}{2}$PS . PS; and consequently PG $= \frac{\frac{1}{2}\text{PS . PS}}{\text{PS}} = \frac{1}{2}$PS; that is, the centre of gravity is in the middle of the axis of any figure whose parallel sections are equal.

227. In other figures, whose parallel sections are not equal, but varying according to some general law, it will not be easy to find the sum of all the PA . s, PB . s', PD . s'', &c, except by the general method of Fluxions; which case therefore will be best reserved, till we come to treat of that doctrine. It will be proper however to add here some examples of another method of finding the centre of gravity of a triangle, or any other right-lined plane figure.

PROPOSITION XLIII.

228. *To find the Centre of Gravity of a Triangle.*



FROM any two of the angles draw lines AD, CE, to bisect the opposite sides, so will their intersection G be the centre of gravity of the triangle.

For, because AD bisects BC, it bisects also all its parallels, namely, all the parallel sections of the figure; therefore AD passes through the centres of gravity of all the parallel sections or component parts of the figure; and consequently the centre of gravity of the whole figure lies in the line AD. For the same reason, it also lies in the line CE. Consequently it is in their common point of intersection G.

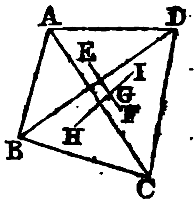
229. *Corol.* The distance of the point G, is AG $= \frac{2}{3}$AD, and CG $= \frac{2}{3}$CE : or AG = 2GD, and CG = 2GE.

For, draw BF parallel to AD, and produce CE to meet it in F. Then the triangles AEG, BEF are similar, and also equal, because AE = BE; consequently AG = BF. But the triangles CDG, CBF are also equiangular, and CB being = 2CD, therefore BF = 2GD. But BF is also = AG; consequently AG = 2GD or $\frac{2}{3}$AD. In like manner, CG = 2GE or $\frac{2}{3}$CE.

PROPOSITION

PROPOSITION XLIV.

230. *To find the Centre of Gravity of a Trapezium.*



DIVIDE the trapezium ABCD into two triangles, by the diagonal BD, and find E, F, the centres of gravity of these two triangles; then shall the centre of gravity of the trapezium lie in the line EF connecting them. And therefore if EF be divided, in G, in the alternate ratio of the two triangles, namely, EG : GF :: triangle BCD : triangle ABD, then G will be the centre of gravity of the trapezium.

231. Or, having found the two points E, F, if the trapezium be divided into two other triangles BAC, DAC, by the other diagonal AC, and the centres of gravity H and I of these two triangles be also found; then the centre of gravity of the trapezium will also lie in the line HI.

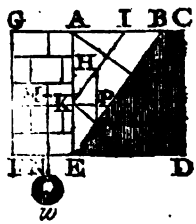
So that, lying in both the lines, EF, HI, it must necessarily lie in their intersection G.

232. And thus we are to proceed for a figure of any greater number of sides, finding the centres of their component triangles and trapeziums, and then finding the common centre of every two of these, till they be all reduced into one only.

Of the use of the place of the centre of gravity, and the nature of forces, the following practical problems are added; viz, to find the force of a bank of earth pressing against a wall, and the force of the wall to support it; also the push of an arch, with the thickness of the piers necessary to support it; also the strength and stress of beams and bars of timber and metal, &c.

PROPOSITION XLV.

233. *To determine the Force with which a Bank of Earth, or such like, presses against a Wall, and the Dimensions of the Wall necessary to Support it.*



LET ACDE be a vertical section of a bank of earth; and suppose, that if it were not supported, a triangular part of it, as ABE, would slide down, leaving it at what is called the natural slope BE; but that, by means of a wall AEFG, it is supported, and kept in its place.—It is required to find the force of ABE, to slide down, and the dimensions of the wall AEFG, to support it.

Let

Let H be the centre of gravity of the triangle ABE, through which draw KHI parallel to the slope face of the earth BE. Now the centre of gravity H may be accounted the place of the triangle ABE, or the point into which it is all collected. Draw HL parallel, and KP perpendicular to AE, also KL perp. to IK or BE. Then if HL represent the force of the triangle ABE in its natural direction HL, HK will denote its force in its direction HK, and PK the same force in the direction PK, perpendicular to the lever EK, on which it acts. Now the three triangles EAB, HKL, HKP are all similar; therefore EB : EA :: (HL : HK ::) w the weight of the triangle EAB : $\frac{\text{EA}}{\text{EB}} w$, which will be the force of the triangle in the direction HK. Then, to find the effect of this force in the direction PK, it will be, as HK : PK :: EB : AB :: $\frac{\text{EA}}{\text{EB}} w : \frac{\text{EA} \cdot \text{AB}}{\text{EB}^2} w$, the force at K, in direction PK, perpendicularly on the lever EK, which is equal to $\frac{1}{3}$AE. But $\frac{1}{2}$AE . AB is the area of the triangle ABE; and if m be the specific gravity of the earth, then $\frac{1}{2}$AE . AB . m is as its weight. Therefore

$\frac{\text{EA} \cdot \text{AB}}{\text{EB}^2} \cdot \frac{1}{2}\text{AE} \cdot \text{AB} = \frac{\text{EA}^2 \ \text{AB}^2}{2\text{EB}^2} m$ is the force acting at K in direction PK. And the effect of this pressure to overturn the wall, is also as the length of the lever KE or $\frac{1}{3}$AE* : con-

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* The principle now employed in the solution of this 45th prop. is a little different from that formerly used; viz, by considering the triangle of earth ABE as acting by lines IK, &c, parallel to the face of the slope BE, instead of acting in directions parallel to the horizon AB; an alteration which gives the length of the lever EK, only the half of what it was in the former way, viz. EK = $\frac{1}{3}$AE instead of $\frac{2}{3}$AE: but every thing else remaining the same as before. Indeed this problem has formerly been treated on a variety of different hypotheses, by Mr. Muller, &c, in this country, and by many French and other authors in other countries. And this has been chiefly owing to the uncertain way in which loose earth may be supposed to act in such a case; which on account of its various circumstances of tenacity, friction, &c, will not perhaps admit of a strict mechanical certainty. On these accounts it seems probable that it is to good experiments only, made on different kinds of earth and walls, that we may probably hope for a just and satisfactory solution of the problem.

The above solution is given only in the most simple case of the problem. But the same principle may easily be extended to any other case that may be required, either in theory or practice, either with walls or banks of earth of different figures, and in different situations.

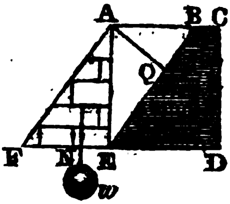
sequently its effect is $\frac{EA^3 \cdot AB^2}{6EB^3} m$, for the perpendicular force against E, to overset the wall AEFG. Which must be balanced by the counter resistance of the wall, in order that it may at least be supported.

Now, if M be the centre of gravity of the wall, into which its whole matter may be supposed to be collected, and acting in the direction MNW, its effect will be the same as if a weight W were suspended from the point N of the lever FN. Hence, if A be put for the area of the wall AEFG, and n its specific gravity; then A . n will be equal to the weight W, and A . n . FN its effect on the lever to prevent it from turning about the point F. And as this effort must be equal to that of the triangle of earth, that it may just support it, which was before found equal to $\frac{EA^3 \cdot AB^2}{6EB^3} m$; therefore A . n . FN = $\frac{AE^3 \cdot AB^2}{6EB^3} m$, in case of an equilibrium.

234. But now, both the breadth of the wall FE, and the lever FN, or place of the centre of gravity M, will depend on the figure of the wall. If the wall be rectangular, or as broad at top as bottom; then FN = $\frac{1}{2}$FE, and the area A = AE . FE; consequently the effort of the wall A . n . FN is = $\frac{1}{2}FE^2 \cdot AE \cdot n$; which must be = $\frac{AE^3 \cdot AB^2}{6EB^2} m$, the effort of the earth. And the resolution of this equation gives the breadth of the wall FE = $\frac{AB \cdot AE}{EB} \sqrt{\frac{m}{3n}} = AQ \sqrt{\frac{m}{3n}}$, drawing AQ perp. to EB. So that the breadth of the wall is always proportional to the perp. depth AQ of the triangle ABE. But the breadth must be made a little more than the above value of it, that it may be more than a bare balance to the earth.— If the angle of the slope E be 45°, as it is nearly in most cases; then FE = $\frac{AE}{\sqrt{2}} \sqrt{\frac{m}{3n}} = AE \sqrt{\frac{m}{6n}} = \frac{2}{5} AE \sqrt{\frac{m}{n}}$ very nearly.

235. If the wall be of brick, its specific gravity is about 2000, and that of earth about 1984; namely, m to n as 1984 to 2000; or they may be taken as equal; then $\sqrt{\frac{m}{n}} = 1$ very nearly; and hence FE = $\frac{4}{10}$AE, or $\frac{2}{5}$AE nearly. That is, whenever a brick rectangular wall is made to support earth, its thickness must be at least $\frac{2}{5}$ or $\frac{4}{10}$ of its height. But if the

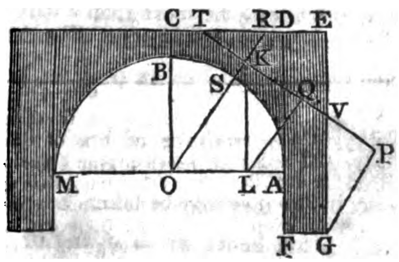
the wall be of stone, whose specific gravity is about 2520; then $\frac{m}{n} = \frac{4}{5}$, and $\sqrt{\frac{m}{n}} = \sqrt{\frac{4}{5}} = \cdot 895$; hence FE $= \cdot 358$ AE $= \frac{5}{14}$AE: that is, when the rectangular wall is of stone, the breadth must be at least $\frac{5}{14}$ of its height.



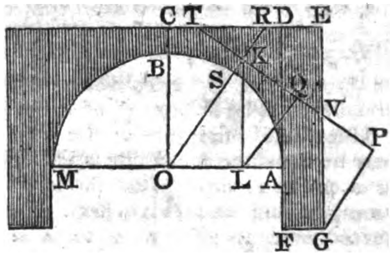
236. But if the figure of the wall be a triangle, the outer side tapering to a point at top. Then the lever FN $= \frac{1}{3}$FE, and the area A $= \frac{1}{2}$FE . AE; consequently its effort A . n . FN is $= \frac{1}{6}$FE^2 . AE . n; which being put $= \frac{AE^3 . AB^2}{6BE^2}m$, the equation gives FE $= \frac{AB . AE}{EB}\sqrt{\frac{m}{2n}} =$ AQ $\sqrt{\frac{m}{2n}}$ for the breadth of the wall at the bottom, for an equilibrium in this case also. —If the angle of the slope E be 45°; then will FE be $= \frac{AE}{\sqrt{2}}\sqrt{\frac{m}{2n}} = \frac{1}{2}AE\sqrt{\frac{m}{n}}$. And when this wall is of brick, then FE $= \frac{1}{2}$AE nearly. But when it is of stone; then $\frac{1}{2}\sqrt{\frac{m}{n}} = \cdot 447 = \frac{4}{9}$ nearly; that is, the triangular stone wall must have its thickness at bottom equal to $\frac{4}{9}$ of its height. And in like manner, for other figures of the wall and also for other figures of the earth.

PROPOSITION XLVI.

237. *To determine the Thickness of a Pier, necessary to support a given Arch.*



LET ABCD be half the arch, and DEFG the pier. From the centre of gravity K of the half arch draw KL perp. OA; also OKR, and TKQP perp. to it; also draw LQ and GP perp. to TP, or parallel to OKR. Then if KL represent the weight of the arch BCDA, in the direction of gravity, this will resolve into KQ, the force acting against the pier perp. to the joint SR, and LQ the part of the force parallel to the same. Now KQ denotes



notes the only force acting perp. on the arm GP, of the crooked lever FGP, to turn the pier about the point G; conseq. KQ × GP will denote the efficacious force of the arch to overturn the pier.

Again, the weight of the pier is as the area DF × FG; therefore DF . FG . $\frac{1}{2}$FG, or $\frac{1}{2}$DF . FG^2, is its effect on the lever $\frac{1}{2}$FG, to prevent the pier from being overset; supposing the length of the pier, from point to point, to be no more than the thickness of the arch.

But that the pier and the arch may be in equilibrio, these two efforts must be equal. Therefore we have $\frac{1}{2}DF . FG^2 = \frac{KQ . GP \; A}{KL}$, an equation, by which will be determined the thickness of the pier FG; A denoting the area of the half arch BCDA*.

Example 1. Suppose the arc ABM to be a semicircle; and that CD or OA or OB = 45, BC = 7 feet, AF = 20. Hence AD = 52, DF = GE = 72. Also by measurement are found OK = 50·3, KL = 40·6, LO = 29·7, TD = 30·87, KQ = 24, the area BCDA = 750 = A; and putting FG = x the breadth of the pier.

Then TE = TD + DE = 30·87 + x, and KL : LO : : TE : EV = 22·58 + 0·73x,

then GE — EV = GV = 49·42 — ·73x,

lastly OK : KL : : GV : GP = 39·89 — ·59x.

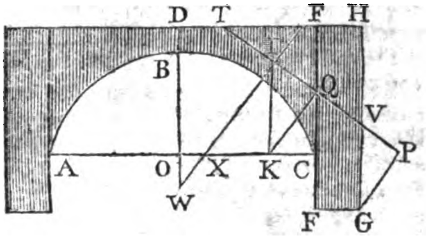
These values being now substituted in the theorem $\frac{1}{2}DF . FG^2 = \frac{KQ . GP . A}{KL}$, give $36x^2 = 17665 - 261{\cdot}5x$, or $x^2 +$

* *Note.* As it is commonly a troublesome thing to calculate the place of the centre of gravity K of the half arch ADCB, it may be easily, and sufficiently near, found mechanically in the manner described in art. 211, thus: Construct that space ADCB accurately by a scale to the given dimensions, on a plate of any uniform flat substance, or even card paper; then cut it nicely out by the extreme lines, and balance it over any edge or the sides of a table in two positions, and the intersection of the two places will give the situation of the point K; then the distances or lines may be measured by the scale, except those depending on the breadth of the pier FG, viz. the lines as mentioned in the examples.

7·26x

$7{\cdot}26x = 490{\cdot}7$; the root of which quadratic equation gives $x = 18{\cdot}8$ feet $=$ DE or FG, the thickness of the pier sought.

Example 2. Suppose the span to be 100 feet, the height 40 feet, the thickness at the top 6 feet, and the height of the pier to the springer 20 feet, as before.



Here the fig. may be considered as a circular segment, having the versed sine OB $= 40$, and the right sine OA or OC $= 50$; also BD $= 6$, CF $= 20$, and EF $= 66$. Now, by the nature of the circle, whose centre is W, the radius WB $=$
$\frac{OB^2 + OC^2}{2OB} = \frac{40^2 + 50^2}{80} = 51\frac{1}{4}$; hence OW $= 51\frac{1}{4} - 40 = 11\frac{1}{4}$;
and the area of the semi-segment OBC is found to be 1491; which taken from the rectangle ODEC $=$ OD . OC $= 46 \times 50 = 2300$, there remains $809 = A$, the area of the space BDECB. Hence, by the method of balancing this space, and measuring the lines, there will be found, KC $= 18$, IK $= 34{\cdot}6$, IX $= 42$, KX $= 24$, OX $= 8$, IQ $= 19{\cdot}4$, TE $= 35{\cdot}6$, and TH $= 35{\cdot}6 + x$, putting $x =$ EH, the breadth of the pier. Then IK : KX :: TH : HV $= 24{\cdot}7 + 0{\cdot}7x$; hence GH $-$ HV $= 41{\cdot}3 - 0{\cdot}7 =$ GV, and IX : IK :: GV : GP $= 34{\cdot}02 - 0{\cdot}58x$. These values being now substituted in the theorem $\frac{1}{2}EF \cdot EG^2 = \frac{IQ \cdot GP \cdot A}{IK}$, gives $33x^2 = 15431{\cdot}47 - 263x$, or $x^2 + 8x = 467{\cdot}62$, the root of which quadratic equation gives $x = 18 =$ EH or FG, the breadth of the pier, and which is probably very near the truth.

ON THE STRENGTH AND STRESS OF BEAMS OR BARS OF TIMBER AND METAL, &c.

238. Another use of the centre of gravity, which may be here considered, is in determining the strength and the stress of beams and bars of timber and metal, &c, in different positions; that is, the force or resistance which a beam or bar makes, to oppose any exertion or endeavour made to break it: and the force or exertion tending to break it; both

both of which will be different, according to the place and position of the centres of gravity.

PROPOSITION XLVII.

239. *The Absolute Strength of any Bar in the Direction of its Length, is Directly Proportional to the Area of its Transverse Section.*

SUPPOSE the bar to be suspended by one end, and hanging freely in the manner of a pendulum; and suppose it to be strained in direction of its length, by any force, or a weight acting at the lower part, in the direction of that length, sufficient to break the bar, or to separate all its particles. Now, as the straining force acts in the direction of the length, all the particles in the transverse section of the body, where it breaks, are equally strained at the same time; and they must all separate or break together, as the bar is supposed to be of uniform texture. Thus then, the particles all adhering and resisting with equal force, the united strength of the whole, will be proportional to the number of them, or as the transverse section at the fracture.

240. *Corol.* 1. Hence the various shapes of bars make no difference in their absolute strength; this depending only on the area of the section, and must be the same in all equal areas, whether round, or square, or oblong, or solid, or hollow, &c.

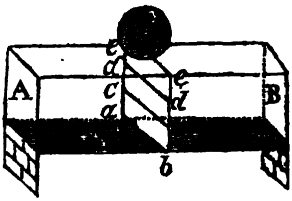
241. *Corol.* 2. Hence also, the absolute strengths of different bars, of the same materials, are to each other as their transverse sections, whatever their shape or form may be.

242. *Corol.* 3. The bar is of equal strength in every part of it, when of any uniform thickness, or prismatic shape, and is equally liable to be drawn asunder at any part of its length, whatever that length may be, by a weight acting at the bottom, independent of the weight of the bar itself; but when considered with its own weight, it is the more disposed to break, and with the less additional appended weight, the longer the bar is, on account of its own weight increasing with its length. And, for the same reason, it will be more and more liable to be broken at every point of its length, all the way in ascending or counting from the bottom to the top, where it may always be expected to part asunder. And hence we see the reason why longer bars are, in this way, more liable to break than shorter ones, or with less appended weights. Hence also we perceive that, by gradually increasing these weights, till the bar separates and breaks, then

then the last or greatest weight, is the proper measure of the absolute strength of the bar. And the same is the case with a rope, or cord, &c.—So much then for the longitudinal strength and stress of bodies. Proceed we now to consider those of their transverse actions.

PROPOSITION XLVIII.

243. *The Strength of a Beam or Bar, of Wood or Metal, &c, in a Lateral or Transverse Direction, to resist a Force acting Laterally, is Proportional to the Area or Section of the Beam in that Place, Drawn into the Distance of its Centre of Gravity from the Place where the Force acts, or where the Fracture will end.*



Let AB represent the beam or bar, supported at its two ends, and on which is laid a weight w, to cause a transverse fracture *abee*. The force w acting downwards there, the fracture will commence or open across the fibres, in the opposite or lowest line *ab*; from thence, as the weight presses down the upper line *ee*, the fracture will open more and more below, and extend gradually upwards, successively to the parallel lines of fibres *cc*, *dd*, &c, till it arrive at, and finally open in the last line of fibres *ee*, where it ends; when the whole fracture is in the form of a wedge, widest at the bottom, and ending in an edge or line *ee* at top. Now the area *ae* contains and denotes the sum of all the fibres to be broken or torn asunder; and as they are supposed to be all equal to one another, in absolute strength, that area will denote the aggregate or whole strength of all the fibres in the longitudinal direction, as in the foregoing proposition. But, with regard to lateral strength, each fibre must be considered as acting at the extremity of a lever whose centre of motion is in the line *ee*: thus, each fibre in the line *ab*, will resist the fracture, by a force proportional to the product of its individual strength into its distance *ae* from the centre of motion; consequently the resistance of all the fibres in *ab*, will be expressed by $ab \times ae$. In like manner, the aggregate resistance of another course of fibres, parallel to *ab*, as *cc*, will be denoted by $cc \times ce$; and a third, as *dd*, by $dd \times de$; and so on throughout the whole fracture. So that the sum of all these products will express the total strength or resistance

ance of all the fibres or of the beam in that part. But, by art. 222, the sum of all these products is equal to the product of the area *aeeb*, into the distance of its centre of gravity from *ee*. Hence the proposition is manifest.

244. *Corol.* 1. Hence it is evident that the lateral strength of a bar, must be considerably less than the absolute longitudinal strength considered in the former proposition, and will be broken by a much less force, than was there necessary to draw the bar asunder lengthways. Because, in the one case the fibres must be all separated at once, in an instant; but in the other, they are overcome and broken successively, one after another, and in some portion of time. For instance, take a walking stick, and stretching it lengthways, it will bear a very great force before it can be drawn asunder; but again taking such a stick, apply the middle of it to the bended knee, and with the two hands drawing the end towards you, the stick is broken across by a small force.

245. *Corol.* 2. In square beams, the lateral strengths are as the cubes of the breadths or depths.

246. *Corol.* 3. And in general, the lateral strengths of any bars, whose sections are similar figures, are as the cubes of the similar sides of the sections.

247. *Corol.* 4. In cylindrical beams, the lateral strengths are as the cubes of the diameters.

248. *Corol.* 5. In rectangular beams, the lateral strengths are to each other, as the breadths and square of the depths.

249. *Corol.* 6. Therefore a joist laid on its narrow edge, is stronger than when laid on its flat side horizontal, in proportion as the breadth exceeds the thickness. Thus if a joist be 10 inches broad, by $2\frac{1}{2}$ thick, then it will bear 4 times more when laid on edge, than when laid flat. Which shows the propriety of the modern method of flooring with very thin, but deep joists.

250. *Corol.* 7. If a beam be fixed firmly by one end into a wall, in a horizontal position, and the fracture be caused by a weight suspended at the other end, the process would be the same, only that the fracture would commence above, and terminate at the lower side; and the prop. and all the corollaries would still hold good.

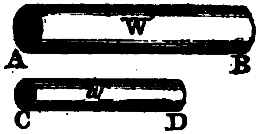
251. *Corol.* 8. When a cylinder or prism is made hollow, it is stronger than when solid, with an equal quantity of materials

rials and length, in the same proportion as its outer diameter is greater. Which shows the wisdom of Providence in making the stalks of corn, and the feathers and bones of animals, &c, to be hollow. Also, if the hollow beam have the hollow or pipe not in the middle, but nearest to that side where the fracture is to end, it will be so much the stronger.

252 *Corol.* 9. If the beam be a triangular prism, it will be strongest when laid with the edge upwards, if the fracture commence or open first on the under side; otherwise with the flat side upwards; because in either case the centre of gravity is the farther from the ending of the fracture. And the same thing is true, and for the same reason, for any other shape of the prism. On the same account also, a square beam is stronger when laid, or when acting angle-wise, than when on a flat side.

PROPOSITION XLIX.

253. *The Lateral Strengths of Prismatic Beams, of the same materials, are Directly as the Areas of the Sections and the Distances of their Centres of Gravity; and Inversely as their Lengths and Weights.*



Let AB and CD represent the two beams fixed horizontally, by their ends, into an upright wall AC. Now, by the last prop. the strength of either beam, considered as without or independent of weight, is as its section drawn into the distance of its centre of gravity from the fixed point, viz. as sc, where s denotes the transverse section at A or C, and c the distance of its centre of gravity above the lowest point A or C. But the effort of their weight, W or w, tending to separate the fibres and break the beam, are, by the principle of the lever, as the weight drawn into the distance of the place where it may be supposed to be collected and applied, which is in the middle of the length of the beam; that is, the effort of the weight upon the beam is as W $\times \frac{1}{2}$AB. Hence the prop. is manifest.

254. *Corol.* 1. Any extraneous weight or force also, anywhere applied to the beam, will have a similar effect to break the beam as its own weight; that is, its effect will be as $w \times d$, as the weight drawn into the length of lever or distance from A where it is applied.

255. *Corol.*

255. *Corol.* 2. When the beam is fixed at both ends, the same property will hold good, with this difference only, that in this case the beam is of the same strength, as another of an equal section, and only half the length, when fixed only at one end For, if the longer beam were bisected, or cut in halves, each half would be in the same circumstances with respect to its fixed end, as the shorter beam of equal length.

256. *Corol.* 3. Square prisms and cylinders have their lateral strengths proportional to the cubes of the depths, or diameters, directly, and to their lengths and weights inversely.

Corol 4. Similar prisms and cylinders have their strengths inversely proportional to their like linear dimensions, the smaller being comparatively larger in that proportion. For their strength increases as the cube of the diameter or of their length; but their stress, from their weight and length of lever, as the 4th power of the length.

257. *Scholium.* From the foregoing deductions it follows that, in similar bodies of the same texture, the force which tends to break them, or to make them liable to injury by accidents, in the larger bodies, increases in a higher proportion than the force which tends to preserve them entire, or to secure them against such accidents; their disadvantage, or tendency to break by their own weight, increasing in the same proportion as their length increases: so that, though a smaller beam may be firm and secure, yet a large and similar one may be so long as to break by its own weight. Hence, it is justly concluded, that what may appear very firm and successful in a model or small machine, may be weak and infirm, or even fall in pieces by its own weight, when it is executed on large dimensions according to the model.

For, in similar bodies, or engines, or in animals, the greater must be always more liable to accidents than the smaller, and have a less relative strength, that is, the greater have not a strength in so great a proportion as their magnitude. A greater column, for instance, is in much more danger of breaking by a fall, than a similar smaller one. A man is in more danger from accidents of this kind than a child. An insect can bear and carry a load many times heavier than itself; whereas a larger animal, as a horse, for instance, can hardly support a burden equal to his own weight.

From the same principle it is also justly inferred, that there are necessarily limits in all the works of nature and art,

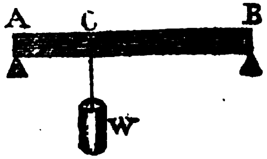
art, which they cannot surpass in magnitude. Thus, for instance, were trees to be of a very enormous size, their branches would break and fall off by their own weight. Large animals have not strength in proportion to their size: and if there were any land animals much larger than those we know, they would hardly be able to move, and would be perpetually subjected to most dangerous accidents.

As to the sea animals indeed, the case is different, as the pressure of the water in a great measure sustains them; and accordingly we find they are vastly larger than land animals.

From what has been said it clearly follows, that to make bodies, or engines, or animals, of equal relative strength, the larger ones must have grosser proportions, or a higher degree of thickness, than they have of length. And this sentiment being suggested to us by continual experience, we naturally join the idea of greater strength and force with the grosser proportions, and of agility with the more delicate ones. In architecture, where the *appearance* of solidity is no less regarded than *real* firmness and strength, in order to satisfy a judicious eye and taste, the various orders of the columns serve to suggest different ideas of strength. But, by the same principle, if we should suppose animals vastly large, from the gross proportions a heaviness and unwieldiness would arise, which would make them useless to themselves, and disagreeable to the eye. In this, as in all other cases, whatever generally pleases taste, not vitiated by prejudice of education, or by fabulous and marvellous relations, may be traced till it appears to have a just foundation in nature.

PROPOSITION L.

258. *If a Weight be placed, or a Force act, on any part of a Horizontal beam, supported at both ends, the Stress upon that part, will be as the Rectangle or Product of its two Distances from the supported ends.*



That is, the stress upon the beam AB, at C, by the weight W, is as AC × BC. For, by the nature of the lever, the effect of the weight W, on the lever AC, is AC . W; and the effect of this force acting at C, on the lever BC, is AC . W . BC = AC . BC . W. And, the weight W being given, the effect or stress is as AC . BC.

259. *Corol.*

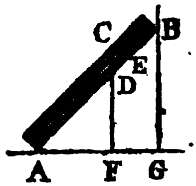
259. *Corol.* 1. The greatest stress is when the weight W is at the middle: for then the rectangle of the two halves, $AC \cdot AC = \frac{1}{2}AB \cdot \frac{1}{2}AB = \frac{1}{4}AB^2$, is the greatest. And, from the middle point, the stress is less and less all the way to the extremities A and B, where it is nothing.

260. *Corol.* 2. The same thing will obtain from the weight of the beam itself, or from any other weight diffused equally all over it; the stress in this case being the half of the former. So that, in all structures, we should avoid as much as possible, placing weights or strains in the middle of beams.

261. *Corol.* 3. If w be the greatest weight that a beam can sustain at its middle point; and it be required to find the place where it will support any greater weight W; that point will be found by making, as $W : w :: \frac{1}{2}AB \cdot \frac{1}{2}AB$, or $\frac{1}{4}AB^2 : AC \cdot BC$ or $AC \times (AB - AC) = AB \cdot AC - AC^2$.

PROPOSITION LI.

262. *When a Beam is placed aslope, its Strength in that position, is to its Strength when Horizontal, to resist a Vertical Force, as the Square of Radius is to the Square of the Cosine of the Elevation.*



Let AB be the beam standing aslope, CF perp. to the horizon AFG; then CD is the vertical section of the beam, and CE, perp. to AB, is the transverse section, and is the same as when in the horizontal position. Now, the strength, in both positions, is as the section drawn into the distance of its centre of gravity from the point C. But the sections, being of the same breadth, are as their depths, CD, CE; and the distances of the centres of gravity are as the same depths; therefore the strengths are as $CD \cdot CD$ to $CE \cdot CE$, or CD^2 to CE^2. But, by the similar triangles CDE, AFD, it is $CD : CE :: AD : AF$, as radius to the cosine of the elevation. Therefore the oblique strength is to the transverse strength, as AD^2 to AF^2, the square of radius to the square of the cosine of elevation.

263. *Corol.* 1. The strength of a beam increases from the horizontal position, where it is least, all the way as it revolves to the vertical position, where it is the greatest.

PRO-

PROPOSITION LII.

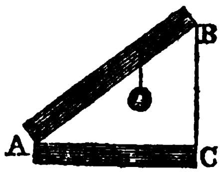
264. *When Beams stand Aslope, or Obliquely, and sustaining Weights, either at the Middle Points, or in any other Similar Situations, or Equally Diffused over their Lengths; the Strains upon them are Directly as the Weights, and the Lengths, and the Cosines of Elevation.*

For, by the inclined plane, the weight is to the pressure on the plane, as AC to AF, as radius to the cosine of elevation: therefore the pressure is as the weight drawn into the cosine of the elevation. Hence the stress will be as the length of the beam and this force; that is, as the weight × length × cosine of elevation.

265. *Corol.* 1. When the lengths and weights of beams are the same, the stress is as the cosine of elevation; and it is therefore the greatest when it lies horizontal.

266. *Corol.* 2. In all similar positions, and the weights varying as the lengths, or the beams uniform; then the stress varies as the squares of the lengths.

267. *Corol.* 3. When the weights are equal, on the oblique beam AB, and the horizontal one AC, and BC is vertical; the stress on both beams is equal. For, the length into the cosine of elevation is the same in both; or AB × cos. A = AC × radius.



268. *Corol.* 4. But if the weights on the beams vary as their lengths; then the stress will also vary in the same ratio.

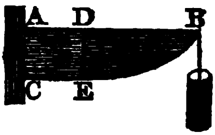
269. *Corol.* 5. And universally, the stress upon any point of an oblique beam, is as the rectangle of the segments of the beam, and the weight, and cosine of inclination, directly; and the length inversely.

PRO-

PROPOSITION LIII.

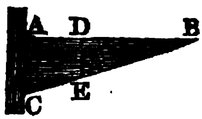
270. *When a Beam is to sustain any Weight, or Pressure, or Force, acting Laterally; then the Strength ought to be as the Stress upon it; that is, the Breadth multiplied by the Square of the Depth, or in similar sections, the Cube of the Diameter, in every place, ought to be proportional to the Length drawn into the Weight or Force acting on it. And the same is true of several Different Pieces of timber compared together.*

For every several piece of timber or metal, as well as every part of the same, ought to have its strength proportioned to the weight, force, or pressure it is to support. And therefore the strength ought to be universally, or in every part as the stress upon it. But the strength is as the breadth into the square of the depth; and the stress is as the weight or force into the distance it acts at. Therefore these must be in constant ratio. This general property will give rise to the effect of different shapes in beams, according to particular circumstances; as in the following corollaries.



271. *Corol.* 1. If ABC be a horizontal beam, fixed at the end AC, and sustaining a weight at the other end B. And if the sections at all places be similar figures; and DE be the diameter at any place D; then BD will be every where as DE^3. So that if ADB be a right line, then BEC will be cubic parabola. In which case $\frac{1}{4}$ of such a beam may be cut away, without any diminution of the strength.—But if the beam be bounded by two parallel planes, perpendicular to the horizon; then BD will be as DE^2; and then BEC will be the common parabola. In which case a 3d part of the beam may be thus cut away.

272. *Corol.* 2. But if a weight press uniformly on every part of AB; and the sections in all points, as D, be similar; then BD^2 will be every where as DE^3: and then BEC is the semicubical parabola.



But, in this disposition of the weight, if the beam be bounded by parallel plains, perpendicular to the horizon; then BD will be always as DE; and BEC a right line, or ABC a wedge. So that then half the beam may be cut away, without diminution of strength.

273. *Corol.*

273. *Corol.* 3. If the beam AB be supported at both ends; and if it sustain a weight at any variable point D, or uniformly on all parts of its length; and if all the sections be similar figures; then will the diameter DE^3 be every where as the rectangle AD . DB.

But if it be bounded by two parallel planes, perpendicular to the horizon; then will DE^2 be every where as the rectangle AD . DB, and the curve AEB an ellipsis.

274. *Corol.* 4. But if a weight be placed at any given point F, and all the sections be similar figures; then will AD be as DE^3, and AG, BG be two cubic parabolas.

But if the beam be bounded by two parallel planes, perpendicular to the horizon; then AD is as DE^2, and AG and BG are two common parabolas.

275. *Scholium.* The relative strengths of several sorts of wood, and of other bodies, as determined by Mr. Emerson, are as follow:

Iron	107
Brass	50
Bone	22
Box, Yew, Plumtree, Oak	11
Elm, Ash	$8\frac{1}{2}$
Walnut, Thorn	$7\frac{1}{2}$
Red fir, Holly, Elder, Plane, Crabtree, Appletree	7
Beech, Cherrytree, Hazle	$6\frac{2}{3}$
Lead	$6\frac{1}{2}$
Alder, Asp, Birch, White fir, Willow	6
Fine freestone	1

A cylindric rod of good clean fir, of 1 inch circumference, drawn lengthways, will bear at extremity 400 lbs; and a spear of fir, 2 inches diameter, will bear about 7 tons in that direction.

A rod of good iron, of an inch circumference, will bear a stretch of near 3 tons weight.

A good hempen rope, of an inch circumference, will bear 1000 lbs at the most.

Hence Mr. Emerson concludes, that if a rod of fir, or of iron,

iron, or a rope of d inches diameter, were to lift $\frac{1}{4}$ of the extreme weight; then

The fir would bear $8\frac{4}{5}\,d^2$ hundred weights.
The rope - - $22\,d^2$ ditto.
The iron - - $6\frac{3}{4}\,d^2$ tons.

Mr. Banks, an ingenious lecturer on mechanics, made many experiments on the strength of wood and metal; whence he concludes, that cast iron is from $3\frac{1}{2}$ to $4\frac{1}{2}$ times stronger than oak of equal dimensions; and from 5 to $6\frac{1}{2}$ times stronger than deal. And that bars of cast iron, an inch square, weighing 9 lbs. to the yard in length, supported at the extremities, bear on an average, a load of 970 lbs. laterally. And they bend about an inch before they break.

Many other experiments on the strength of different materials, and curious results deduced from them, may be seen in Dr. Gregory's and Mr. Emerson's Treatises on Mechanics, as well as some more propositions on the strength and stress of different bars.

ON THE CENTRES OF PERCUSSION, OSCILLATION, AND GYRATION.

276. THE CENTRE of PERCUSSION of a body, or a system of bodies, revolving about a point, or axis, is that point, which striking an immoveable object, the whole mass shall not incline to either side, but rest as it were in equilibrio, without acting on the centre of suspension.

277. The Centre of Oscillation is that point, in a body vibrating by its gravity, in which if any body be placed, or if the whole mass be collected, it will perform its vibrations in the same time, and with the same angular velocity, as the whole body, about the same point or axis of suspension.

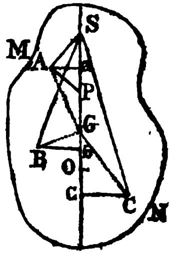
278. The Centre of Gyration, is that point, in which if the whole mass be collected, the same angular velocity will be generated in the same time, by a given force acting at any place, as in the body or system itself.

279. The angular motion of a body, or system of bodies, is the motion of a line connecting any point and the centre or axis of motion; and is the same in all parts of the same revolving body. And in different unconnected bodies, each revolving about a centre, the angular velocity is as the absolute velocity directly, and as the distance from the centre inversely; so that, if their absolute velocities be as their radii or distances, the angular velocities will be equal.

PRO-

PROPOSITION LIV.

280. *To find the Centre of Percussion of a Body, or System of Bodies.*



Let the body revolve about an axis passing through any point S in the line SGO, passing through the centres of gravity and percussion, G and O. Let MN be the section of the body, or the plane in which the axis SGO moves. And conceive all the particles of the body to be reduced to this plane, by perpendiculars let fall from them to the plane: a supposition which will not affect the centres G, O, nor the angular motion of the body.

Let A be the place of one of the particles, so reduced; join SA, and draw AP perpendicular to AS, and Aa perpendicular to SGO: then AP will be the direction of A's motion as it revolves about S; and the whole mass being stopped at O, the body A will urge the point P, forward, with a force proportional to its quantity of matter and velocity, or to its matter and distance from the point of suspension S; that is, as $A \cdot SA$; and the efficacy of this force in a direction perpendicular to SO, at the point P, is as $A \cdot Sa$, by similar triangles; also, the effect of this force on the lever, to turn it about O, being as the length of the lever, is as $A \cdot Sa \cdot PO = A \cdot Sa \cdot (SO - SP) = A \cdot Sa \cdot SO - A \cdot Sa \cdot SP = A \cdot Sa \cdot SO - A \cdot SA^2$. In like manner, the forces of B and C, to turn the system about O, are as

$$B \cdot Sb \cdot SO - B \cdot SB^2, \text{ and}$$
$$C \cdot Sc \cdot SO - C \cdot SC^2, \text{ \&c.}$$

But, since the forces on the contrary sides of O destroy one another, by the definition of this force, the sum of the positive parts of these quantities must be equal to the sum of the negative parts,

that is, $A \cdot Sa \cdot SO + B \cdot Sb \cdot SO + C \cdot Sc \cdot SO \text{ \&c} =$
$$A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2 \text{ \&c; and}$$
$$\text{hence } SO = \frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2 \text{ \&c}}{A \cdot Sa + B \cdot Sb + C \cdot Sc \text{ \&c}},$$

 which

which is the distance of the centre of percussion below the axis of motion.

And here it may be observed that, if any of the points a, b, &c, fall on the contrary side of s, the corresponding product A . sa, or B . sb, &c, must be made negative.

281. *Corol.* 1. Since, by cor. 3, pr. 40, A + B + C &c, or the body $b \times$ the distance of the centre of gravity, SG, is $=$ A . sa + B . sb + C . sc &c, which is the denominator of the value of SO; therefore the distance of the centre of percussion, is $\text{SO} = \frac{\text{A} \cdot \text{SA}^2 + \text{B} \cdot \text{SB}^2 + \text{C} \cdot \text{SC}^2 \text{ \&c}}{\text{SG} \times \text{body } b}$.

282. *Corol.* 2. Since, by Geometry, theor. 36, 37,

it is $\text{SA}^2 = \text{SG}^2 + \text{GA}^2 - 2\text{SG} \cdot \text{Ga}$,

and $\text{SB}^2 = \text{SG}^2 + \text{GB}^2 + 2\text{SG} \cdot \text{Gb}$,

and $\text{SC}^2 = \text{SG}^2 + \text{GC}^2 + 2\text{SG} \cdot \text{GC}$, &c;

and, by cor. 5, pr. 40, the sum of the last terms is nothing, namely, $- 2\text{SG} \cdot \text{Ga} + 2\text{SG} \cdot \text{Gb} + 2\text{SG} \cdot \text{GC} \text{ \&c} = 0$; therefore the sum of the others, or $\text{A} \cdot \text{SA}^2 + \text{B} \cdot \text{SB}^2$ &c

is $= (\text{A} + \text{B} \text{ \&c}) \cdot \text{SO}^2 + \text{A} \cdot \text{GA}^2 + \text{B} \cdot \text{GB}^2 + \text{C} \cdot \text{GC}^2$ &c,

or $= b \cdot \text{SG}^2 + \text{A} \cdot \text{GA}^2 + \text{B} \cdot \text{GB}^2 + \text{C} \cdot \text{GC}^2$ &c;

which being substituted in the numerator of the foregoing value of SO, gives

$$\text{SO} = \frac{b \cdot \text{SG}^2 + \text{A} \cdot \text{GA}^2 + \text{B} \cdot \text{GB}^2 + \text{\&c}}{b \cdot \text{SG}},$$

$$\text{or SO} = \text{SG} + \frac{\text{A} \cdot \text{GA}^2 + \text{B} \cdot \text{GB}^2 + \text{C} \cdot \text{GC}^2 \text{ \&c}}{b \cdot \text{SG}}.$$

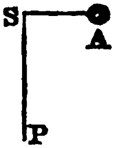
283. *Corol.* 3. Hence the distance of the centre of percussion always exceeds the distance of the centre of gravity, and the excess is always $\text{GO} = \frac{\text{A} \cdot \text{GA}^2 + \text{B} \cdot \text{GB}^2 \text{ \&c}}{b \cdot \text{SG}}$.

284. And hence also, $\text{SG} \cdot \text{GO} = \frac{\text{A} \cdot \text{GA}^2 + \text{B} \cdot \text{GB}^2 \text{ \&c}}{\text{the body } b}$; that is SG . GO is always the same constant quantity, wherever the point of suspension S is placed; since the point G and the bodies A, B, &c, are constant. Or GO is always reciprocally as SG, that is GO is less, as SG is greater; and consequently the point O rises upwards and approaches towords the point G, as the point S is removed to the greater distance; and they coincide when SG is infinite. But when S coincides with G, then GO is infinite, or O is at an infinite distance.

PROPOSITION

PROPOSITION LV.

285. *If a Body* A, *at the Distance* SA *from an axis passing through* S, *be made to revolve about that axis by any Force acting at* P *in the Line* SP, *Perpendicular to the Axis of Motion: It is required to determine the Quantity or Matter of another Body* Q, *which being placed at* P, *the Point where the Force acts, it shall be accelerated in the Same Manner, as when* A *revolved at the Distance* SA; *and consequently, that the Angular Velocity of* A *and* Q *about* S, *may be the Same in Both Cases.*

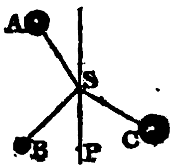


By the nature of the lever, SA : SP : : f : $\frac{SP}{SA} . f$, the effect of the force f, acting at P, on the body at A; that is, the force f acting at P, will have the same effect on the body A, as the force $\frac{SP}{SA} f$, acting directly at the point A.

But as ASP revolves altogether about the axis at S, the absolute velocities of the points A and S, or of the bodies A and Q, will be as the radii SA, SP, of the circle described by them. Here then we have two bodies A and Q, which being urged directly by the forces f and $\frac{SP}{SA} f$, acquire velocities which are as SP and SA. And since the motive forces of bodies are as their mass and velocity: therefore - - - -

$\frac{SP}{SA} f : f : : A . SA : Q . SP$, and $SP^2 : SA^2 : : A : Q = \frac{SA^2}{SP^2} A$, which therefore expresses the mass of matter which, being placed at P, would receive the same angular motion from the action of any force at P, as the body A receives. So that the resistance of any body A, to a force acting at any point P, is directly as the square of its distance SA from the axis of motion, and reciprocally as the square of the distance SP of the point where the force acts.

286. *Corol.* 1. Hence the force which accelerates the point P, is to the force of gravity, as $\frac{f . SP^2}{A . SA^2}$ to 1, or as $f . SP^2$ to $A . SA^2$.



287. *Corol.* 2. If any number of bodies A, B, C, be put in motion, about a fixed axis passing through S, by a force acting at P; the point P will be accelerated in the same manner, and consequently the whole system will have the same angular velocity, if instead of the

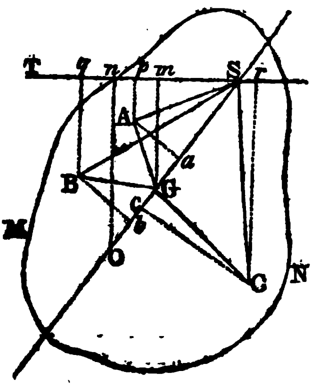
bodies A, B, C, placed at the distances SA, SB, SC, there be substituted the bodies $\frac{SA^2}{SP^2}A$, $\frac{SB^2}{SP^2}B$, $\frac{SC^2}{SP^2}C$; these being collected into the point P. And hence, the moving force being f, and the matter moved being $\frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}{SP^2}$; theref. $\frac{f \cdot SP^2}{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}$ is the accelerating force; which therefore is to the accelerating force of gravity, as $f \cdot SP^2$ to $A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2$.

288. *Corol.* 3. The angular velocity of the whole system of bodies, is as $\frac{f \cdot SP}{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}$. For the absolute velocity of the point P, is as the accelerating force, or directly as the motive force f, and inversely as the mass $\frac{A \cdot SA^2 \text{ \&c}}{SP^2}$: but the angular velocity is as the absolute velocity directly, and the radius SP inversely; therefore the angular velocity of P, or of the whole system, which is the same thing, is as $\frac{f \cdot SP}{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}$.

PROPOSITION LVI.

289. *To determine the Centre of Oscillation of any Compound Mass or Body* MN, *or of any System of Bodies* A, B, C, *&c.*

Let MN be the plane of vibration, to which let all the matter be reduced, by letting fall perpendiculars from every particle, to this plane. Let G be the centre of gravity, and O the centre of oscillation; through the axis S draw SGO, and the horizontal line Sq; then from every particle A, B, C, &c, let fall perpendiculars Aa, Ap, Bb, Bq, Cc, Cr, to these two lines; and join SA, SB, SC; also, draw Gm, On, perpendicular to Sq. Now the forces of the weights A, B, C, to turn the body about the axis, are $A \cdot Sp$, $B \cdot Sq$, $- C \cdot Sr$; therefore, by cor. 3, prop. 55, the angular



motion

motion generated by all these forces is $\frac{A \cdot sp + B \cdot sq - C \cdot sr}{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}$. Also, the angular veloc. any particle p, placed in O, generates in the system, by its weight, is $\frac{p \cdot sn}{p \cdot SO^2}$ or $\frac{sn}{SO^2}$, or $\frac{sm}{SG \cdot SO}$, because of the similar triangles SGm, SOn. But, by the problem, the vibrations are performed alike in both cases, and therefore these two expressions must be equal to each other, that is, $\frac{sm}{SG \cdot SO} = \frac{A \cdot sp + B \cdot sq - C \cdot sr}{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}$; and hence

$$SO = \frac{sm}{SG} \times \frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}{A \cdot sp + B \cdot sq - C \cdot sr}.$$

But, by cor. 2, pr. 41, the sum $A \cdot sp + B \cdot sq - C \cdot sr = (A + B + C) \cdot sm$; therefore the distance SO = - - -

$$\frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}{SG \cdot (A + B + C)} = \frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}{A \cdot sa + B \cdot sb + C \cdot sc}$$

by prop 42, which is the distance of the centre of oscillation O, below the axis of suspension; where any of the products $A \cdot sa$, $B \cdot sb$, must be negative, when a, b &c, lie on the other side of S. So that this is the same expression as that for the distance of the centre of percussion, found in prop. 54.

Hence it appears, that the centres of percussion and of oscillation, are in the very same point. And therefore the properties in all the corollaries there found for the former, are to be here understood of the latter.

290. *Corol.* 1. If p be any particle of a body b, and d its distance from the axis of motion S; also G, O the centres of gravity and oscillation. Then the distance of the centre of oscillation of the body, from the axis of motion, is - - -

$$SO = \frac{\text{sum of all the } pd^2}{SG \times \text{the body } b}.$$

291. *Corol.* 2. If b denote the matter in any compound body, whose centres of gravity and oscillation are G and O; the body P, which being placed at P, where the force acts as in the last proposition, and which receives the same motion from that force as the compound body b, is $P = \frac{SG \cdot SO}{SP^2} \cdot b$.

For, by corol. 2, prop. 54, this body P is = - - - - $\frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}{SP^2}$. But, by corol. 1, prop. 53,

$SG \cdot SO \cdot b = A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2$. therefore $P = \frac{SG \cdot SO}{SP} \cdot b$

SCHOLIUM.

292. By the method of Fluxions, the centre of oscillation, for a regular body, will be found from cor. 1. But for an irregular one; suspend it at the given point; and hang up also a simple pendulum of such a length, that making them both vibrate, they may keep time together. Then the length of the simple pendulum, is equal to the distance of the centre of oscillation of the body, below the point of suspension.

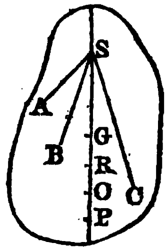
293. Or it will be still better found thus: Suspend the body very freely by the given point, and make it vibrate in small arcs, counting the number of vibrations it makes in any time, as a minute, by a good stop watch; and let that number of vibrations made in a minute be called n: Then shall the distance of the centre of oscillation, be $SO = \frac{140850}{nn}$ inches. For the length of the pendulum vibrating seconds, or 60 times in a minute, being $39\frac{1}{8}$ inches; and the lengths of pendulums being reciprocally as the square of the number of vibrations made in the same time; therefore - - - - $n^2 : 60^2 :: 39\frac{1}{8} : \frac{60^2 \times 39\frac{1}{8}}{nn} = \frac{140850}{nn}$: the length of the pendulum which vibrates n times in a minute, or the distance of the centre of oscillation below the axis of motion.

294. The foregoing determination of the point, into which all the matter of a body being collected, it shall oscillate in the same manner as before, only respects the case in which the body is put in motion by the gravity of its own particles, and the point is the centre of oscillation: but when the body is put in motion by some other extraneous force, instead of its gravity, then the point is different from the former, and is called the Centre of Gyration; which is determined in the following manner:

PRO-

PROPOSITION LVII.

295. *To determine the Centre of Gyration of a Compound Body or of a System of Bodies.*



Let R be the centre of gyration, or the point into which all the particles A, B, C, &c, being collected, it shall receive the same angular motion from a force f acting at P, as the whole system receives.

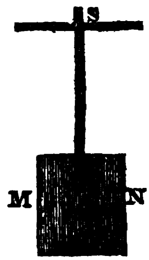
Now, by cor. 3, pr. 54, the angular velocity generated in the system by the force f, is as $\frac{f \cdot SP}{A \cdot SA^2 + B \cdot SB^2 \&c}$, and by the same, the angular velocity of the system placed in R, is $\frac{f \cdot SP}{(A + B + C \&c) \cdot SR^2}$: then, by making these two expressions equal to each other, the equation gives - - - $SR = \sqrt{\frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}{A + B + C}}$, for the distance of the centre of gyration below the axis of motion.

296. *Corol.* 1. Because $A \cdot SA^2 + B \cdot SB^2 \&c = SG \cdot SO \cdot b$; where G is the centre of gravity, O the centre of oscillation, and b the body A + B + C &c; therefore $SR^2 = SG \cdot SO$; that is, the distance of the centre of gyration, is a mean proportional between those of gravity and oscillation.

297. *Corol.* 2. If p denote any particle of a body b, at d distance from the axis of motion; then $SR^2 = \frac{\text{sum of all the } pd^2}{\text{body } b}$.

PROPOSITION LVIII.

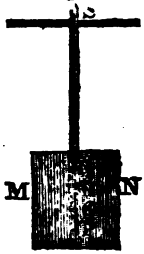
298. *To determine the Velocity with which a Ball moves, which being shot against a Ballistic Pendulum, causes it to vibrate through a given Angle.*



The Ballistic Pendulum is a heavy block of wood MN, suspended vertically by a strong horizontal iron axis at S, to which it is connected by a firm iron stem. This problem is the application of the last proposition, or of prop. 54, and was invented by the very ingenious Mr. Robins, to determine the initial velocities of military projectiles; a circumstance very useful in that science; and it is the best method yet known for determining them with any degree of accuracy.

Let

Let G, R, O be the centres of gravity, gyration, and oscillation, as determined by the foregoing propositions; and let P be the point where the ball strikes the face of the pendulum; the momentum of which, or the product of its weight and velocity, is expressed by the force f, acting at P, in the foregoing propositions. Now,



Put p = the whole weight of the pendul.
b = the weight of the ball,
g = SG the dist. of the cen. of grav.
o = SO the dist. of the cen. of oscilla.
r = SR = $\sqrt{go}$ the dist. of cen. of gyr.
i = SP the dist. of the point of impact,
v = the velocity of the ball,
u = that of the point of impact P,
c = chord of the arc described by O.

By prop. 56, if the mass p be placed all at R, the pendulum will receive the same motion from the blow in the point P: and as $SP^2 : SR^2 :: p : \frac{SR^2}{SP^2} . p$ or $\frac{r^2}{i^2}p$ or $\frac{go}{ii}p$, (prop. 54), the mass which being placed at P, the pendulum will still receive the same motion as before. Here then are two quantities of matter, namely, b and $\frac{go}{ii}p$, the former moving with the velocity v, and striking the latter at rest; to determine their common velocity u, with which they will jointly proceed forward together after the stroke. In which case, by the law of the impact of non-elastic bodies, we have $\frac{go}{ii}p + b : b :: v : u$, and therefore $v = \frac{bii + gop}{bii}u$ the velocity of the ball in terms of u, the velocity of the point P, and the known dimensions and weights of the bodies.

But now to determine the value of u, we must have recourse to the angle through which the pendulum vibrates; for when the pendulum descends down again to the vertical position, it will have acquired the same velocity with which it began to ascend, and, by the laws of falling bodies, the velocity of the centre of oscillation is such, as a heavy body would acquire by freely falling through the versed sine of the arc described by the same centre O. But the chord of that arc is c, and its radius is o; and, by the nature of the circle, the chord is a mean proportional between the versed sine and diameter, therefore $2o : c :: c : \frac{cc}{2o}$, the versed sine of the arc described by O. Then, by the laws of falling bodies

√16

$\sqrt{16\frac{1}{12}} : \sqrt{\frac{cc}{2o}} :: 32\frac{1}{6} : c\sqrt{\frac{2a}{o}}$, the velocity acquired by the point o in descending through the arc whose chord is c, where $a = 16\frac{1}{12}$ feet: and therefore $o : i :: c\sqrt{\frac{2a}{o}} : \frac{ci}{o}\sqrt{\frac{2a}{o}}$, which is the velocity u, of the point P.

Then, by substituting this value for u, the velocity of the ball before found, becomes $v = \frac{bii + gop}{bio} \times c\sqrt{\frac{2a}{o}}$. So that the velocity of the ball is directly as the chord of the arc described by the pendulum in its vibration.

SCHOLIUM.

299. In the foregoing solution, the change in the centre of oscillation is omitted, which is caused by the ball lodging in the point P. But the allowance for that small change, and that of some other small quantities, may be seen in my Tracts, where all the circumstances of this method are treated at full length.

300. For an example in numbers of this method, suppose the weights and dimensions to be as follow: namely,

$p = 570$lb,	Then
$b = 18$oz 1d r	$\frac{bii + gop}{bio} \times c = \frac{1{\cdot}131 \times 94{\cdot}3^2 + 78\frac{1}{2} \times 84\frac{7}{9} \times 570}{1{\cdot}131 \times 94\frac{3}{10} \times 84\frac{7}{9}}$
$= 1{\cdot}131$lb,	
$g = 78\frac{1}{2}$ inc.	$\times \frac{18{\cdot}73}{12} = 656{\cdot}56$,
$o = 84\frac{7}{9}$ inc.	
$= 7{\cdot}065$ feet	
$i = 94\frac{3}{10}$ inc.	And $\sqrt{\frac{2a}{o}} = \sqrt{\frac{32\frac{1}{6}}{7{\cdot}065}} = \sqrt{\frac{193}{42{\cdot}39}} = 2{\cdot}1337$.
$c = 18{\cdot}73$ inc.	

Therefore $656{\cdot}56 \times 2{\cdot}1337$ or 1401 feet, is the velocity, per second, with which the ball moved when it struck the pendulum.

OF HYDROSTATICS.

301. HYDROSTATICS is the science which treats of the pressure, or weight, and equilibrium of water and other fluids, especially those that are non-elastic.

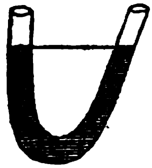
302. A fluid is elastic, when it can be reduced into a less volume by compression, and which restores itself to its former bulk again when the pressure is removed; as air. And it is non-elastic, when it is not compressible by such force; as water, &c.

PROPOSITION LIX.

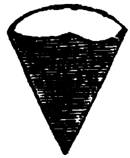
303. *If any Part of a Fluid be raised higher than the rest, by any Force, and then left to itself; the higher Parts will descend to the lower Places, and the Fluid will not rest, till its Surface be quite even and level.*

For, the parts of a fluid being easily moveable every way, the higher parts will descend by their superior gravity, and raise the lower parts, till the whole come to rest in a level or horizontal plane.

304. *Corol.* 1. Hence, water that communicates with other water, by means of a close canal or pipe, will stand at the same height in both places. Like as water in the two legs of a syphon.



305. *Corol.* 2. For the same reason, if a fluid gravitate towards a centre; it will dispose itself into a spherical figure, the centre of which is the centre of force. Like the sea in respect of the earth.



PROPOSITION LX.

306. *When a Fluid is at Rest in a Vessel, the Base of which is Parallel to the Horizon; Equal Parts of the Base are Equally Pressed by the Fluid.*

For, on every equal part of this base there is an equal column of the fluid supported by it. And as all the columns are of equal height, by the last proposition they are of equal weight, and therefore they press the base equally; that is, equal parts of the base sustain an equal pressure.

307. *Corol.* 1. All parts of the fluid press equally at the same depth. For, if a plane parallel to the horizon be conceived to be drawn at that depth; then the pressure being the same in any part of that plane, by the proposition, therefore the parts of the fluid, instead of the plane, sustain the same pressure at the same depth.

308. *Corol.* 2. The pressure of the fluid at any depth, is as the depth of the fluid. For the pressure is as the weight, and the weight is as the height of the fluid.

309. *Corol.*

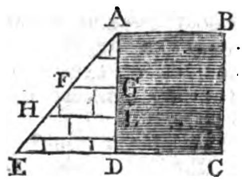
309. *Corol.* 5. The pressure of the fluid on any horizontal surface or plane, is equal to the weight of a column of the fluid, whose base is equal to that plane, and altitude is its depth below the upper surface of the fluid.

PROPOSITION LXI.

310. *When a Fluid is Pressed by its own Weight, or by any other Force; at any Point it Presses Equally, in all Directions whatever.*

THIS arises from the nature of fluidity, by which it yields to any force in any direction. If it cannot recede from any force applied, it will press against other parts of the fluid in the direction of that force. And the pressure in all directions will be the same: for if it were less in any part, the fluid would move that way, till the pressure be equal every way.

311. *Corol.* 1. In a vessel containing a fluid; the pressure is the same against the bottom, as against the sides, or even upwards at the same depth.



312. *Corol.* 2. Hence, and from the last proposition, if ABCD be a vessel of water, and there be taken, in the base produced, DE, to represent the pressure at the bottom; joining AE, and drawing any parallels to the base, as FG, HI; then shall FG represent the pressure at the depth AG, and HI the pressure at the depth AI, and so on; because the parallels - FG, HI, ED, by sim. triangles are as the depths AG, AI, AD: which are as the pressures, by the proposition.

And hence the sum of all the FG, HI, &c, or area of the triangle ADE, is as the pressure against all the points G, I, &c, that is, against the line AD. But as every point in the line CD is pressed with a force as DE, and that thence the pressure on the whole line CD is as the rectangle ED . DC, while that against the side is as the triangle ADE or $\frac{1}{2}$AD . DE; therefore the pressure on the horizontal line DC, is to the pressure against the vertical line DA, as DC to $\frac{1}{2}$DA. And hence, if the vessel be an upright rectangular one, the pressure on the bottom, or whole weight of the fluid, is to the pressure against one side, as the base is to half that side. Therefore the weight of the fluid is to the pressure against all

all the four upright sides, as the base is to half the upright surface. And the same holds true also in any upright vessel, whatever the sides be, or in a cylindrical vessel. Or in the cylinder, the weight of the fluid, is to the pressure against the upright surface, as the radius of the base is to double the altitude.

Also, when the rectangular prism becomes a cube, it appears that the weight of the fluid on the base, is double the pressure against one of the upright sides, or half the pressure against the whole upright surface.

313. *Corol.* 3. The pressure of a fluid against any upright surface, as the gate of a sluice or canal, is equal to half the weight of a column of the fluid whose base is equal to the surface pressed, and its altitude the same as the altitude of that surface. For the pressure on a horizontal base equal to the upright surface, is equal to that column; and the pressure on the upright surface, is but half that on the base, of the same area.

So that, if b denote the breadth, and d the depth of such a gate or upright surface; then the pressure against it, is equal to the weight of the fluid whose magnitude is $\frac{1}{2}bd^2 = \frac{1}{2}\text{AB} \cdot \text{AD}^2$. Hence, if the fluid be water, a cubic foot of which weighs 1000 ounces, or $62\frac{1}{2}$ pounds; and if the depth AD be 12 feet, the breadth AB 20 feet; then the content, or $\frac{1}{2}\text{AB} \cdot \text{AD}^2$, is 1440 feet; and the pressure is 1440000 ounces, or 90000 pounds, or 40^1 tons weight nearly.

PROPOSITION LXII.

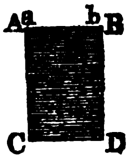
314. *The pressure of a Fluid on a Surface any how immersed in it, either Perpendicular, or Horizontal, or Oblique; is Equal to the Weight of a Column of the Fluid, whose Base is equal to the Surface pressed, and its Altitude equal to the Depth of the Centre of Gravity of the Surface pressed below the Top or Surface of the Fluid.*

For, conceive the surface pressed to be divided into innumerable sections parallel to the horizon; and let s denote any one of those horizontal sections, also d its distance or depth below the top surface of the fluid. Then, by art. 309, the pressure of the fluid on the section is equal to the weight of ds; consequently the total pressure on the whole surface is equal to all the weights ds. But, if b denote the whole surface pressed, and g the depth of its centre of gravity below the top of the fluid; then, by art. 256 or 259, bg is equal to

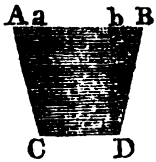
to the sum of all the ds. Consequently the whole pressure of the fluid on the body or surface b, is equal to the weight of the bulk bg of the fluid, that is, of the column whose base is the given surface b, and its height is g the depth of the centre of gravity in the fluid.

PROPOSITION LXIII.

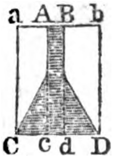
315. *The Pressure of a Fluid, on the Base of the Vessel in which it is contained, is as the Base and Perpendicular Altitude; whatever be the Figure of the Vessel that contains it.*



If the sides of the base be upright, so that it be a prism of a uniform width throughout; then the case is evident; for then the base supports the whole fluid, and the pressure is just equal to the weight of the fluid.



But if the vessel be wider at top than bottom; then the bottom sustains or is pressed by, only the part contained within the upright lines ac, bD; because the parts ACa, BDb are supported by the sides AC, BD; and those parts have no other effect on the part abDC than keeping it in its position, by the lateral pressure against ac and bD, which does not alter its perpendicular pressure downwards. And thus the pressure on the bottom is less than the weight of the contained fluid.



And if the vessel be widest at bottom; then the bottom is still pressed with a weight which is equal to that of the whole upright column abDC. For, as the parts of the fluid are in equilibrio, all the parts have an equal pressure at the same depth; so that the parts within cC and dD press equally as those in cd, and therefore equally the same as if the sides of the vessel had gone upright to a and b, the defect of fluid in the parts ACa and BD*b* being exactly compensated by the downward pressure or resistance of the sides AC and BD against the contiguous fluid. And thus the pressure on the base may be made to exceed the weight of the contained fluid, in any proportion whatever.

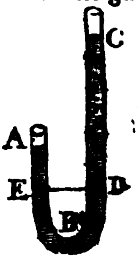
So that, in general, be the vessels of any figure whatever, regular or irregular, upright or sloping, or variously wide and narrow in different parts, if the bases and perpendicular altitudes be but equal, the bases always sustain the same pressure. And as that pressure, in the regular upright vessel,

vessel, is the whole column of the fluid, which is as the base and altitude; therefore the pressure in all figures is in that same ratio.

316. *Corol.* 1. Hence, when the heights are equal, the pressures are as the bases. And when the bases are equal, the pressure is as the height. But when both the heights and bases are equal, the pressures are equal in all, though their contents be ever so different.

317. *Corol.* 2. The pressure on the base of any vessel, is the same as on that of a cylinder, of an equal base and height.

318. *Corol.* 3. If there be an inverted syphon, or bent tube, ABC, containing two different fluids CD, ABD, that balance each other, or rest in equilibrio; then their heights in the two legs, AE, CD, above the point of meeting will be reciprocally as their densities.

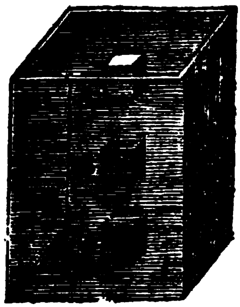


For if they do not meet at the bottom, the part BD balances the part BE, and therefore the part CD balances the part AE; that is, the weight of CD is equal to the weight of AE. And as the surface at D is the same where they act against each other, therefore AE : CD : : density of CD : density of AE.

So, if CD be water, and AE quicksilver, which is near 14 times heavier; then CD will be = 14AE; that is, if AE be 1 inch, CD will be 14 inches; if AE be 2 inches, CD will be 28 inches; and so on.

PROPOSITION LXIV.

319. *If a Body be Immersed in a Fluid of the same Density or Specific Gravity; it will Rest in any Place where it is put. But a Body of Greater Density will Sink; and one of a Less Density will Rise to the Top, and Float.*



The body, being of the same density, or of the same weight with the like bulk of the fluid, will press the fluid under it, just as much as if its space was filled with the fluid itself. The pressure then all around it will be the same as if the fluid were in its place; consequently there is no force, neither upward nor downward, to put the body out of its place. And therefore it will remain wherever it is put.

But

But if the body be lighter; its pressure downward will be less than before, and less than the water upward at the same depth; therefore the great force will overcome the less, and push the body upward to A.

And if the body be heavier than the fluid, the pressure downward will be greater than the fluid at the same depth; therefore the greater force will prevail, and carry the body down to the bottom at C.

320. *Corol.* 1. A body immersed in a fluid, loses as much weight, as an equal bulk of the fluid weighs. And the fluid gains the same weight. Thus, if the body be of equal density with the fluid, it loses all its weight, and so requires no force but the fluid to sustain it. If it be heavier, its weight in the water will be only the difference between its own weight and the weight of the same bulk of water; and it requires a force to sustain it just equal to that difference. But if it be lighter, it requires a force equal to the same difference of weights to keep it from rising up in the fluid.

321. *Corol.* 2. The weights lost, by immerging the same body in different fluids, are as the specific gravities of the fluids. And bodies of equal weight, but different bulk, lose, in the same fluid, weights which are reciprocally as the specific gravities of the bodies, or directly as their bulks.

322. *Corol.* 3. The whole weight of a body which will float in a fluid, is equal to as much of the fluid, as the immersed part of the body takes up, when it floats. For the pressure under the floating body, is just the same as so much of the fluid as is equal to the immersed part; and therefore the weights are the same.

323. *Corol.* 4. Hence the magnitude of the whole body, is to the magnitude of the part immersed, as the specific gravity of the fluid, is to that of the body. For, in bodies of equal weight, the densities, or specific gravities, are reciprocally as their magnitudes.

324. *Corol.* 5. And because when the weight of a body taken in a fluid, is subtracted from its weight out of the fluid, the difference is the weight of an equal bulk of the fluid; this therefore is to its weight in the air, as the specific gravity of the fluid, is to that of body.

Therefore, if W be the weight of a body in air,
w its weight in water, or any fluid,
S the specific gravity of the body, and
s the specific gravity of the fluid;

then

then $W - w : W :: s : S$, which proportion will give either of those specific gravities, the one from the other.

Thus $S = \frac{W}{W - w}s$, the specific gravity of the body;

and $s = \frac{W - w}{W}S$; the specific gravity of the fluid.

So that the specific gravities of bodies, are as their weights in the air directly, and their loss in the same fluid inversely.

325. *Corol.* 6. And hence, for two bodies connected together, or mixed together into one compound, of different specific gravities, we have the following equations, denoting their weights and specific gravities, as below, viz.

H = weight of the heavier body in air, h = weight of the same in water, } S its spec. gravity;
L = weight of the lighter body in air, l = weight of the same in water, } s its spec. gravity;
C = weight of the compound in air, c = weight of the same in water, } f its spec. gravity;
w = the specific gravity of water. Then,

1st, $(H - h)S = Hw$,
2d, $(L - l)s = Lw$,
3d, $(C - c)f = Cw$,
4th, $H + L = C$,
5th, $h + l = c$,
6th, $\frac{H}{S} + \frac{L}{s} = \frac{C}{f}$

From which equations may be found any of the above quantities, in terms of the rest.

Thus, from one of the first three equations, is found the specific gravity of any body, as $s = \frac{Lw}{L - l}$, by dividing the absolute weight of the body by its loss in water, and multiplying by the specific gravity of water.

But if the body L be lighter than water; then l will be negative, and we must divide by $L + l$ instead of $L - l$, and to find l we must have recourse to the compound mass C; and because, from the 4th and 5th equations, $L - l = \overline{C - c} - \overline{H - h}$, therefore $s = \frac{Lw}{(C - c) - (H - h)}$; that is, divide the absolute weight of the light body, by the difference between the losses in water, of the compound and heavier body, and multiply by the specific gravity of water. Or thus, $s = \frac{SfL}{CS - Hf}$, as found from the last equation.

Also, if it were required to find the quantities of two ingredients mixed in a compound, the 4th and 6th equations would give their values as follows, viz.

$H =$

$$H = \frac{(f-s)S}{(S-s)f}C, \text{ and } L = \frac{(S-f)s}{(S-s)f}C,$$

the quantities of the two ingredients H and L, in the compound C. And so for any other demand.

PROPOSITION LXV.

To find the Specific Gravity of a Body.

326. CASE I.—*When the body is heavier than water:* weigh it both in water and out of water, and take the difference, which will be the weight lost in water. Then, by corol. 6, prop. 64, $s = \frac{Bw}{B-b}$, where B is the weight of the body out of water, b its weight in water, s its specific gravity, and w the specific gravity of water. That is,

As the weight lost in water,
Is to the whole or absolute weight,
So is the specific gravity of water,
To the specific gravity of the body.

EXAMPLE. If a piece of stone weigh 10 lb, but in water only $6\frac{3}{4}$ lb, required its specific gravity, that of water being 1000? Ans. 3077.

327. CASE II.—*When the body is lighter than water, so that it will not sink:* annex to it a piece of another body, heavier than water, so that the mass compounded of the two may sink together. Weigh the denser body and the compound mass, separately, both in water and out of it; then find how much each loses in water, by subtracting its weight in water from its weight in air; and subtract the less of these remainders from the greater. Then say, by proportion,

As the last remainder,
Is to the weight of the light body in air,
So is the specific gravity of water,
To the specific gravity of the body.

That is, the specific gravity is $s = \frac{Lw}{(C-c)-(H-h)}$, by cor. 6, prop. 64.

EXAMPLE. Suppose a piece of elm weighs 15 lb in air; and that a piece of copper, which weighs 18 lb in air and 16 lb in water, is affixed to it, and that the compound weighs 6 lb in water; required the specific gravity of the elm?

Ans. 600.

328. Case III.—*For a fluid of any sort.*—Take a piece of a body of known specific gravity; weigh it both in and out of the fluid, finding the loss of weight by taking the difference of the two; then say,

As the whole or absolute weight,
Is to the loss of weight,
So is the specific gravity of the solid,
To the specific gravity of the fluid.

That is, the spec. grav. $w = \frac{B - b}{B} s$, by cor. 6, pr. 64.

Example. A piece of cast iron weighed $35\frac{61}{100}$ ounces in a fluid, and 40 ounces out of it; of what specific gravity is that fluid? Ans. 1000.

PROPOSITION LXVI.

329. *To find the Quantities of Two Ingredients in a Given Compound.*

Take the three differences of every pair of the three specific gravities, namely, the specific gravities of the compound and each ingredient; and multiply each specific gravity by the difference of the other two. Then say, by proportion,

As the greatest product,
Is to the whole weight of the compound,
So is each of the other two products,
To the weights of the two ingredients.

That is, $H = \frac{(f - s)\,S}{(S - s)f} C$ = the one, and $L = \frac{(S - f)\,s}{(S - s)f} C$, the other, by cor. 6, prop. 64.

Example. A composition of 112 lb being made of tin and copper, whose specific gravity is found to be 8784; required the quantity of each ingredient, the specific gravity of tin being 7320, and that of copper 9000?

Answer, there is 100 lb of copper, and consequently 12 lb of tin, } in the composition.

SCHOLIUM.

330. The specific gravities of several sorts of matter, as found from experiments, are expressed by the numbers annexed to their names in the following Table:

A Table

A Table of Specific Gravities of Bodies.

Platina (pure) - - - -	23000	Clay - - - - - - - -	2160
Fine gold - - - - -	19400	Brick - - - - - - - -	2000
Standard gold - - - -	17724	Common earth - - - -	1984
Quicksilver (pure) -	14000	Nitre - - - - - - - -	1900
Quicksilver (common)	13600	Ivory - - - - - - - -	1825
Lead - - - - - - - -	11325	Brimstone - - - - -	1810
Fine silver - - - - -	11091	Solid gunpowder - -	1745
Standard silver - - - -	10535	Sand - - - - - - - -	1520
Copper - - - - - - -	9000	Coal - - - - - - - -	1250
Copper halfpence - -	8915	Box-wood - - - - - -	1030
Gun metal - - - - -	8784	Sea-water - - - - - -	1030
Cast brass - - - - -	8000	Common-water - - - -	1000
Steel - - - - - - - -	7850	Oak - - - - - - - -	925
Iron - - - - - - - -	7645	Gunpowder, close shaken	937
Cast iron - - - - -	7425	Ditto, in a loose heap -	836
Tin - - - - - - - -	7320	Ash - - - - - - - -	800
Clear crystal glass - -	3150	Maple - - - - - - - -	755
Granite - - - - - -	3000	Elm - - - - - - - -	600
Marble and hard stone	2700	Fir - - - - - - - - -	550
Common green glass -	2600	Charcoal - - - - - -	
Flint - - - - - - - -	2570	Cork - - - - - - - -	240
Common stone - - - -	2520	Air at a mean state - -	$1\frac{2}{9}$

331. *Note.* The several sorts of wood are supposed to be dry. Also, as a cubic foot of water weighs just 1000 ounces avoirdupois, the numbers in this table express, not only the specific gravities of the several bodies, but also the weight of a cubic foot of each, in avoirdupois ounces; and therefore, by proportion, the weight of any other quantity, or the quantity of any other weight, may be known, as in the next two propositions.

PROPOSITION LXVII.

332. *To find the Magnitude of any Body, from its Weight.*

As the tabular specific gravity of the body,
Is to its weight in avoirdupois ounces,
So is one cubic foot, or 1728 cubic inches,
To its content in feet, or inches, respectively.

Example 1. Required the content of an irregular block of common stone, which weighs 1 cwt, or 112 lb?

Ans. $1228\frac{2016}{2520}$ cubic inches.

Example 2. How many cubic inches of gunpowder are there in 1 lb. weight? Ans. $29\frac{1}{2}$ cubic inches nearly.

Example 3.

Example 3. How many cubic feet are there in a ton weight of dry oak? Ans. $38\frac{138}{185}$ cubic feet.

PROPOSITION LXVIII.

333. *To find the Weight of a Body from its Magnitude.*

As one cubic foot, or 1728 cubic inches,
Is to the content of the body,
So is the tabular specific gravity,
To the weight of the body.

Example 1. Required the weight of a block of marble, whose length is 63 feet, and breadth and thickness each 12 feet; being the dimensions of one of the stones in the walls of Balbeck?

Ans. $683\frac{4}{10}$ ton, which is nearly equal to the burden of an East-India ship.

Example 2. What is the weight of 1 pint, ale measure, of gunpowder? Ans. 19 oz. nearly.

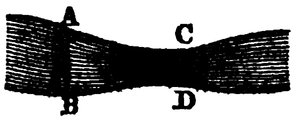
Example 3. What is the weight of a block of dry oak, which measures 10 feet in length, 3 feet broad, and $2\frac{1}{4}$ feet deep or thick? Ans. $4335\frac{15}{16}$ lb.

OF HYDRAULICS.

334. HYDRAULICS is the science which treats of the motion of fluids, and the forces with which they act upon bodies.

PROPOSITION LXIX.

335. *If a Fluid Run through a Canal or River, or Pipe of various Widths, always filling it; the Velocity of the Fluid in different Parts of it* AB, CD, *will be reciprocally as the Transverse Sections in those Parts.*



THAT is, veloc. at A : veloc. at C : : CD : AB; where AB and CD denote, not the diameters at A and B, but the areas or sections there.

For, as the channel is always equally full, the quantity of water running through AB is equal to the quantity running through CD, in the same time; that is, the column through AB

AB is equal to the column through CD, in the same time; or AB × length of its column = CD × length of its column; therefore AB : CD :: length of column through CD : length of column through AB. But the uniform velocity of the water, is as the space run over, or length of the columns; therefore AB : CD :: velocity through CD : velocity through AB.

336. *Corol.* Hence, by observing the velocity at any place AB, the quantity of water discharged in a second, or any other time, will be found, namely, by multiplying the section AB by the velocity there.

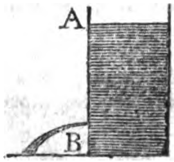
But if the channel be not a close pipe or tunnel, kept always full, but an open canal or river; then the velocity in all parts of the section will not be the same, because the velocity towards the bottom and sides will be diminished by the friction against the bed or channel; and therefore a medium among the three ought to be taken. So if the velocity at the top be - 100 feet per minute,
that at the bottom - 60
and that at the sides - 50

3) 210 sum;

dividing their sum by 3 gives 70 for the mean velocity, which is to be multiplied by the section, to give the quantity discharged in a minute.

PROPOSITION LXX.

337. *The Velocity with which a Fluid Runs out by a Hole in the Bottom or Side of a Vessel, is Equal to that which is Generated by Gravity through the Height of the Water above the Hole; that is, the Velocity of a Heavy Body acquired by Falling freely through the Height* AB.



Divide the altitude AB into a great number of very small parts, each being 1, their number a, or a = the altitude AB.

Now, by prop. 61, the pressure of the fluid against the hole B, by which the motion is generated, is equal to the weight of the column of fluid above it, that is the column whose height is AB or a, and base the area of the hole B. Therefore the pressure on the hole, or small part of the fluid 1, is to its weight, or the natural force of gravity, as a to 1. But, by art. 28, the velocities generated in the same body in any time,

time are as those forces; and because gravity generates the velocity 2 in descending through the small space 1, therefore $1 : a :: 2 : 2a$, the velocity generated by the pressure of the column of fluid in the same time. But $2a$ is also, by corol. 1, prop. 6, the velocity generated by gravity in descending through a or AB. That is, the velocity of the issuing water, is equal to that which is acquired by a body in falling through the height AB.

The same otherwise.

Because the momenta, or quantities of motion generated in two given bodies, by the same force, acting during the same or an equal time, are equal. And as the force in this case, is the weight of the superincumbent column of the fluid over the hole. Let the one body to be moved, be that column itself, expressed by ah, where a denotes the altitude AB, and h the area of the hole; and the other body is the column of the fluid that runs out uniformly in one second suppose, with the middle or medium velocity of that interval of time, which is $\frac{1}{2}hv$, if v be the whole velocity required. Then the mass $\frac{1}{2}hv$, with the velocity v, gives the quantity of motion $\frac{1}{2}hv \times v$ or $\frac{1}{2}hv^2$, generated in one second, in the spouting water: also $2g$, or $32\frac{1}{6}$ feet, is the velocity generated in the mass ah, during the same interval of one second; consequently $ah \times 2g$, or $2ahg$, is the motion generated in the column ah in the same time of one second. But as these two momenta must be equal, this gives $\frac{1}{2}hv^2 = 2ahg$: hence then $v^2 = 4ag$, and $v = 2\sqrt{ag}$, for the value of the velocity sought; which therefore is exactly the same as the velocity generated by the gravity in falling through the space a, or the whole height of the fluid.

For example, if the fluid were air, of the whole height of the atmosphere, supposed uniform, which is about $5\frac{1}{4}$ miles, or 27720 feet $= a$. Then $2\sqrt{ag} = 2\sqrt{27720 \times 16\frac{1}{12}} =$ 1335 feet $= v$ the velocity, that is, the velocity with which common air would rush into a vacuum.

338. *Corol.* 1. The velocity, and quantity run out, at different depths, are as the square roots of the depths. For the velocity acquired in falling through AB, is as $\sqrt{AB}$.

339. *Corol.* 2. The fluid spouts out with the same velocity, whether it be downward or upward, or sideways; because the pressure of fluids is the same in all directions, at the same depth. And therefore, if an adjutage be turned upward, the jet will ascend, to the height of the surface of the water in the vessel. And this is confirmed by experience, by which it is found that jets really ascend nearly to the height

height of the reservoir, abating a small quantity only, for the friction against the sides, and some resistance from the air and from the oblique motion of the fluid in the hole.

340. *Corol.* 3. The quantity run out in any time, is equal to a column or prism, whose base is the area of the hole, and its length the space described in that time by the velocity acquired by falling through the altitude of the fluid. And the quantity is the same, whatever be the figure of the orifice, if it is of the same area.

Therefore, if a denote the altitude of the fluid,
and h the area of the orifice,
also $g = 16\frac{1}{12}$ feet, or 193 inches;
then $2h\sqrt{ag}$ will be the quantity of water discharged in a second of time; or nearly $8\frac{1}{48}h\sqrt{a}$ cubic feet, when a and h are taken in feet.

So, for example, if the height a be 25 inches, and the orifice $h = 1$ square inch; then $2h\sqrt{ag} = 2\sqrt{25 \times 193} = 139$ cubic inches, which is the quantity that would be discharged per second.

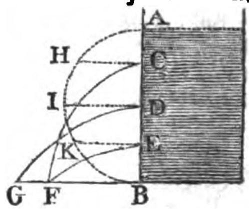
SCHOLIUM.

341. When the orifice is in the side of the vessel, then the velocity is different in the different parts of the hole, being less in the upper parts of it than in the lower. However, when the hole is but small, the difference is inconsiderable, and the altitude may be estimated from the centre of the hole, to obtain the mean velocity. But when the orifice is pretty large, then the mean velocity is to be more accurately computed by other principles, given in the next proposition.

342. It is not to be expected that experiments, as to the quantity of water run out, will exactly agree with this theory, both on account of the resistance of the air, the resistance of the water against the sides of the orifice, and the oblique motion of the particles of the water in entering it. For, it is not merely the particles situated immediately in the column over the hole, which enter it and issue forth, as if that column only were in motion; but also particles from all the surrounding parts of the fluid, which is in a commotion quite around; and the particles thus entering the hole in all directions, strike against each other, and impede one another's motion: from which it happens, that it is the particles in the centre of the hole only that issue out with the whole velocity due to the entire height of the fluid, while the other particles towards the sides of the orifices pass out with decreased velocities; and hence the medium velocity through the orifice, is somewhat less than that of a single body only, urged with the same pressure of the superincumbent column of

of the fluid. And experiments on the quantity of water discharged through apertures, show that the quantity must be diminished, by those causes, rather more than the fourth part, when the orifice is small, or such as to make the mean velocity nearly equal to that in a body falling through $\frac{1}{2}$ the height of the fluid above the orifice.

343. Experiments have also been made on the extent to which the spout of water ranges on a horizontal plane, and compared with the theory, by calculating it as a projectile discharged with the velocity acquired by descending through the height of the fluid. For, when the aperture is in the side of the vessel, the fluid spouts out horizontally with a uniform velocity, which, combined with the perpendicular velocity from the action of gravity, causes the jet to form the curve of a parabola. Then the distances to which the jet will spout on the horizontal plane BG, will be as the roots of the rectangles of the segments AC . CB, AD . DB, AE . EB. For the spaces BF, BG, are as the times and horizontal velocities; but the velocity is as $\sqrt{AC}$; and the time of the fall, which is the same as the time of moving, is as $\sqrt{CB}$; therefore the distance BF is as $\sqrt{AC . CB}$; and the distance BG as $\sqrt{AD . DB}$. And hence, if two holes are made equidistant from the top and bottom, they will project the water to the same distance; for if AC = EB, then the rectangle AC . CB is equal the rectangle AE . EB: which makes EF the same for both. Or, if on the diameter AB a semicircle be described; then, because the squares of the ordinates CH, DI, EK are equal to the rectangles AC . EB, &c; therefore the distances BF, BG are as the ordinates CH, DI. And hence also it follows, that the projection from the middle point D will be farthest, for DI is the greatest ordinate.



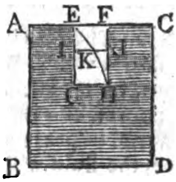
These are the proportions of the distances: but for the absolute distances, it will be thus. The velocity through any hole C, is such as will carry the water horizontally through a space equal to 2AC in the time of falling through AC: but, after quitting the hole, it describes a parabola, and comes to F in the time a body will fall through CB; and to find this distance, since the times are as the roots of the spaces, therefore $\sqrt{AC} : \sqrt{CB} :: 2AC : 2\sqrt{AC . CB} =$ 2CH

2CH = BF, the space ranged on the horizontal plane. And the greatest range BG = 2DI, or 2AD, or equal to AB.

And as these ranges answer very exactly to the experiments, this confirms the theory, as to the velocity assigned.

PROPOSITION LXXI.

344. *If a Notch or Slit* EH *in form of a Parallelogram, be cut in the Side of a Vessel, Full of Water,* AD *; the Quantity of Water flowing through it, will be* $\frac{2}{3}$ *of the Quantity flowing through an equal Orifice, placed at the Whole Depth* EG, *or at the Base* GH, *in the Same Time ; it being supposed that the Vessel is always kept full.*



For the velocity at GH is to the velocity at IL, as $\sqrt{EG}$ to $\sqrt{EI}$; that is, as GH or IL to IK, the ordinate of a parabola EKH, whose axis is EG. Therefore the sum of the velocities at all the points I, is to as many times the velocity at G, as the sum of all the ordinates IK, to the sum of all the IL's ; namely, as the area of the parabola EGH, is to the area EGHF ; that is, the quantity running through the notch EH, is to the quantity running through an equal horizontal area placed at GH, as EGHKE, to EGHF, or as 2 to 3 ; the area of a parabola being $\frac{2}{3}$ of its circumscribing parallelogram.

345. *Corol.* 1. The mean velocity of the water in the notch, is equal to $\frac{2}{3}$ of that at GH.

346. *Corol.* 2. The quantity flowing through the hole IGHL, is to that which would flow through an equal orifice placed as low as GH, as the parabolic frustum IGHK, is to the rectangle IGHL. As appears from the demonstration.

OF PNEUMATICS.

347. Pneumatics is the science which treats of the properties of air, or elastic fluids.

PROPOSITION LXXII.

348. *Air is a Heavy Fluid Body ; and it Surrounds the Earth, and Gravitates on all Parts of its Surface.*

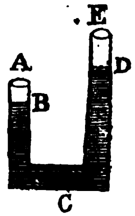
These properties of air are proved by experience.— That it is a fluid, is evident from its easily yielding to any

the least force impressed on it, without making a sensible resistance.

But when it is moved briskly, by any means, as by a fan or a pair of bellows; or when any body is moved very briskly through it; in these cases we become sensible of it as a body, by the resistance it makes in such motions, and also by its impelling or blowing away any light substances. So that, being capable of resisting, or moving other bodies, by its impulse, it must itself be a body, and be heavy, like all other bodies in proportion to the matter it contains; and therefore it will press on all bodies that are placed under it.

Also, as it is a fluid, it spreads itself all over on the earth; and, like other fluids, it gravitates and presses everywhere on the earth's surface.

349. The gravity and pressure of the air is also evident from many experiments. Thus, for instance, if water, or quicksilver, be poured into the tube ACE, and the air be suffered to press on it, in both ends of the tube, the fluid will rest at the same height in both legs: but if the air be drawn out of one end as E, by any means; then the air pressing on the other end A, will press down the fluid in this leg at B, and raise it up in the other to D, as much higher than at B, as the pressure of the air is equal to. From which it appears, not only that the air does really press, but also how much the intensity of that pressure is equal to. And this is the principle of the barometer.



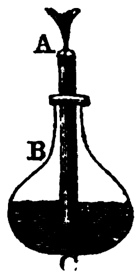
PROPOSITION LXXIII.

350. *The Air is also an Elastic Fluid, being Condensible and Expansible. And the Law it observes is this, that its Density and Elasticity are proportional to the Force or Weight which Compresses it.*

This property of the air is proved by many experiments. Thus, if the handle of a syringe be pushed inward, it will condense the inclosed air into less space, thereby showing its condensibility. But the included air, thus condensed, is felt to act strongly against the hand, resisting the force compressing it more and more; and, on withdrawing the hand, the handle is pushed back again to where it was at first. Which shows that the air is elastic.

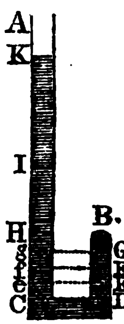
351. Again,

351. Again, fill a strong bottle half full of water; then insert a small glass tube into it, putting its lower end down near to the bottom, and cementing it very close round the mouth of the bottle. Then, if air be strongly injected through the pipe, as by blowing with the mouth or otherwise, it will pass through the water from the lower end, ascending into the parts before occupied with air at B, and the whole mass of air become there condensed, because the water is not compressible into a less space. But, on removing the force which injected the air at A, the water will begin to rise from thence in a jet, being pushed up the pipe by the increased elasticity of the air B, by which it presses on the surface of the water, and forces it through the pipe, till as much be expelled as there was air forced in; when the air at B will be reduced to the same density as at first, and, the balance being restored, the jet will cease.



352. Likewise, if into a jar of water AB, be inverted an empty glass tumbler CD, or such-like, the mouth downward; the water will enter it, and partly fill it, but not near so high as the water in the jar, compressing and condensing the air into a less space in the upper parts C, and causing the glass to make a sensible resistance to the hand in pushing it down. Then, on removing the hand, the elasticity of the internal condensed air throws the glass up again. All these showing that the air is condensible and elastic.

353. Again, to show the rate or proportion of the elasticity to the condensation: take a long crooked glass tube, equally wide throughout, or at least in the part BD, and open at A, but close at the other end B. Pour in a little quicksilver at A, just to cover the bottom to the bend at CD, and to stop the communication between the external air and the air in BD. Then pour in more quicksilver, and mark the corresponding heights at which it stands in the two legs: so, when it rises to H in the open leg AC, let it rise to E in the close one, reducing its included air from the natural bulk BD to the contracted space BE,

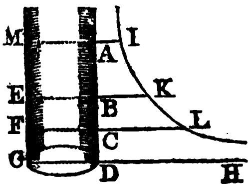


by

by the pressure of the column HE; and when the quicksilver stands at I and K, in the open leg, let it rise to F and G in the other, reducing the air to the respective spaces BF, BG, by the weights of the columns I*f*, K*g*. Then it is always found, that the condensations and elasticities are as the compressing weights or columns of the quicksilver, and the atmosphere together. So, if the natural bulk of the air BD be compressed into the spaces BE, BF, BG, which are $\frac{3}{4}$, $\frac{2}{4}$, $\frac{1}{4}$ of BD, or as the numbers 3, 2, 1; then the atmosphere, together with the corresponding columns H*e*, I*f*, K*g*, are also found to be in the same proportion reciprocally, viz. as $\frac{1}{3}$, $\frac{1}{2}$, $\frac{1}{1}$, or as the numbers 2, 3, 6. And then H*e* = $\frac{1}{3}$A, I*f* = A, and K*g* = 3A; where A is the weight of atmosphere. Which show, that the condensations are directly as the compressing forces. And the elasticities are in the same ratio, since the columns in AC are sustained by the elasticities in BD.

From the foregoing principles may be deduced many useful remarks, as in the following corollaries, viz.

354. *Corol.* 1. The space which any quantity of air is confined in, is reciprocally as the force that compresses it. So, the forces which confine a quantity of air in the cylindrical spaces AG, BG, CG, are reciprocally as the same, or reciprocally as the heights AD, BD, CD. And therefore if to the two perpendicular lines DA, DH, as asymptotes, the hyperbola IKL be described, and the ordinates AI, BK, CL be drawn; then the forces which confine the air in the spaces AG, BG, CG, will be directly as the corresponding ordinates AI, BK, CL, since these are reciprocally as the abscisses AD, BD, CD, by the nature of the hyperbola.



355. *Corol.* 2. All the air near the earth is in a state of compression, by the weight of the incumbent atmosphere.

356. *Corol.* 3. The air is denser near the earth, than in high places; or denser at the foot of a mountain, than at the top of it. And the higher above the earth, the less dense it is.

357. *Corol.* 4. The spring or elasticity of the air, is equal to the weight of the atmosphere above it; and they will produce the same effects: since they always sustain and balance each other.

358. *Corol.* 5.

358. *Corol.* 5. If the density of the air be increased, preserving the same heat or temperature, its spring or elasticity is also increased, and in the same proportion.

359. *Corol.* 6. By the pressure and gravity of the atmosphere, on the surface of fluids, the fluids are made to rise in any pipes or vessels, when the spring or pressure within is decreased or taken off.

PROPOSITION LXXIV.

360. *Heat Increases the Elasticity of the Air, and Cold Diminishes it. Or, Heat Expands, and Cold Condenses the Air.*

This property is also proved by experience.

361. Thus, tie a bladder very close with some air in it; and lay it before the fire: then as it warms, it will more and more distend the bladder, and at last burst it, if the heat be continued and increased high enough. But if the bladder be removed from the fire, as it cools it will contract again, as before. And it was on this principle that the first air-balloons were made by Montgolfier: for, by heating the air within them, by a fire beneath, the hot air distends them to a size which occupies a space in the atmosphere, whose weight of common air exceeds that of the balloon.

362. Also, if a cup or glass, with a little air in it, be inverted into a vessel of water; and the whole be heated over the fire, or otherwise; the air in the top will expand till it fill the glass, and expel the water out of it; and part of the air itself will follow, by continuing or increasing the heat.

Many other experiments, to the same effect, might be adduced, all proving the properties mentioned in the proposition.

SCHOLIUM.

363. So that, when the force of the elasticity of air is considered, regard must be had to its heat or temperature; the same quantity of air being more or less elastic, as its heat is more or less. And it has been found, by experiment, that the elasticity is increased by the 435th part, for each degree of heat, of which there are 180 between the freezing and boiling heat of water.

364. *N. B.* Water expands about the $\frac{3}{20000}$ part, with each degree of heat. (Sir Geo. Shuckburgh, Philos. Trans. 1777, p. 560, &c.)

Also,

Also, the

Spec. grav. of air	1·201 or $1\frac{1}{5}$	}	when the barom. is 29·5,
water	1000	}	and the therm. is 55°
mercury	13592	}	which are their mean heights in this country.

Or thus, air	1·222 or $1\frac{2}{9}$	}	when the barom. is 30,
water	1000	}	and thermometer 55.
mercury	13600	}	

PROPOSITION LXXV.

365. *The Weight or Pressure of the Atmosphere, on any Base at the Earth's Surface, is Equal to the Weight of a Column of Quicksilver, of the Same Base, and the Height of which is between* 28 *and* 31 *inches.*

This is proved by the barometer, an instrument which measures the pressure of the air, and which is described below. For, at some seasons, and in some places, the air sustains and balances a column of mercury, of about 28 inches: but at other times it balances a column of 29, or 30, or near 31 inches high; seldom in the extremes 28 or 31, but commonly about the means 29 or 30. A variation which depends partly on the different degrees of heat in the air near the surface of the earth, and partly on the commotions and changes in the atmosphere, from winds and other causes, by which it is accumulated in some places, and depressed in others, being thereby rendered denser and heavier, or rarer and lighter; which changes in its state are almost continually happening in any one place. But the medium state is commonly about $29\frac{1}{2}$ or 30 inches.

366. *Corol.* 1. Hence the pressure of the atmosphere on every square inch at the earth's surface, at a medium, is very near 15 pounds avoirdupois, or rather $14\frac{3}{4}$ pounds. For, a cubic foot of mercury weighing 13600 ounces nearly, an inch of it will weigh 7·866 or almost 8 ounces, or nearly half a pound, which is the weight of the atmosphere for every inch of the barometer on a base of a square inch; and therefore 30 inches, or the medium height, weighs very near $14\frac{3}{4}$ pounds.

367. *Corol.* 2. Hence also the weight or pressure of the atmosphere, is equal to that of a column of water from 32 to 35 feet high, or on a medium 33 or 34 feet high. For, water and quicksilver are in weight nearly as 1 to 13·6; so

so that the atmosphere will balance a column of water 13·6 times as high as one of quicksilver; consequently

13·6 times 28 inches = 381 inches, or $31\frac{3}{4}$ feet,
13·6 times 29 inches = 394 inches, or $32\frac{5}{6}$ feet,
13·6 times 30 inches = 408 inches, or 34 feet,
13·6 times 31 inches = 422 inches, or $35\frac{1}{6}$ feet.

And hence a common sucking pump will not raise water higher than about 33 or 34 feet. And a siphon will not run, if the perpendicular height of the top of it be more than about 33 or 34 feet.

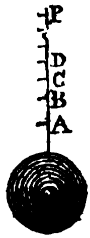
368. *Corol.* 3. If the air were of the same uniform density at every height up to the top of the atmosphere, as at the surface of the earth; its height would be about $5\frac{1}{4}$ miles at a medium. For, the weights of the same bulk of air and water, are nearly as 1·222 to 1000; therefore as 1·222 : 1000 :: $33\frac{3}{4}$ feet : 27600 feet, or $5\frac{1}{4}$ miles nearly. And so high the atmosphere would be, if it were all of uniform density, like water. But, instead of that, from its expansive and elastic quality, it becomes continually more and more rare, the farther above the earth, in a certain proportion, which will be treated of below, as also the method of measuring heights by the barometer, which depends on it.

369. *Corol.* 4. From this proposition and the last it follows, that the height is always the same, of an uniform atmosphere above any place, which shall be all of the uniform density with the air there, and of equal weight or pressure with the real height of the atmosphere above that place, whether it be at the same place, at different times, or at any different places or heights above the earth; and that height is always about $5\frac{1}{4}$ miles, or 27600 feet, as above found. For, as the density varies in exact proportion to the weight of the column, therefore it requires a column of the same height in all cases, to make the respective weights or pressures. Thus, if W and w be the weights of atmosphere above any places, D and d their densities, and H and h the heights of the uniform columns, of the same densities and weights; Then $H \times D = W$, and $h \times d = w$; therefore $\frac{W}{D}$ or H is equal to $\frac{w}{d}$ or h. The temperature being the same.

PROPOSITION

PROPOSITION LXXVI.

370. *The Density of the Atmosphere, at Different Heights above the Earth, Decreases in such Sort, that when the Heights Increase in Arithmetical Progression, the Densities Decrease in Geometrical Progression.*



LET the indefinite perpendicular line AP, erected on the earth, be conceived to be divided into a great number of very small equal parts, A, B, C, D, &c, forming so many thin strata of air in the atmosphere, all of different density, gradually decreasing from the greatest at A: then the density of the several strata, A, B, C, D, &c, will be in geometrical progression decreasing.

For, as the strata A, B, C, &c, are all of equal thickness, the quantity of matter in each of them, is as the density there; but the density in any one, being as the compressing force, is as the weight or quantity of all the matter from that place upward to the top of the atmosphere; therefore the quantity of matter in each stratum, is also as the whole quantity from that place upward. Now, if from the whole weight at any place as B, the weight or quantity in the stratum B be subtracted, the remainder is the weight at the next stratum C; that is, from each weight subtracting a part which is proportional to itself, leaves the next weight; or, which is the same thing, from each density subtracting a part which is proportional to itself, leaves the next density. But when any quantities are continually diminished by parts which are proportional to themselves, the remainders form a series of continued proportionals: consequently these densities are in geometrical progression.

Thus, if the first density be D, and from each be taken its nth part; there will then remain its $\frac{n-1}{n}$ part, or the $\frac{m}{n}$ part, putting m for $n-1$; and therefore the series of densities will be D, $\frac{m}{n}$ D, $\frac{m^2}{n^2}$ D, $\frac{m^3}{n^3}$ D, $\frac{m^4}{n^4}$ D, &c, the common ratio of the series being that of n to m.

SCHOLIUM.

371. Because the terms of an arithmetical series, are proportional to the logarithms of the terms of a geometrical series: therefore different altitudes above the earth's surface,

face, are as the logarithms of the densities, or of the weights of air, at those altitudes.

So that, if D denote the density at the altitude A,
and d - the density at the altitude a;
then A being as the log. of D, and a as the log. of d,
the dif. of alt. $A - a$ will be as the log. $D -$ log. d. or log. $\frac{D}{d}$.

And if $A = 0$, or D the density at the surface of the earth; then any altitude above the surface a, is as the log. of $\frac{D}{d}$.

Or, in general, the log. of $\frac{D}{d}$ is as the altitude of the one place above the other, whether the lower place be at the surface of the earth, or any where else.

And from this property is derived the method of determining the heights of mountains and other eminences, by the barometer, which is an instrument that measures the pressure or density of the air at any place. For, by taking, with this instrument, the pressure or density, at the foot of a hill for instance, and again at the top of it, the difference of the logarithms of these two pressures, or the logarithm of their quotient, will be as the difference of altitude, or as the height of the hill; supposing the temperatures of the air to be the same at both places, and the gravity of air not altered by the different distances from the earth's centre.

372. But as this formula expresses only the relations between different altitudes with respect to their densities, recourse must be had to some experiment, to obtain the real altitude which corresponds to any given density, or the density which corresponds to a given altitude. And there are various experiments by which this may be done. The first, and most natural, is that which results from the known specific gravity of air, with respect to the whole pressure of the atmosphere on the surface of the earth. Now, as the altitude a is always as log. $\frac{D}{d}$; assume h so that $a = h \times$ log. $\frac{D}{d}$, where h will be of one constant value for all altitudes; and to determine that value, let a case be taken in which we know the altitude a corresponding to a known density d; as for instance, take $a = 1$ foot, or 1 inch, or some such small altitude; then, because the density D may be measured by the pressure of the atmosphere, or the uniform column of 27600 feet, when the temperature is 55°; therefore 27600 feet will

 denote

denote the density D at the lower place, and 27599 the less density d at 1 foot above it; consequently $1 = h \times \log. \frac{27600}{27599}$; which, by the nature of logarithms, is nearly $= h \times \frac{\cdot 43429448}{27600}$ $= \frac{h}{63551}$ nearly; and hence $h = 63551$ feet; which gives, for any altitude in general, this theorem, viz. $a = 63551 \times \log. \frac{D}{d}$, or $= 63551 \times \log. \frac{M}{m}$ feet, or $10592 \times \log. \frac{M}{m}$ fathoms; where M is the column of mercury which is equal to the pressure or weight of the atmosphere at the bottom, and m that at the top of the altitude a; and where M and m may be taken in any measure, either feet or inches, &c.

373. Note, that this formula is adapted to the mean temperature of the air 55°. But, for every degree of temperature different from this, in the medium between the temperatures at the top and bottom of the altitude a, that altitude will vary by its 435th part; which must be added, when that medium exceeds 55°, otherwise subtracted.

374. Note, also, that a column of 30 inches of mercury varies its length by about the $\frac{1}{320}$ part of an inch for every degree of heat, or rather $\frac{1}{9600}$ of the whole volume.

375. But the formula may be rendered much more convenient for use, by reducing the factor 10592 to 10000, by changing the temperature proportionally from 55°; thus, as the diff. 592 is the 18th part of the whole factor 10592; and as 18 is the 24th part of 435; therefore the corresponding change of temperature is 24°, which reduces the 55° to 31°. So that the formula is, $a = 10000 \times \log. \frac{M}{m}$ fathoms, when the temperature is 31 degrees; and for every degree above that, the result is to be increased by so many times its 435th part.

376. *Exam.* 1. To find the height of a hill when the pressure of the atmosphere is equal to 29·68 inches of mercury at the bottom, and 25·28 at the top; the mean temperature being 50°? Ans. 4378 feet, or 730 fathoms.

377. *Exam.* 2. To find the height of a hill when the atmosphere weighs 29·45 inches of mercury at the bottom, and 26·82 at the top, the mean temperature being 33°?
Ans. 2385 feet, or 397½ fathoms.

378. *Exam.* 3.

378. *Exam.* 3. At what altitude is the density of the atmosphere only the 4th part of what it is at the earth's surface? Ans. 6020 fathoms.

By the weight and pressure of the atmosphere, the effect and operations of pneumatic engines may be accounted for, and explained; such as siphons, pumps, barometers, &c; of which it may not be improper here to give a brief description.

OF THE SIPHON.

379. The Siphon, or Syphon, is any bent tube, having its two legs either of equal or of unequal length.

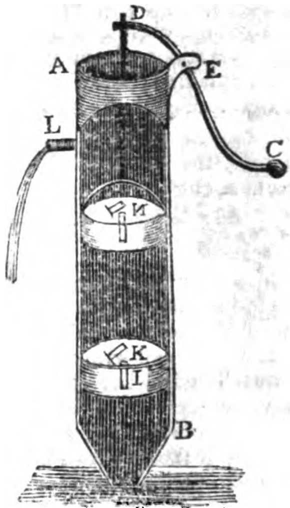
If it be filled with water, and then inverted, with the two open ends downward, and held level in that position; the water will remain suspended in it, if the two legs be equal. For the atmosphere will press equally on the surface of the water in each end, and support them, if they are not more than 34 feet high; and the legs being equal, the water in them is an exact counterpoise by their equal weights; so that the one has no power to move more than the other; and they are both supported by the atmosphere.

But if now the siphon be a little inclined to one side, so that the orifice of one end be lower than that of the other; or if the legs be of unequal length, which is the same thing; then the equilibrium is destroyed, and the water will all descend out by the lower end, and rise up in the higher. For, the air pressing equally, but the two ends weighing unequally, a motion must commence where the power is greatest, and so continue till all the water has run out by the lower end. And if the shorter leg be immersed into a vessel of water, and the siphon be set a running as above, it will continue to run till all the water be exhausted out of the vessel, or at least as low as that end of the siphon. Or, it may be set a running without filling the siphon as above, by only inverting it, with its shorter leg into the vessel of water; then, with the mouth applied to the lower orifice A, suck the air out; and the water will presently follow, being forced up into the siphon by the pressure of the air on the water in the vessel.

Of

OF THE PUMP.

380. There are three sorts of pumps: the Sucking, the Lifting, and the Forcing Pump. By the first, water can be raised only to about 34 feet, viz. by the pressure of the atmosphere; but by the others, to any height; but then they require more apparatus and power.



The annexed figure represents a common sucking pump. AB is the barrel of the pump, being a hollow cylinder, made of metal, and smooth within, or of wood for very common purposes. CD is the handle, moveable about the pin E, by moving the end C up and down. DF an iron rod turning about a pin D, which connects it to the end of the handle. This rod is fixed to the piston, bucket, or sucker, FG, by which this is moved up and down within the barrel, which it must fit very tight and close, that no air or water may pass between the piston and the sides of the barrel; and for this purpose it is commonly armed with leather. The piston is made hollow, or it has a perforation through it, the orifice of which is covered by a valve H opening upwards. I is a plug firmly fixed in the lower part of the barrel, also perforated, and covered by a valve K opening upwards.

381. When the pump is first to be worked, and the water is below the plug I; raise the end C of the handle, then the piston descending, compresses the air in HI, which by its spring shuts fast the valve K, and pushes up the valve H, and so enters into the barrel above the piston. Then putting the end C of the handle down again, raises the piston or sucker, which lifts up with it the column of air above it, the external atmosphere by its pressure keeping the valve H shut: the air in the barrel being thus exhausted, or rarefied, is no longer a counterpoise to that which presses on the surface of the water in the well; this is forced up the pipe, and through the valve K, into the barrel of the pump. Then pushing the piston down again into this water, now in the

barrel,

barrel, its weight shuts the lower valve K, and its resistance forces up the valve of the piston, and enters the upper part of the barrel, above the piston. Then, the bucket being raised, lifts up with it the water which had passed above its valve, and it runs out by the cock L; and taking off the weight below it, the pressure of the external atmosphere on the water in the well again forces it up through the pipe and lower valve close to the piston, all the way as it ascends, thus keeping the barrel always full of water. And thus, by repeating the strokes of the piston, a continued discharge is made at the cock L.

OF THE AIR-PUMP.

382. Nearly on the same principles as the water-pump, is the invention of the Air-pump, by which the air is drawn out of any vessel, like as water is drawn out by the former. A brass barrel is bored and polished truly cylindrical, and exactly fitted with a turned piston, so that no air can pass by the sides of it, and furnished with a proper valve opening upward. Then, by lifting up the piston, the air in the close vessel below it follows the piston, and fills the barrel; and being thus diffused through a larger space than before, when it occupied the vessel or receiver only, but not the barrel, it is made rarer than it was before, in proportion as the capacity of the barrel and receiver together exceeds the receiver alone. Another stroke of the piston exhausts another barrel of this now rarer air, which again rarefies it in the same proportion as before. And so on, for any number of strokes of the piston, still exhausting in the same geometrical progression, of which the ratio is that which the capacity of the receiver and barrel together exceeds the receiver, till this is exhausted to any proposed degree, or as far as the nature of the machine is capable of performing; which happens when the elasticity of the included air is so far diminished, by rarefying, that it is too feeble to push up the valve of the piston, and escape.

383. From the nature of this exhausting, in geometrical progression, we may easily find how much the air in the receiver is rarefied by any number of strokes of the piston; or what number of such strokes is necessary, to exhaust the receiver to any given degree. Thus, if the capacity of the receiver and barrel together, be to that of the receiver alone,

as c to r, and 1 denote the natural density of the air at first: then

$c : r :: 1 : \frac{r}{c}$, the density after one stroke of the piston;

$c : r :: \frac{r}{c} : \frac{r^2}{c^2}$, the density after 2 strokes,

$c : r :: \frac{r^2}{c^2} : \frac{r^3}{c^3}$, the density after three strokes.

&c, and $\frac{r^n}{c^n}$, the density after n strokes.

So, if the barrel be equal to $\frac{1}{4}$ of the receiver; then $c : r ::$ $5 : 4$; and $\frac{4^n}{5^n} = 0{\cdot}8^n$ is $= d$ the density after n turns. And if n be 20, then $0{\cdot}8^{20} = {\cdot}0115$ is the density of the included air after 20 strokes of the piston; which being the $86\frac{7}{10}$ part of 1, or the first density, it follows that the air is $86\frac{7}{10}$ times rarefied by the 20 strokes.

384. Or, if it were required to find the number of strokes necessary to rarefy the air any number of times; because $\frac{r^n}{c^n}$ is $=$ the proposed density d; therefore, taking the logarithms, $n \times \log. \frac{r}{c} = \log. d$, and $n = \frac{\log. d}{\mathrm{l}.\, r - \mathrm{l}.\, c}$, the number of strokes required. So if r be $\frac{4}{5}$ of c, and it be required to rarefy the air 100 times: then $d = \frac{1}{100}$ or ${\cdot}01$; and hence $n = \frac{\log. 100}{\mathrm{l}.\, 5 - \mathrm{l}.\, 4} = 20\frac{2}{3}$ nearly. So that in $20\frac{2}{3}$ strokes the air will be rarefied 100 times.

OF THE DIVING BELL & CONDENSING MACHINE.

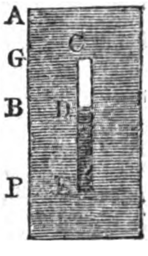
385. On the same principles too depend the operations and effect of the Condensing Engine, by which air may be condensed to any degree, instead of rarefied as in the air-pump. And, like as the air-pump rarefies the air, by extracting always one barrel of air after another; so, by this other machine, the air is condensed, by throwing in or adding always one barrel of air after another; which it is evident may be done by only turning the valves of the piston and barrel, that is, making them to open the contrary way, and working the piston in the same manner; so

so that, as they both open upward or outward in the air-pump, or rarefier, they will both open downward or inward in the condenser.

386. And on the same principles, namely, of the compression and elasticity of the air, depends the use of the Diving Bell, which is a large vessel, in which a person descends to the bottom of the sea, the open end of the vessel being downward; only in this case the air is not condensed by forcing more of it into the same space, as in the condensing engine; but by compressing the same quantity of air into a less space in the bell, by increasing always the force which compresses it.

387. If a vessel of any sort be inverted into water, and pushed or let down to any depth in it; then by the pressure of the water some of it will ascend into the vessel, but not so high as the water without, and will compress the air into less space, according to the difference between the heights of the internal and external water; and the density and elastic force of the air will be increased in the same proportion, as its space in the vessel is diminished.

So, if the tube CE be inverted, and pushed down into water, till the external water exceed the internal, by the height AB, and the air of the tube be reduced to the space CD; then that air is pressed both by a column of water of the height AB, and by the whole atmosphere which presses on the upper surface of the water; consequently the space CD is to the whole space CE, as the weight of the atmosphere, is to the weights both of the atmosphere and the column of water AB. So that if AB be about 34 feet, which is equal to the force of the atmosphere, then CD will be equal to $\frac{1}{2}$CE; but if AB be double of that, or 68 feet, then CD will be $\frac{1}{3}$CE; and so on. And hence, by knowing the depth AF, to which the vessel is sunk, we can easily find the point D, to which the water will rise within it at any time. For let the weight of the atmosphere at that time be equal to that of 34 feet of water; also, let the depth AF be 20 feet, and the length of the tube CE 4 feet: then putting the height of the internal water DE $= x$,



it is $34 + AB : 34 :: CE : CD$,

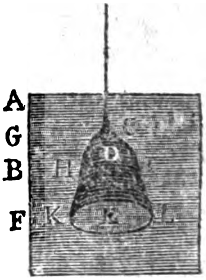
that is $34 + AF - DE : 34 :: CE : CE - DE$,

or $54 - x : 34 :: 4 : 4 - x$;

hence, multiplying extremes and means, $216 - 58x + x^2 = 136$,

$= 136$, and the root is $x = \sqrt{2}$ very nearly $= 1{\cdot}414$ of a foot, or 17 inches nearly; being the height DE to which the water will rise within the tube.

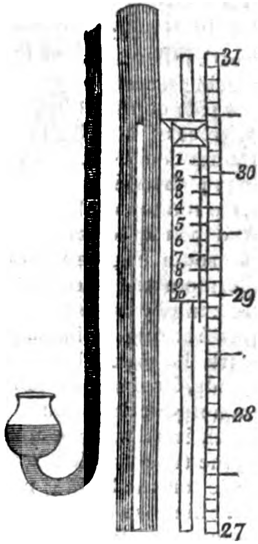
388. But if the vessel be not equally wide throughout, but of any other shape, as of a bell-like form, such as is used in diving; then the altitudes will not observe the proportion above, but the spaces or bulks only will respect that proportion, namely, 34 + AB : 34 : : capacity CKL : capacity CHI, if it be common or fresh water; and 33 + AB : 33 : : capacity CKL : capacity CHI, if it be sea-water. From which proportion, the height DE may be found, when the nature or shape of the vessel or bell CKL is known.



OF THE BAROMETER.

389. THE BAROMETER is an instrument for measuring the pressure of the atmosphere, and elasticity of the air, at any time. It is commonly made of a glass tube, of near 3 feet long, close at one end, and filled with mercury. When the tube is full, by stopping the open end with the finger, then inverting the tube, and immersing that end with the finger into a bason of quicksilver, on removing the finger from the orifice, the fluid in the tube will descend into the bason, till what remains in the tube be of the same weight with a column of the atmosphere, which is commonly between 28 and 31 inches of quicksilver; and leaving an entire vacuum in the upper end of the tube above the mercury. For, as the upper end of the tube is quite void of air, there is no pressure downwards but from the column of quicksilver, and therefore that will be an exact balance to the counter pressure of the whole column of atmosphere, acting on the orifice of the tube by the quicksilver in the bason. The upper 3 inches of the tube, namely, from 28 to 31 inches, have a scale attached to them, divided into inches, tenths, and hundredths, for measuring the length of the column at all times, by observing which division of the scale the top of the quicksilver is opposite to; as it ascends and descends within these limits, according to the state of the atmosphere.

So

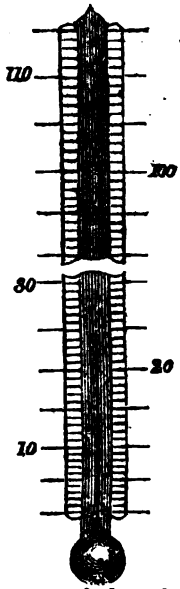


So that the weight of the quicksilver in the tube, above that in the bason, is at all times equal to the weight or pressure of the column of atmosphere above it, and of the same base with the tube; and hence the weight of it may at all times be computed; being nearly at the rate of half a pound avoirdupoise for every inch of quicksilver in the tube, on every square inch of base; or more exactly it is $\frac{59}{120}$ of a pound on the square inch, for every inch in the altitude of the quicksilver weighs just $\frac{59}{120}$lb, or nearly ½ a pound, in the mean temperature of 55° of heat. And consequently, when the barometer stands at 30 inches, or 2½ feet high, which is nearly the medium or standard height, the whole pressure of the atmosphere is equal to 14¾ pounds, on every square inch of the base; and so in proportion for other heights.

OF THE THERMOMETER.

390. THE THERMOMETER is an instrument for measuring the temperature of the air, as to heat and cold.

It is found by experience, that all bodies expand by heat, and contract by cold; and hence the degrees of expansion become the measure of the degrees of heat. Fluids are more convenient for this purpose than solids: and quicksilver is now most commonly used for it. A very fine glass tube, having a pretty large hollow ball at the bottom, is filled about half way up with quicksilver: the whole being then heated very hot till the quicksilver rise quite to the top, the top is then hermetically sealed, so as perfectly to exclude all communication with the outward air. Then, in cooling, the quicksilver contracts, and consequently its surface descends in the tube, till it come to a certain point, correspondent to the temperature or heat of the air. And when the weather becomes warmer, the quicksilver expands,



and its surface rises in the tube; and again contracts and descends when the weather becomes cooler. So that, by placing a scale of any divisions against the side of the tube, it will show the degrees of heat by the expansion and contraction of the quicksilver in the tube; observing at what division of the scale the top of the quicksilver stands. And the method of preparing the scale, as used in England, is thus:—Bring the thermometer into the temperature of freezing, by immersing the ball in water just freezing, or in ice just thawing, and mark the scale where the mercury then stands, for the point of freezing. Next, immerge it in boiling water; and the quicksilver will rise to a certain height in the tube; which mark also on the scale for the boiling point, or the heat of boiling water. Then the distance between these two points, is divided into 180 equal divisions, or degrees; and the like equal degrees are also continued to any extent below the freezing point, and above the boiling point. The divisions are then numbered as follows; namely, at the freezing point is set the number 32, and consequently 212 at the boiling point; and all the other numbers in their order.

This division of the scale is commonly called Fahrenheit's. According to this division, 55 is at the mean temperature of the air in this country; and it is in this temperature, and in an atmosphere which sustains a column of 30 inches of quicksilver in the barometer, that all measures and specific gravities are taken, unless when otherwise mentioned; and in this temperature and pressure the relative weights, or specific gravities of air, water, and quicksilver, are as

$1\frac{2}{9}$ for air,
1000 for water,
13600 for mercury;
{ and these also are the weights of a cubic foot of each, in avoirdupois ounces, in that state of the barometer and thermometer. For other states of the thermometer, each of these bodies expands or contracts according to the following rate, with each degree of heat, viz.

Air about - $\frac{1}{435}$ part of its bulk,
Water about $\frac{1}{6666}$ part of its bulk,
Mercury about $\frac{1}{9600}$ part of its bulk.

ON

ON THE MEASUREMENT OF ALTITUDES BY THE BAROMETER AND THERMOMETER.

391. FROM the principles laid down in the scholium to prop. 76, concerning the measuring of altitudes by the barometer, and the foregoing descriptions of the barometer and thermometer, we may now collect together the precepts for the practice of such measurements, which are as follow:

First. Observe the height of the barometer at the bottom of any height, or depth, intended to be measured; with the temperature of the quicksilver, by means of a thermometer attached to the barometer, and also the temperature of the air in the shade by a detached thermometer.

Secondly. Let the same thing be done also at the top of the said height or depth, and at the same time, or as near the same time as may be. And let those altitudes of barometer be reduced to the same temperature, if it be thought necessary, by correcting either the one or the other, that is, augment the height of the mercury in the colder temperature, or diminish that in the warmer, by its $\frac{1}{9600}$ part for every degree of difference of the two.

Thirdly. Take the difference of the common logarithms of the two heights of the barometer, corrected as above if necessary, cutting off 3 figures next the right hand for decimals, when the log-tables go to 7 figures, or cut off only 2 figures when the tables go to 6 places, and so on; or in general remove the decimal point 4 places more towards the right hand, those on the left hand being fathoms in whole numbers.

Fourthly. Correct the number last found for the difference of temperature of the air, as follows; Take half the sum of the two temperatures, for the mean one: and for every degree which this differs from the temperature 31°, take so many times the $\frac{1}{435}$ part of the fathoms above found, and add them if the mean temperature be above 31°, but subtract them if the mean temperature be below 31°; and the sum or difference will be the true altitude in fathoms: or, being multiplied by 6, it will be the altitude in feet.

392. *Example* 1. Let the state of the barometers and thermometers be as follows; to find the altitude, viz.

Barom.	Thermom. attach.	Thermom. detach.
Lower 29·68	57	57
Upper 25·28	43	42

Ans. the alt. is $719\frac{1}{3}$ fathoms.

393. *Exam.*

393. *Exam.* 2. To find the altitude, when the state of the barometers and thermometers is as follows, viz.

Barom.	Thermom. attach	Thermom. detach.	Ans. the alt. is
Lower 29·45	38	31	$409\frac{8}{13}$ fathoms:
Upper 26·82	41	35	or 2458 feet.

ON THE RESISTANCE OF FLUIDS, WITH THEIR FORCES AND ACTIONS ON BODIES.

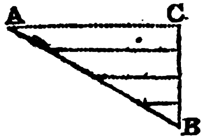
PROPOSITION LXXVII.

394. *If any Body Move through a Fluid at Rest, or the Fluid Move against the Body at Rest; the Force or Resistance of the Fluid against the Body, will be as the Square of the Velocity and the Density of the Fluid. That is,* $R \propto dv^2$.

For, the force or resistance is as the quantity of matter or particles struck, and the velocity with which they are struck. But the quantity or number of particles struck in any time, are as the velocity and the density of the fluid. Therefore the resistance, or force of the fluid, is as the density and square of the velocity.

395. *Corol.* 1. The resistance to any plane, is also more or less, as the plane is greater or less; and therefore the resistance on any plane, is as the area of the plane a, the density of the medium, and the square of the velocity. That, is, $R \propto adv^2$.

396. *Corol.* 2. If the motion be not perpendicular, but oblique to the plane, or to the face of the body; then the resistance, in the direction of motion, will be diminished in the triplicate ratio of radius to the sine of the angle of inclination of the plane to the direction of the motion, or as the cube of radius to the cube of the sine of that angle. So that $R \propto adv^2s^3$, putting $1 =$ radius, and $s =$ sine of the angle of inclination CAB.



For, if AB be the plane, AC the direction of motion, and BC perpendicular to AC; then no more particles meet the plane than what meet the perpendicular BC, and therefore their number is diminished as AB to BC or as 1 to s. But the force of each par-

ticle,

ticle, striking the plane obliquely in the direction CA, is also diminished as AB to BC, or as 1 to s; therefore the resistance, which is perpendicular to the face of the plane by art. 52, is as 1^2 to s^2. But again, this resistance in the direction perpendicular to the face of the plane, is to that in the direction AC, by art. 51, as AB to BC, or as 1 to s. Consequently, on all these accounts, the resistance to the plane when moving perpendicular to its face, is to that when moving obliquely, as 1^3 to s^3, or 1 to s^3. That is, the resistance in the direction of the motion, is diminished as 1 to s^3, or in the triplicate ratio of radius to the sine of inclination.

PROPOSITION LXXVIII.

397. *The Real Resistance to a Plane, by a Fluid acting in a Direction perpendicular to its Face, is equal to the Weight of a Column of the Fluid, whose Base is the Plane, and Altitude equal to that which is due to the Velocity of the Motion, or through which a Heavy Body must fall to acquire that Velocity.*

The resistance to the plane moving through a fluid, is the same as the force of the fluid in motion with the same velocity, on the plane at rest. But the force of the fluid in motion, is equal to the weight or pressure which generates that motion; and this is equal to the weight or pressure of a column of the fluid, whose base is the area of the plane, and its altitude that which is due to the velocity.

398. *Corol.* 1. If a denote the area of the plane, v the velocity, n the density or specific gravity of the fluid, and $g = 16\frac{1}{12}$ feet, or 193 inches. Then the altitude due to the velocity v being $\frac{v^2}{4g}$, therefore $a \times n \times \frac{v^2}{4g} = \frac{anv^2}{4g}$ will be the whole resistance, or motive force R.

399. *Corol.* 2. If the direction of motion be not perpendicular to the face of the plane, but oblique to it, in any angle, whose sine is s. Then the resistance to the plane will be $\frac{anv^2s^3}{4g}$.

400. *Corol.* 3. Also, if w denote the weight of the body, whose plane face a is resisted by the absolute force R; then the retarding force f, or $\frac{R}{w}$ will be $\frac{anv^2s^3}{4gw}$.

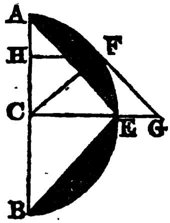
401. *Corol.* 4. And if the body be a cylinder, whose face or

or end is a, and radius r, moving in the direction of its axis; because then $s = 1$, and $a = pr^2$, where $p = 3\cdot1416$; then $\frac{pnv^2r^2}{4g}$ will be the resisting force R, and $\frac{pnv^2r^2}{4gw}$ the retarding force f.

402. *Coral.* 5. This is the value of the resistance when the end of the cylinder is a plane perpendicular to its axis, or to the direction of motion. But were its face an elliptic section, or a conical surface, or any other figure everywhere equally inclined to the axis, or direction of motion, the sine or inclination being s: then, the number of particles of the fluid striking the face being still the same, but the force of each, opposed to the direction of motion, diminished in the duplicate ratio of radius to the sine of inclination, the resisting force R would be $\frac{pnr^2v^2s^2}{4g}$.

PROPOSITION LXXIX.

403. *The Resistance to a Sphere moving through a Fluid, is but Half the Resistance to its Great Circle, or to the End of a Cylinder of the same Diameter, moving with an Equal Velocity.*



LET AFEB be half the sphere, moving in the direction CEG. Describe the paraboloid AIEKB on the same base. Let any particle of the medium meet the semicircle in F, to which draw the tangent FG, the radius FC, and the ordinate FIH. Then the force of any particle on the surface at F, is to its force on the base at H, as the square of the sine of the angle G, or its equal the angle FCH, to the square of radius, that is, as HF^2 to CF^2. Therefore the force of all the particles, or the whole fluid, on the whole surface, is to its force on the circle of the base, as all the HF^2 to as many times CF^2. But CF^2 is $= CA^2 = AC \cdot CB$, and $HF^2 = AH \cdot HB$ by the nature of the circle: also, $AH \cdot HB : AC \cdot CB :: HI : CE$ by the nature of the parabola; consequently the force on the spherical surface, is to the force on its circular base, as all the HI's to as many CE's, that is, as the content of the paraboloid to the content of its circumscribed cylinder, namely, as 1 to 2.

404. *Corol.* Hence, the resistance to the sphere is $R = \frac{pnv^2r^2}{8g}$, being the half of that of a cylinder of the same diameter.

diameter. For example. a 9lb iron ball, whose diameter is 4 inches, when moving through the air with a velocity of 1600 feet per second, would meet a resistance which is equal to a weight of $132\frac{2}{3}$lb, over and above the pressure of the atmosphere, for want of the counterpoise behind the wall.

PRACTICAL EXERCISES CONCERNING SPECIFIC GRAVITY.

The Specific Gravities of Bodies are their relative weights contained under the same given magnitude; as a cubic foot, or a cubic inch, &c.

The specific gravities of several sorts of matter, are expressed by the numbers annexed to their names in the Table of Specific Gravities, at page 211; from which the numbers are to be taken, when wanted.

Note. The several sorts of wood are supposed to be dry. Also, as a cubic foot of water weighs just 1000 ounces avoirdupois, the numbers in the table express, not only the specific gravities of the several bodies, but also the weight of a cubic foot of each in avoirdupois ounces; and hence, by proportion, the weight of any other quantity, or the quantity of any other weight, may be known, as in the following problems.

PROBLEM I.

To find the Magnitude of any Body, from its Weight.

As the tabular specific gravity of the body,
Is to its weight in avoirdupois ounces,
So is one cubic foot, or 1728 cubic inches,
To its content in feet, or inches, respectively.

EXAMPLES.

Exam. 1. Required the content of an irregular block of common stone, which weighs 1cwt. or 112lb.

Ans. $1228\frac{4}{5}$ cubic inches.

Exam. 2. How many cubic inches of gunpowder are there in 1lb weight? Ans. $29\frac{1}{2}$ cubic inches nearly.

Exam. 3. How many cubic feet are there in a ton weight of dry oak? Ans. $38\frac{138}{185}$ cubic feet.

PROBLEM

PROBLEM II.

To find the Weight of a Body from its Magnitude.

As one cubic foot, or 1728 cubic inches,
Is to the content of the body,
So is its tabular specific gravity,
To the weight of the body.

EXAMPLES.

Exam. 1. Required the weight of a block of marble, whose length is 63 feet, and breadth and thickness each 12 feet; being the dimensions of one of the stones in the walls of Balbeck?

Ans. $683\frac{7}{16}$ ton, which is nearly equal to the burden of an East-India ship.

Exam. 2. What is the weight of 1 pint, ale measure, of gunpowder? Ans. 19 oz. nearly.

Exam. 3. What is the weight of a block of dry oak, which measures 10 feet in length, 3 feet broad, and $2\frac{1}{2}$ feet deep?

Ans. $4335\frac{15}{16}$lb.

PROBLEM III.

To find the Specific Gravity of a Body.

Case 1. When the body is heavier than water, weigh it both in water and out of water, and take the difference, which will be the weight lost in water. Then say,

As the weight lost in water,
Is to the whole weight,
So is the specific gravity of water,
To the specific gravity of the body.

EXAMPLE.

A piece of stone weighed 10lb, but in water only $6\frac{1}{6}$lb, required its specific gravity? Ans. 2609.

Case 2. When the body is lighter than water, so that it will not quite sink, affix to it a piece of another body, heavier than water, so that the mass compounded of the two may sink together. Weigh the denser body and the compound mass separately, both in water and out of it; then find how much each loses in water, by subtracting its weight in water from its weight in air; and subtract the less of these remainders from the greater. Then say,

As

As the last remainder,
Is to the weight of the light body in air,
So is the specific gravity of water,
To the specific gravity of the body.

EXAMPLE.

Suppose a piece of elm weighs 15lb in air; and that a piece of copper which weighs 18lb in air, and 16lb in water, is affixed to it, and that the compound weighs 6lb in water; required the specific gravity of the elm? Ans. 600.

PROBLEM IV.

To find the Quantities of Two Ingredients, in a Given Compound.

Take the three differences of every pair of the three specific gravities, namely, the specific gravities of the compound and each ingredient; and multiply the difference of every two specific gravities by the third. Then say, as the greatest product, is to the whole weight of the compound, so is each of the other products, to the two weights of the ingredients.

EXAMPLE.

A composition of 112lb being made of tin and copper, whose specific gravity if found to be 8784; required the quantity of each ingredient, the specific gravity of tin being 7320, and of copper 9000?

Ans. there is 100lb of copper } in the composition.
and consequently 12lb of tin }

OF THE WEIGHT AND DIMENSIONS OF BALLS AND SHELLS.

THE weight and dimensions of Balls and Shells might be found from the problems last given, concerning specific gravity. But they may be found still easier by means of the experimented weight of a ball of a given size, from the known proportion of similar figures, namely, as the cubes of their diameters.

PROBLEM I.

To find the Weight of an Iron Ball, from its Diameter.

An iron ball of 4 inches diameter weighs 9lb, and the weights being as the cubes of the diameters, it will be, as 64

(which is the cube of 4) is to 9 its weight, so is the cube of the diameter of any other ball, to its weight. Or, take $\frac{9}{64}$ of the cube of the diameter, for the weight. Or, take $\frac{1}{8}$ of the cube of the diameter, and $\frac{1}{8}$ of that again, and add the two together, for the weight.

EXAMPLES.

EXAM. 1. The diameter of an iron shot being 6·7 inches, required its weight? Ans. 42·294lb.

EXAM. 2. What is the weight of an iron ball, whose diameter is 5·54 inches? Ans 24lb nearly.

PROBLEM II.

To find the Weight of a Leaden Ball.

A leaden ball of 1 inch diameter weighs $\frac{3}{14}$ of a lb; therefore as the cube of 1 is to $\frac{3}{14}$, or as 14 is to 3, so is the cube of the diameter of a leaden ball, to its weight. Or, take $\frac{3}{14}$ of the cube of the diameter, for the weight, nearly.

EXAMPLES.

EXAM. 1. Required the weight of a leaden ball of 6·6 inches diameter? Ans 61·606lb.

EXAM. 2. What is the weight of a leaden ball of 5·30 inches diameter? Ans. 32lb nearly.

PROBLEM III.

To find the Diameter of an Iron Ball.

MULTIPLY the weight by $7\frac{1}{9}$, and the cube root of the product will be the diameter.

EXAMPLES.

EXAM. 1. Required the diameter of a 42lb iron ball?
Ans. 6·685 inches.

EXAM. 2. What is the diameter of a 24lb iron ball?
Ans. 5·54 inches.

PROBLEM IV.

To find the Diameter of a Leaden Ball.

MULTIPLY the weight by 14, and divide the product by 3; then the cube root of the quotient will be the diameter.

EXAMPLES.

EXAMPLES.

Exam. 1. Required the diameter of a 64lb leaden ball? Ans. 6·684 inches.

Exam. 2. What is the diameter of an 8lb leaden ball? Ans. 3·343 inches.

PROBLEM V.

To find the Weight of an Iron Shell.

Take $\frac{9}{64}$ of the difference of the cubes of the external and internal diameter, for the weight of the shell.

That is, from the cube of the external diameter, take the cube of the internal diameter, multiply the remainder by 9, and divide the product by 64.

EXAMPLES.

Exam. 1. The outside diameter of an iron shell being 12·8, and the inside diameter 9·1 inches; required its weight? Ans. 188·941lb.

Exam. 2. What is the weight of an iron shell, whose external and internal diameters are 9·8 and 7 inches? Ans. $84\frac{1}{8}$lb.

PROBLEM VI.

To find how much Powder will fill a Shell.

Divide the cube of the internal diameter, in inches, by 57·3, for the lbs of powder.

EXAMPLES.

Exam. 1. How much powder will fill the shell whose internal diameter is 9·1 inches? Ans. $13\frac{3}{4}$lb nearly.

Exam. 2. How much powder will fill a shell whose internal diameter is 7 inches? Ans. 6lb.

PROBLEM VII.

To find how much Powder will fill a Rectangular Box.

Find the content of the box in inches, by multiplying the length, breadth, and depth all together. Then divide by 30 for the pounds of powder.

EXAMPLES.

Exam. 1. Required the quantity of powder that will fill a box, the length being 15 inches, the breadth 12, and the depth 10 inches? Ans. 60lb

Exam. 2

Exam. 2. How much powder will fill a cubical box whose side is 12 inches? Ans. 57$\frac{3}{5}$lb.

PROBLEM VIII.

To find how much Powder will fill a Cylinder.

Multiply the square of the diameter by the length, then divide by 38·2 for the pounds of powder.

EXAMPLES.

Exam. 1. How much powder will the cylinder hold, whose diameter is 10 inches, and length 20 inches? Ans. 52$\frac{1}{3}$lb nearly.

Exam. 2. How much powder can be contained in the cylinder whose diameter is 4 inches, and length 12 inches? Ans. 5$\frac{5}{191}$lb.

PROBLEM IX.

To find the Size of a Shell to contain a Given Weight of Powder.

Multiply the pounds of powder by 57·3, and the cube root of the product will be the diameter in inches.

EXAMPLES.

Exam. 1. What is the diameter of a shell that will hold 13$\frac{1}{4}$ of powder? Ans. 9·1 inches.

Exam. 2. What is the diameter of a shell to contain 6lb of powder? Ans. 7 inches.

PROBLEM X.

To find the Size of a Cubical Box, to contain a given Weight of Powder.

Multiply the weight in pounds by 30, and the cube root of the product will be the side of the box in inches.

EXAMPLES.

Exam. 1. Required the side of a cubical box, to hold 50lb of gunpowder? Ans. 11·44 inches.

Exam. 2. Required the side of a cubical box, to hold 400lb of gunpowder? Ans. 22·89 inches.

PROBLEM XI.

To find what Length of a Cylinder will be filled by a given Weight of Gunpowder.

Multiply the weight in pounds by 38·2, and divide the product by the square of the diameter in inches, for the length.

Examples.

EXAMPLES.

Exam. 1. What length of a 36-pounder gun, of $6\frac{2}{3}$ inches diameter, will be filled with 12lb of gunpowder?

Ans. 10·314 inches.

Exam. 2. What length of a cylinder, of 8 inches diameter, may be filled with 20lb of powder? Ans. $11\frac{15}{16}$ inches.

OF THE PILING OF BALLS AND SHELLS.

Iron Balls and Shells are commonly piled by horizontal courses, either in a pyramidical or in a wedge-like form; the base being either an equilateral triangle, or a square, or a rectangle. In the triangle and square, the pile finishes in a single ball; but in the rectangle, it finishes in a single row of balls, like an edge.

In triangular and square piles, the number of horizontal rows, or courses, is always equal to the number of balls in one side of the bottom row. And in rectangular piles, the number of rows is equal to the number of balls in the breadth of the bottom row. Also, the number in the top row, or edge, is one more than the difference between the length and breadth of the bottom row.

PROBLEM I.

To find the Number of Balls in a Triangular Pile.

Multiply continually together the number of balls in one side of the bottom row, and that number increased by 1, also the same number increased by 2; then $\frac{1}{6}$ of the last product will be the answer.

That is, $\frac{n \cdot \overline{n+1} \cdot \overline{n+2}}{6}$ is the number or sum, where n is the number in the bottom row.

EXAMPLES.

Exam. 1. Required the number of balls in a triangular pile, each side of the base containing 30 balls? Ans 4960.

Exam. 2. How many balls are in the triangular pile, each side of the base containing 20? Ans. 1540.

PROBLEM

PROBLEM II.

To find the Number of Balls in a Square Pile.

MULTIPLY continually together the number in one side of the bottom course, that number increased by 1, and double the same number increased by 1; then $\frac{1}{6}$ of the last product will be the answer.

That is, $\frac{n \cdot \overline{n+1} \cdot \overline{2n+1}}{6}$ is the number.

EXAMPLES.

EXAM. 1. How many balls are in a square pile of 30 rows?
Ans. 9455.

EXAM. 2. How many balls are in a square pile of 20 rows?
Ans. 2870.

PROBLEM III.

To find the Number of Balls in a Rectangular Pile.

FROM 3 times the number in the length of the base row, subtract one less than the breadth of the same, multiply the remainder by the same breadth, and the product by one more than the same, and divide by 6 for the answer.

That is, $\frac{b \cdot \overline{b+1} \cdot \overline{3l-b+1}}{6}$ is the number; where l is the length, and b the breadth of the lowest course.

Note. In all the piles the breadth of the bottom is equal to the number of courses. And in the oblong or rectangular pile, the top row is one more than the difference between the length and breadth of the bottom.

EXAMPLES.

EXAM. 1. Required the number of balls in a rectangular pile, the length and breadth of the base row being 46 and 15?
Ans. 4960.

EXAM. 2. How many shot are in a rectangular complete pile, the length of the bottom course being 59, and its breadth 20?
Ans. 11060.

PROBLEM IV.

To find the Number of Balls in an Incomplete Pile.

FROM the number in the whole pile, considered as complete, subtract the number in the upper pile which is wanting

ing at the top, both computed by the rule for their proper form; and the remainder will be the number in the frustum, or incomplete pile.

EXAMPLES.

Exam. 1. To find the number of shot in the incomplete triangular pile, one side of the bottom course being 40, and the top course 20? Ans. 10150.

Exam. 2. How many shot are in the incomplete triangular pile, the side of the base being 24, and of the top 8? Ans. 2516.

Exam. 3. How many balls are in the incomplete square pile, the side of the base being 24, and of the top 8? Ans. 4760.

Exam. 4. How many shot are in the incomplete rectangular pile, of 12 courses, the length and breadth of the base being 40 and 20? Ans. 6146.

OF DISTANCES BY THE VELOCITY OF SOUND.

By various experiments it has been found, that sound flies, through the air, uniformly at the rate of about 1142 feet in 1 second of time, or a mile in $4\frac{2}{3}$ or $\frac{14}{3}$ seconds. And therefore, by proportion, any distance may be found corresponding to any given time; namely, multiplying the given time, in seconds, by 1142, for the corresponding distance in feet; or taking $\frac{3}{14}$ of the given time for the distance in miles. Or dividing any given distance by these numbers, to find the corresponding time.

Note. The time for the passage of sound in the interval between seeing the flash of a gun, or lightning, and hearing the report, may be observed by a watch, or a small pendulum. Or, it may be observed by the beats of the pulse in the wrist, counting, on an average, about 70 to a minute for persons in moderate health, or $5\frac{1}{2}$ pulsations to a mile; and more or less according to circumstances.

EXAMPLES.

Exam. 1. After observing a flash of lightning, it was 12 seconds before the thunder was heard; required the distance of the cloud from whence it came? Ans. $2\frac{4}{7}$ miles.

Exam. 2. How long, after firing the Tower guns, may

the report be heard at Shooter's-Hill, supposing the distance to be 8 miles in a straight line? Ans. $37\frac{1}{3}$ seconds.

EXAM. 3. After observing the firing of a large cannon at a distance, it was 7 seconds before the report was heard; what was its distance? Ans. $1\frac{1}{2}$ mile.

EXAM. 4. Perceiving a man at a distance hewing down a tree with an axe, I remarked that 6 of my pulsations passed between seeing him strike and hearing the report of the blow; what was the distance between us, allowing 70 pulses to a minute? Ans. 1 mile and 198 yards.

EXAM. 5. How far off was the cloud from which thunder issued, whose report was 5 pulsations after the flash of lightning; counting 75 to a minute? Ans. 1523 yards.

EXAM. 6. If I see the flash of a cannon, fired by a ship in distress at sea, and hear the report 33 seconds after, how far is she off? Ans. $7\frac{1}{14}$ miles.

PRACTICAL EXERCISES IN MECHANICS, STATICS, HYDROSTATICS, SOUND, MOTION, GRAVITY, PROJECTILES, AND OTHER BRANCHES OF NATURAL PHILOSOPHY.

QUESTION 1. REQUIRED the weight of a cast iron ball of 3 inches diameter, supposing the weight of a cubic inch of the metal to be 0·258lb avoirdupois? Ans. 3·64739lb.

QUEST. 2. To determine the weight of a hollow spherical iron shell, 5 inches in diameter, the thickness of the metal being one inch? Ans. 13·2387lb.

QUEST. 3. Being one day ordered to observe how far a battery of cannon was from me, I counted, by my watch, 17 seconds between the time of seeing the flash and hearing the report; what then was the distance? Ans. $3\frac{2}{5}$ miles.

QUEST. 4. It is proposed to determine the proportional quantities of matter in the earth and moon; the density of the former being to that of the latter, as 10 to 7, and their diameters as 7930 to 2160. Ans. as 71 to 1 nearly.

QUEST. 5. What difference is there, in point of weight, between a block of marble, containing 1 cubic foot and a half, and another of brass of the same dimensions?

Ans. 496lb 14oz.

QUEST. 6. In the walls of Balbeck in Turkey, the ancient Heliopolis, there are three stones laid end to end, now in sight, that

that measure in length 61 yards; one of which in particular is 21 yards or 63 feet long, 12 feet thick, and 12 feet broad: now if this block be marble, what power would balance it, so as to prepare it for moving?

Ans. $683\frac{7}{16}$ tons, the burden of an East-India ship.

QUEST. 7. The battering-ram of Vespasian weighed, suppose 10,000 pounds; and was moved, let us admit, with such a velocity, by strength of hand, as to pass through 20 feet in one second of time; and this was found sufficient to demolish the walls of Jerusalem. The question is, with what velocity a 32lb ball must move, to do the same execution? Ans. 6250 feet.

QUEST. 8. There are two bodies, of which the one contains 25 times the matter of the other, or is 25 times heavier; but the less moves with 1000 times the velocity of the greater: in what proportion then are the momenta, or forces, with which they moved?

Ans. the less moves with a force 40 times greater.

QUEST. 9. A body, weighing 20lb, is impelled by such a force, as to send it through 100 feet in a second; with what velocity then would a body of 8lb weight move, if it were impelled by the same force? Ans. 250 feet per second.

QUEST. 10. There are two bodies, the one of which weighs 100lb, the other 60; but the less body is impelled by a force 8 times greater than the other; the proportion of the velocities, with which these bodies move, is required?

Ans. the velocity of the greater to that of the less, as 3 to 40.

QUEST. 11. There are two bodies, the greater contains 8 times the quantity of matter in the less, and is moved with a force 48 times greater; the ratio of the velocities of these two bodies is required?

Ans. the greater is to the less, as 6 to 1.

QUEST. 12. There are two bodies, one of which moves 40 times swifter than the other; but the swifter body has moved only one minute, whereas the other has been in motion 2 hours: the ratio of the spaces described by these two bodies is required?

Ans. the swifter is to the slower, as 1 to 3.

QUEST. 13. Supposing one body to move 30 times swifter than another, as also the swifter to move 12 minutes, the other only 1: what difference will there be between the spaces described by them, supposing the last has moved 5 feet? Ans. 1795 feet.

QUEST. 14. There are two bodies, the one of which has passed over 50 miles, the other only 5; and the first had

moved with 5 times the celerity of the second; what is the ratio of the times they have been in describing those spaces?

Ans. as 2 to 1.

QUEST. 15. If a lever, 40 effective inches long, will, by a certain power thrown successively on it, in 13 hours, raise a weight 104 feet; in what time will two other levers, each 18 effective inches long, raise an equal weight 73 feet?

Ans. 10 hours $8\frac{1}{4}$ minutes.

QUEST. 16. What weight will a man be able to raise, who presses with the force of a hundred and a half, on the end of an equipoised handspike, 100 inches long, meeting with a convenient prop exactly $7\frac{1}{2}$ inches from the lower end of the machine?

Ans. 2072lb.

QUEST. 17. A weight of $1\frac{1}{2}$lb, laid on the shoulder of a man, is no greater burden to him than its absolute weight, or 24 ounces: what difference will he feel between the said weight applied near his elbow, at 12 inches from the shoulder, and in the palm of his hand, 28 inches from the same; and how much more must his muscles then draw, to support it at right angles, that is, having his arm stretched right out?

Ans. 24lb avoirdupois.

QUEST. 18. What weight hung on at 70 inches from the centre of motion of a steel-yard, will balance a small gun of $9\frac{1}{2}$ cwt, freely suspended at 2 inches distance from the said centre on the contrary side?

Ans. $30\frac{2}{5}$lb.

QUEST. 19. It is proposed to divide the beam of a steel-yard, or to find the points of division where the weights of 1, 2, 3, 4, &c, lb, on the one side, will just balance a constant weight of 95lb at the distance of 2 inches on the other side of the fulcrum; the weight of the beam being 10lb, and its whole length 36 inches?

Ans. 30, 15, 10, $7\frac{1}{2}$, 6, 5, $4\frac{2}{7}$, $3\frac{3}{4}$, $3\frac{1}{3}$, 3, $2\frac{8}{11}$, $2\frac{1}{2}$, &c.

QUEST. 20. Two men carrying a burden of 200lb weight between them, hung on a pole, the ends of which rest on their shoulders; how much of this load is borne by each man, the weight hanging 6 inches from the middle, and the whole length of the pole being 4 feet?

Ans. 125lb and 75lb.

QUEST. 21. If, in a pair of scales, a body weigh 90lb in one scale, and only 40lb in the other; required its true weight, and the proportion of the lengths of the two arms of the balance beam, on each side of the point of suspension?

Ans. the weight 60lb, and the proportion 3 to 2.

QUEST. 22. To find the weight of a beam of timber, or other body, by means of man's own weight, or any other weight. For instance, a piece of tapering timber, 24 feet long, being laid over a prop, or the edge of another beam, is found to balance itself when the prop is 13 feet from the less

less end ; but removing the prop a foot nearer to the said end, it takes a man's weight of 210lb. standing on the less end, to hold it in equilibrium. Required the weight of the tree ? Ans. 2520lb.

QUEST. 23. If AB be a cane or walking-stick, 40 inches long, suspended by a string SD fastened to the middle point D : now a body being hung on at E, 6 inches distance from D, is balanced by a weight of 2lb, hung on at the larger end A ; but removing the body to F, one inch nearer to D, the 2lb weight on the other side is moved to G, within 8 inches of D, before the cane will rest in equilibrio. Required the weight of the body ? Ans. 24lb.

QUEST. 24. If AB, BC be two inclined planes, of the lengths of 30 and 40 inches, and moveable about the joint at B : what will be the ratio of two weights P, Q, in equilibrio on the planes, in all positions of them : and what will be the altitude BD of the angle B above the horizontal plane AC, when this is 50 inches long ?

Ans. BD = 24 ; and P to Q as AB to BC, or as 3 to 4.

QUEST. 25. A lever, of 6 feet long, is fixed at right angles in a screw, whose threads are one inch asunder, so that the lever turns just once round in raising or depressing the screw one inch. If then this lever be urged by a weight or force of 50lb, with what force will the screw press ?

Ans. $22619\frac{1}{2}$lb.

QUEST. 26. If a man can draw a weight of 150lb up the side of a perpendicular wall, of 20 feet high ; what weight will he be able to raise along a smooth plank of 30 feet long, laid aslope from the top of the wall ? Ans. 225lb.

QUEST. 27. If a force of 150lb be applied on the head of a rectangular wedge, its thickness being 2 inches, and the length of its side 12 inches ; what weight will it raise or balance perpendicular to its side ? Ans. 900lb.

QUEST. 28. If a round pillar of 30 feet diameter be raised on a plane, inclined to the horizon in an angle of 75°, or the shaft inclining 15 degrees out of the perpendicular : what length will it bear before it overset ?

Ans. $30\,(2 + \sqrt{3})$ or 111·9615 feet.

QUEST. 29. If the greatest angle at which a bank of natural earth will stand be 45° ; it is proposed to determine what thickness an upright wall of stone must be made throughout, just to support a bank of 12 feet high ; the specific gravity of the stone being to that of earth, as 5 to 4.

Ans. $4\frac{4}{5}\sqrt{\frac{4}{5}}$, or 4·29325 feet.

QUEST. 30. If the stone wall be made like a wedge, or having its upright section a triangle ; tapering to a point at top,

top, but its side next the bank of earth perpendicular to the horizon; what is its thickness at the bottom, so as to support the same bank? Ans. $12\sqrt{\frac{1}{5}}$ or 5·36656 feet.

QUEST. 31. But if the earth will only stand at an angle of 30 degrees to the horizontal line; it is required to determine the thickness of wall in both the preceding cases?

Ans. the breadth of the rectangle $12\sqrt{\frac{1}{5}}$, or 5·36656,
but the base of the triangular bank $12\sqrt{\frac{3}{16}}$, or 6·53667.

QUEST. 32. To find the thickness of an upright rectangular wall, necessary to support a body of water; the water being 10 feet deep, and the wall 12 feet high; also the specific gravity of the wall to that of the water, as 11 to 7.

Ans. 4·204374 feet.

QUEST. 33. To determine the thickness of the wall at the bottom, when the section of it is triangular, and the altitudes as before. Ans. 5·1492866 feet.

QUEST. 34. Supposing the distance of the earth from the sun to be 95 millions of miles; I would know at what distance from him another body must be placed, so as to receive light and heat quadruple to that of the earth.

Ans. at half the distance, or $47\frac{1}{2}$ millions.

QUEST. 35. If the mean distance of the sun from us be 106 of his diameters; how much hotter is it at the surface of the sun, than under our equator?

Ans. 11236 times hotter.

QUEST. 36. The distance between the earth and the sun being accounted 95 millions of miles, and between Jupiter and the sun 495 millions; the degree of light and heat received by Jupiter, compared with that of the earth, is required?

Ans. $\frac{361}{9801}$, or nearly $\frac{1}{27}$ of the earth's light and heat.

QUEST. 37. A certain body on the surface of the earth weighs a cwt, or 112lb; the question is whither this body must be carried, that it may weigh only 10lb?

Ans. either at 3·3466 semi-diameters, or $\frac{5}{56}$ of a semi-diameter, from the centre.

QUEST. 38. If a body weigh 1 pound, or 16 ounces, on the surface of the earth; what will its weight be at 50 miles above it, taking the earth's diameter at 7930 miles?

Ans. 15oz. $9\frac{2}{3}$dr. nearly.

QUEST. 39. Whereabouts, in the line between the earth and moon is their common centre of gravity; supposing the earth's diameter to be 7930 miles, and the moon's 2160; also the

the density of the former to that of the latter, as 99 to 68, or as 10 to 7 nearly, and their mean distance 30 of the earth's diameters?

Ans. at $\frac{105}{281}$ parts of a diameter from the earth's centre, or $\frac{41}{502}$ parts of a diameter, or 648 miles below the surface.

Quest. 40. Whereabouts, between the earth and moon, are their attractions equal to each other? Or where must another body be placed, so as to remain suspended in equilibrio, not being more attracted to the one than to the other or having no tendency to fall either way? Their dimensions being as in the last question.

Ans. From the earth's centre $26\frac{9}{11}$ } of the earth's
From the moon's centre $3\frac{2}{11}$ } diameters.

Quest. 41. Suppose a stone dropt into an abyss, should be stopped at the end of the 11th second after its delivery; what space would it have gone through?

Ans. 1946 $\frac{1}{12}$ feet.

Quest. 42. What is the difference between the depths of two wells, into each of which should a stone be dropped at the same instant, the one will strike the bottom at 6 seconds, the other at 10? Ans. $1029\frac{1}{3}$ feet.

Quest. 43. If a stone be $19\frac{1}{2}$ seconds in descending from the top of a precipice to the bottom, what is its height?

Ans. $6115\frac{11}{16}$ feet.

Quest. 44. In what time will a musket ball, dropped from the top of Salisbury steeple, said to be 400 feet high, reach the bottom? Ans. 5 seconds nearly.

Quest. 45. If a heavy body be observed to fall through 100 feet in the last second of time, from what height did it fall, and how long was it in motion?

Ans. time $3\frac{235}{386}$ sec. and height 209 $\frac{4273}{9264}$ feet.

Quest. 46. A stone being let fall into a well, it was observed that, after being dropped, it was 10 seconds before the sound of the fall at the bottom reached the ear. What is the depth of the well? Ans. 1270 feet nearly.

Quest. 47. It is proposed to determine the length of a pendulum vibrating seconds, in the latitude of London, where a heavy body falls through $16\frac{1}{12}$ feet in the first second of time? Ans. 39·11 inches.

By experiment this length is found to be $39\frac{1}{8}$ inches.

Quest. 48.

Quest. 48. What is the length of a pendulum vibrating in 2 seconds; also in half a second, and in a quarter second?

Ans. the 2 second pendulum $156\frac{1}{2}$
the $\frac{1}{2}$ second pundulum $9\frac{25}{32}$
the $\frac{1}{4}$ second pendulum $2\frac{57}{128}$ inches.

Quest. 49. What difference will there be in the number of vibrations, made by a pendulum of 6 inches long, and another of 12 inches long, in an hour's time? Ans. $2692\frac{1}{2}$.

Quest. 50. Observed that while a stone was descending, to measure the depth of a well, a string and plummet, that from the point of suspension, or the place where it was held, to the centre of oscillation, measured just 18 inches, had made 8 vibrations, when the sound from the bottom returned. What was the depth of the well?

Ans. 412·61 feet.

Quest. 51. If a ball vibrate in the arch of a circle, 10 degrees on each side of the perpendicular; or a ball roll down the lowest 10 degrees of the arch; required the velocity at the lowest point? the radius of the circle, or length of the pendulum, being 20 feet. Ans. 4·4213 feet per second.

Quest. 52. If a ball descend down a smooth inclined plane, whose length is 100 feet, and altitude 10 feet; how long will it be in descending, and what will be the last velocity?

Ans. the veloc. 25 364 feet per sec. and time 7·8852 sec.

Quest. 53. If a cannon ball, of 1lb weight, be fired against a pendulous block of wood, and, striking the centre of oscillation, cause it to vibrate an arc whose chord is 30 inches; the radius of that arc, or distance from the axis to the lowest point of the pendulum, being 118 inches, and the pendulum vibrating in small arcs 40 oscillations per minute. Required the velocity of the ball, and the velocity of the centre of oscillation of the pendulum, at the lowest point of the arc; the whole weight of the pendulum being 500lb?

Ans. veloc. ball 1956·6054 feet per sec.
and veloc. cent. oscil. 3·9054 feet per sec.

Quest. 54. How deep will a cube of oak sink in common water; each side of the cube being 1 foot?

Ans. $11\frac{1}{10}$ inches.

Quest. 55. How deep will a globe of oak sink in water; the diameter being 1 foot? Ans. 9·9867 inches.

Quest.

QUEST. 56. If a cube of wood, floating in common water, have three inches of it dry above the water, and $4\frac{8}{103}$ inches dry when in sea water; it is proposed to determine the magnitude of the cube, and what sort of wood it is made of?

Ans. the wood is oak, and each side 40 inches.

QUEST 57. An irregular piece of lead ore weighs, in air 12 ounces, but in water only 7; and another fragment weighs in air $14\frac{1}{2}$ ounces, but in water only 9; required their comparative densities, or specific gravities?

Ans. as 145 to 132.

QUEST. 58. An irregular fragment of glass, in the scale, weighs 171 grains, and another of magnet 102 grains; but in water the first fetches up no more than 120 grains, and the other 79: what then will their specific gravities turn out to be?

Ans. glass to magnet as 3933 to 5202
or nearly as 10 to 13.

QUEST. 59. Hiero, king of Sicily, ordered his jeweller to make him a crown, containing 63 ounces of gold. The workmen thought that substituting part silver was only a proper perquisite; which taking air, Archimedes was appointed to examine it; who on putting it into a vessel of water, found it raised the fluid 8·2245 cubic inches: and having discovered that the inch of gold more critically weighed 10·36 ounces, and that of silver but 5·85 ounces, he found by calculation what part of the king's gold had been changed. And you are desired to repeat the process.

Ans. 28·8 ounces.

QUEST. 60. Supposing the cubic inch of common glass weigh 1·4921 ounces troy, the same of sea-water ·59542, and of brandy ·5368; then a seaman having a gallon of this liquor in a glass bottle, which weighs 3·84lb out of water, and, to conceal it from the officers of the customs, throws it overboard. It is proposed to determine, if it will sink, how much force will just buoy it up?

Ans. 14·1496 ounces.

QUEST. 61. Another person has half an anker of brandy, of the same specific gravity as in the last question; the wood of the cask suppose measures $\frac{1}{8}$ of a cubic foot; it is proposed to assign what quantity of lead is just requisite to keep the cask and liquor under water?

Ans. 89·743 ounces.

QUEST. 62. Suppose, by measurement, it be found that a man-of-war, with its ordnance, rigging, and appointments, sinks

sinks so deep as to displace 50000 cubic feet of fresh water; what is the whole weight of the vessel?

Ans. $1395\frac{1}{10}$ tons.

Quest. 63. It is required to determine what would be the height of the atmosphere, if it were every where of the same density as at the surface of the earth, when the quicksilver in the barometer stands at 30 inches; and also, what would be the height of a water barometer at the same time?

Ans. height of the air $29166\frac{2}{3}$ feet, or 5·5240 miles, height of water 35 feet.

Quest. 64. With what velocity would each of those three fluids, viz. quicksilver, water, and air, issue through a small orifice in the bottom of vessels, of the respective heights of 30 inches, 35 feet, and 5·5240 miles, estimating the pressure by the whole altitudes, and the air rushing into a vacuum?

Ans. the veloc. of quicksilver 12·681 feet,
the veloc. of water - 47·447
the veloc. of air - - 1369·8

Quest. 65. A very large vessel of 10 feet high (no matter what shape) being kept constantly full of water, by a large supplying cock at the top; if 9 small circular holes, each $\frac{1}{2}$ of an inch diameter, be opened in its perpendicular side at every foot of the depth: it is required to determine the several distances to which they will spout on the horizontal plane of the base, and the quantity of water discharged by all of them in 10 minutes?

Ans. the distances are
$\sqrt{36}$ or 6·00000
$\sqrt{64}$ - 8·00000
$\sqrt{84}$ - 9·16515
$\sqrt{96}$ - 9·79796
$\sqrt{100}$ - 10·00000
$\sqrt{96}$ - 9·79796
$\sqrt{84}$ - 9·16515
$\sqrt{64}$ - 8·00000
$\sqrt{36}$ - 6·00000
and the quantity discharged in 10 min. 123·8849 gallons.

Note. In this solution, the velocity of the water is supposed to be equal to that which is acquired by a heavy body in falling through the whole height of the water above the orifice, and that it is the same in every part of the holes.

Quest.

QUEST. 66. If the inner axis of a hollow globe of copper, exhausted of air, be 100 feet; what thickness must it be of, that it may just float in the air?

Ans. ·02688 of an inch thick.

QUEST. 67. If a spherical balloon of copper, of $\frac{1}{100}$ of an inch thick, have its cavity of 100 feet diameter, and be filled with inflammable air, of $\frac{1}{10}$ of the gravity of common air, what weight will just balance it, and prevent it from rising up into the atmosphere? Ans. 21273lb.

QUEST. 68. If a glass tube, 36 inches long, close at top, be sunk perpendicularly into water, till its lower or open end be 30 inches below the surface of the water; how high will the water rise within the tube, the quicksilver in the common barometer at the same time standing at 29½ inches?

Ans. 2·26545 inches.

QUEST. 69. If a diving bell, of the form of a parabolic conoid, be let down into the sea to the several depths of 5, 10, 15, and 20 fathoms; it is required to assign the respective heights to which the water will rise within it: its axis and the diameter of its base being each 8 feet, and the quicksilver in the barometer standing at 30·9 inches?

Ans. at 5 fathoms deep the water rises 2·03546 feet,
at 10 - - - - 3·06393
at 15 - - - - 3·70267
at 20 - - - 4·14653

ON THE NATURE AND SOLUTION OF EQUATIONS IN GENERAL.

1. In order to investigate the general properties of the higher equations, let there be assumed between an unknown quantity x, and given quantities a, b, c, d, an equation constituted of the continued product of uniform factors; thus

$$(x-a)\times(x-b)\times(x-c)\times(x-d)=0.$$

This, by performing the multiplications, and arranging the final product according to the powers or dimensions of x, becomes

$$x^4 \left.\begin{matrix}-a\\-b\\-c\\-d\end{matrix}\right\} x^3 \left.\begin{matrix}+ab\\+ac\\+ad\\+bc\\+bd\\+cd\end{matrix}\right\} x^2 \left.\begin{matrix}-abc\\-abd\\-acd\\-bcd\end{matrix}\right\} x + abcd = 0\ldots.(A)$$

Now it is obvious that the assemblage of terms which compose the first side of this equation may become equal to nothing in four different ways; namely, by supposing either $x=a$, or $x=b$, or $x=c$, or $x=d$; for in either case one or other of the factors $x-a$, $x-b$, $x-c$, $x-d$, will be equal to nothing, and nothing multiplied by any quantity whatever will give *nothing* for the product. If any other value e be put for x, then none of the factors $e-a$, $e-b$, $e-c$, $e-d$, being equal to nothing, their continued product cannot be equal to nothing. There are therefore, in the proposed equation, four roots or values of x; and that which characterizes these roots is, that on substituting each of them successively instead of x, the aggregate of the terms of the equation vanishes by the opposition of the signs $+$ and $-$.

The preceding equation is only of the fourth power or degree; but it is manifest that the above remark applies to equations of higher or lower dimensions: viz, that in general an equation of any degree whatever has as many roots as there are units in the exponent of the highest power of the unknown quantity, and that each root has the property of rendering, by its substitution in place of the unknown quantity, the aggregate of all the terms of the equation equal to nothing.

It must be observed that we cannot have all at once $x=a$, $x=b$, $x=c$, &c, for the roots of the equation; but that the particular equations $x-a=0$, $x-b=0$, $x-c=0$, &c, obtain only in a *disjunctive* sense. They exist as factors in the

the same equation, because algebra gives, by one and the same formula, not only the solution of the particular problem from which that formula may have originated, but also the solution of all problems which have similar conditions. The different roots of the equation satisfy the respective conditions: and those roots may differ from one another, by their *quantity*, and by their *mode* of existence.

It is true, we say frequently that the roots of an equation are $x = a$, $x = b$, $x = c$, &c, as though those values of x existed conjunctively; but this manner of speaking is an abbreviation, which it is necessary to understand in the sense explained above.

2. In the equation A, all the roots are positive; but if the factors which constitute the equation had been $x + a$, $x + b$, $x + c$, $x + d$, the roots would have been negative or subtractive. Thus

$$x^4 \left.\begin{array}{l}+a\\+b\\+c\\+d\end{array}\right\} x^3 \left.\begin{array}{l}+ab\\+ac\\+ad\\+bc\\+bd\\+cd\end{array}\right\} x^2 \left.\begin{array}{l}+abc\\+abd\\+acd\\+bcd\end{array}\right\} x + abcd = 0 \ldots (\mathrm{B})$$

has negative roots, those roots being $x = -a$, $x = -b$, $x = -c$, $x = -d$: and here again we are to apply them disjunctively.

3. Some equations have their roots in part positive, in part negative. Such is the following:

$$x^3 \left.\begin{array}{l}-a\\-b\\+c\end{array}\right\} x^2 \left.\begin{array}{l}+ab\\-ac\\-bc\end{array}\right\} x + abc = 0 \ldots\ldots\ldots (\mathrm{C})$$

Here are the two positive roots, viz, $x = a$, $x = b$; and one negative root, viz, $x = -c$: the equation being constituted of the continued product of the three factors, $x - a = 0$, $x - b = 0$, $x + c = 0$.

From an inspection of the equations A, B, C, it may be inferred, that a complete equation consists of a number of terms exceeding by *unity* the number of its roots.

4. The preceding equations have been considered as formed from equations of the first degree, and then each of them contains so many of those constituent equations as there are units in the exponent of its degree. But an equation which exceeds the second dimension, may be considered as composed of one or more equations of the second degree, or of the third, &c, combined, if it be necessary, with equations of the first degree, in such manner, that the product of all those constituent equations shall form the proposed equation. Indeed

deed, when an equation is formed by the successive multiplication of several simple equations, quadratic equations, cubic equations, &c, are formed; which of course may be regarded as factors of the resulting equation.

5. It sometimes happens that an equation contains imaginary roots; and then they will be found also in its constituent equations. This class of roots always enters an equation by pairs; because they may be considered as containing, in their expression at least, one *even* radical placed before a negative quantity, and because an *even* radical is necessarily preceded by the double sign $\pm$. Let, for example, the equation be $x^4 - (2a-2c)x^3 + (a^2+b^2-4ac+c^2+d^2)x^2 + (2a^2c + 2b^2c - 2ac^2 - 2ad^2)x + (a^2+b^2).(c^2+d^2) = 0$. This may be regarded as constituted of the two subjoined quadratic equations, $x^2 - 2ax + a^2 + b^2 = 0$, $x^2 + 2cx + c^2 + d^2 = 0$: and each of these quadratics contains two imaginary roots; the first giving $x = a \pm b\sqrt{-1}$, and the second $x = -c \pm d\sqrt{-1}$.

In the equation resulting from the product of these two quadratics, the coefficients of the powers of the unknown quantity, and of the last term of the equation, are real quantities, though the constituent equations contain imaginary quantities; the reason is, that these latter disappear by means of addition and multiplication.

The same will take place in the equation $(x-a).(x+b).(x^2+2cx+c^2+d^2) = 0$, which is formed of two equations of the first degree, and one equation of the second whose roots are imaginary.

These remarks being premised, the subsequent general theorems will be easily established.

THEOREM I.

Whatever be the Species of the Roots of an Equation, when the Equation is arranged according to the Powers of the Unknown Quantity, if the First Term be positive, and have unity for its Coefficient, the following Properties may be traced:

I. The first term of the equation is the unknown quantity raised to the power denoted by the number of roots.

II. The second term contains the unknown quantity raised to a power less than the former by unity, with a coefficient equal to the sum of the roots taken with contrary signs.

III. The third term contains the unknown quantity raised to a power less by 2 than that of the first term, with a coefficient equal to the sum of all the products which can be formed by multiplying all the roots two and two.

IV. The fourth term contains the unknown quantity raised to a power less by 3 than that of the first term, with a coefficient equal to the sum of all the products which can be made by multiplying any three of the roots with contrary signs.

V. And so on to the last term, which is the continued product of all the roots taken with contrary signs.

All this is evident from inspection of the equations exhibited in arts. 1, 2, 3, 5.

Cor. 1. Therefore an equation having all its roots real, but some positive, the others negative, will want its second term when the sum of the positive roots is equal to the sum of the negative roots. Thus, for example, the equation c will want its second term, if $a + b = c$.

Cor. 2. An equation whose roots are all imaginary, will want the second term, if the sum of the real quantities which enter into the expression of the roots, is partly positive, partly negative, and has the result reduced to nothing, the imaginary parts mutually destroying each other by addition in each pair of roots. Thus, the first equation of art. 5 will want the second term if $-2a + 2c = 0$, or $a = c$. The second equation of the same article, which has its roots partly real, partly imaginary, will want the second term if $b - a + 2c = 0$, or $a - b = 2c$.

Cor. 3. An equation will want its third term, if the sum of the products of the roots taken two and two, is partly positive, partly negative, and these mutually destroy each other.

Remark. An *incomplete* equation may be thrown into the form of *complete* equations, by introducing, with the coefficient *a cypher*, the absent powers of the unknown quantity: thus, for the equation $x^3 + r = 0$, may be written $x^3 + 0\,x^2 + 0\,x + r = 0$. This in some cases will be useful.

Cor. 4. An equation with positive roots may be transformed into another which shall have negative roots of the same value, and reciprocally. In order to this, it is only necessary to change the signs of the alternate terms, beginning with the second. Thus, for example, if instead of the equation $x^3 - 8x^2 + 17x - 10 = 0$, which has three positive roots 1, 2, and 5, we write $x^3 + 8x^2 + 17x + 10 = 0$, this latter equation will have three negative roots $x = -1, x = -2, x = -5$. In like manner, if instead of the equation $x^3 + 2x^2 - 13x + 10 = 0$, which has two positive roots $x = 1, x = 2$, and one negative root $x = -5$, there be taken $x^3 - 2x^2 - 13x - 10 = 0$, this latter equation will have two negative roots, $x = -1, x = -2$, and one positive root $x = 5$.

In general, if there be taken the two equations, $(x - a) \times (x - b) \times (x - c) \times (x - d) \times$ &c $= 0$, and $(x + a) \times (x + b) \times$ $(x +$

$(x+c) \times (x+d) \times$ &c $= 0$, of which the roots are the same in magnitude, but with different signs: if these equations be developed by actual multiplication, and the terms arranged according to the powers of x, as in arts. 1, 2; it will be seen that the second terms of the two equations will be affected with different signs, the third terms with like signs, the fourth terms with different signs, &c.

When an equation has not all its terms, the deficient terms must be supplied by cyphers, before the preceding rule can be applied.

Cor. 5. The sum of the roots of an equation, the sum of their squares, the sum of their cubes, &c, may be found without knowing the roots themselves. For, let an equation of any degree or dimension, m, be $x^m + fx^{m-1} + gx^{m-2} + hx^{m-3} +$ &c $= 0$, its roots being a, b, c, d, &c. Then we shall have,

1st. The sum of the first powers of the roots, that is, of the roots themselves, or $a + b + c +$ &c $= -f$; since the coefficient of the unknown quantity in the second term, is equal to the sum of the roots taken with different signs.

2dly. The sum of the squares of the roots, is equal to the square of the coefficient of the second term made less by twice the coefficient of the third term: viz, $a^2 + b^2 + c^2 +$ &c $= f^2 - 2g$. For, if the polynomial $a + b + c +$ &c, be squared, it will be found that the square contains the sum of the squares of the terms, a, b, c, &c, *plus* twice the sum of the products formed by multiplying two and two all the roots a, b, c, &c. That is, $(a+b+c+\&c)^2 = a^2 + b^2 + c^2 + \&c + 2(ab + ac + bc + \&c)$. But it is obvious, from equa. A, B, that $(a + b + c + \&c)^2 = f^2$, and $(ab + ac + bc + \&c) = g$. Thus we have $f^2 = (a^2 + b^2 + c^2 + \&c) + 2g$; and consequently $a^2 + b^2 + c^2 + \&c = f^2 - 2g$.

3dly. The sum of the cubes of the roots, is equal to 3 times the rectangle of the coefficient of the second and third terms, made less by the cube of the co-efficient of the second term, and 3 times the coefficient of the fourth term: viz, $a^3 + b^3 + c^3 + \&c = -f^3 + 3fg - 3h$. For we shall by actual involution, have $(a + b + c + \&c)^3 = a^3 + b^3 + c^3 + \&c + 3(a + b + c) \times (ab + ac + bc) - 3abc$. But $(a+b+c+\&c)^3 = -f^3$, $(a + b + c + \&c) \times (ab + ac + bc + \&c) = -fg$, $abc = -h$. Hence therefore, $-f^3 = a^3 + b^3 + c^3 + \&c - 3fg + 3h$; and consequently, $a^3 + b^3 + c^3 + \&c = -f^3 + 3fg - 3h$. And so on, for other powers of the roots.

THEOREM

THEOREM II.

In Every Equation, which contain only Real Roots:

I. If all the roots are positive, the terms of the equation will be $+$ and $-$ alternately.

II. If all the roots are negative, all the terms will have the sign $+$.

III. If the roots are partly positive, partly negative, there will be as many positive roots as there are *variations* of signs, and as many negative roots as there are *permanencies* of signs; these variations and permanencies being observed from one term to the following through the whole extent of the equation.

In all these, either the equations are complete in their terms, or they are made so.

The first part of this theorem is evident from the examination of equation A; and the second from equation B.

To demonstrate the third, we revert to the equation C (art. 3), which has two positive roots, and one negative. It may happen that either $c > a + b$, or $c < a + b$.

In the first case, the second term is positive, and the third is negative; because, having $c > a + b$, we shall have $ac + bc > (a + b)^2 > ab$. And, as the last term is positive, we see that from the first to the second there is a permanence of signs; from the second to the third a variation of signs; and from the third to the fourth another variation of signs. Thus there are two variations and one permanence of signs; that is, as many variations as there are positive roots, and as many permanencies as there are negative roots.

In the second case, the second term of the equation is negative, and the third may be either positive or negative. If that term is positive, there will be from the first to the second a variation of signs; from the second to the third another variation; from the third to the fourth a permanence; making in all two variations and one permanence of signs. If the third term be negative; there will be one variation of signs from the first to the second; one permanence from the second to the third; and one variation from the third to the fourth: thus making again two variations and one permanence. The number of variations of signs therefore in this case, as well as in the former, is the same as that of the positive roots; and the number of permanencies, the same as that of the negative roots.

Corol. Whence it follows, that if it be known, by any means whatever, that an equation contains only real roots, it is

is also known how many of them are positive, and how many negative. Suppose, for example, it be known that, in the equation $x^5 + 3x^4 - 23x^3 - 27x^2 + 166x - 120 = 0$, all the roots are real: it may immediately be concluded that there are *three* positive and *two* negative roots. In fact this equation has the three positive roots $x = 1$, $x = 2$, $x = 3$; and two negative roots, $x = -4$, $x = -5$.

If the equation were incomplete, the absent terms must be supplied by adopting cyphers for coefficients, and those terms must be marked with the ambiguous sign $\pm$. Thus, if the equation were

$$x^5 - 20x^3 + 30x^2 + 19x - 30 = 0,$$

all the roots being real, and the second term wanting. It must be written thus:

$$x^5 \pm 0x^4 - 20x^3 + 30x^2 + 19x - 30 = 0.$$

Then it will be seen, that, whether the second term be positive or negative, there will be 3 variations and 2 permanencies of signs: and consequently the equation has 3 positive and 2 negative roots. The roots in fact are, 1, 2, 3, -1, -5.

This rule only obtains with regard to equations whose roots are real. If, for example, it were inferred that, because the equation $x^2 + 2x + 5 = 0$ had two permanencies of signs, it had two negative roots, the conclusion would be erroneous: for both the roots of this equation are imaginary.

THEOREM III.

Every Equation may be Transformed into Another whose Roots shall be Greater or Less by a Given Quantity.

In any equation whatever, of which x is unknown, (the equations A, B, C, for example) make $x = z + m$, z being a new unknown quantity, m any given quantity, positive or negative: then substituting, instead of x and its powers, their values resulting from the hypothesis that $x = z + m$; so shall there arise an equation, whose roots shall be greater or less than the roots of the primitive equation, by the assumed quantity m.

Corol. The principal use of this transformation is, to take away any term out of an equation. Thus, to transform an equation into one which shall want the *second* term, let m be so assumed that $nm - a = 0$, or $m = \frac{a}{n}$, n being the index of the highest power of the unknown quantity, and a the coefficient of the second term of the equation, with its sign changed: then, if the roots of the transformed equation can be found, the roots of the original equation may also be found, because $x = z + \frac{a}{n}$.

THEOREM

THEOREM IV.

Every Equation may be Transformed into Another, whose Roots shall be Equal to the Roots of the First Multiplied or Divided by a Given Quantity.

1. Let the equation be $z^3 + az^2 + bz + c = 0$: if we put $fz = x$, or $z = \frac{x}{f}$, the transformed equation will be $x^3 + fax^2 + f^2bx + f^3c = 0$, of which the roots are the respective products of the roots of the primitive equation multiplied into the quantity f.

By means of this transformation, an equation with fractional quantities, may be changed into another which shall be free from them. Suppose the equation were $z^3 + \frac{az^2}{g} + \frac{bz}{h} + \frac{d}{k} = 0$: multiplying the whole by the product of the denominators, there would arise $ghkz^3 + hkaz^2 + gkbz + ghd = 0$: then assuming $ghkz = x$, or $z = \frac{x}{ghk}$, the transformed equa. would be $x^3 + hkax^2 + g^2k^2hbx + g^3k^3h^2d = 0$.

The same transformation may be adopted, to exterminate the radical quantities which affect certain terms of an equation. Thus, let there be given the equation $z^3 + az^2\sqrt{k} + bz + c\sqrt{k}$: make $z\sqrt{k} = x$; then will the transformed equation be $x^3 + akx^2 + bkx + ck^2 = 0$, in which there are no radical quantities.

2. Take, for one more example, the equation $z^3 + az^2 + bz + c = 0$. Make $\frac{z}{f} = x$; then will the equation be transformed to $x^3 + \frac{ax^2}{f} + \frac{bx}{f^2} + \frac{c}{f^3} = 0$, in which the roots are equal to the quotients of those of the primitive equations divided by f.

It is obvious that, by analogous methods, an equation may be transformed into another, the roots of which shall be to those of the proposed equation, in any required ratio. But the subject need not be enlarged on here. The preceding succinct view will suffice for the usual purposes, so far as relates to the nature and chief properties of equations. We shall therefore conclude this chapter with a summary of the most useful rules for the solution of equations of different degrees, besides those already given in the first volume.

I. *Rules for the Solution of Quadratics by Tables of Sines and Tangents.*

1. If the equation be of the form $x^2 + px = q$:

Make $\tan A = \frac{2}{p}\sqrt{q}$; then will the two roots be,

$x = +\tan \frac{1}{2}A\sqrt{q}$ ..... $x = -\cot \frac{1}{2}A\sqrt{q}$.

2. For quadratics of the form $x^2 - px = q$.

Make, as before, $\tan A = \frac{2}{p}\sqrt{q}$: then will

$x = -\tan \frac{1}{2}A\sqrt{q}$ ..... $x = +\cot \frac{1}{2}A\sqrt{q}$.

3. For quadratics of the form $x^2 + px = -q$.

Make $\sin A = \frac{2}{p}\sqrt{q}$: then will

$x = -\tan \frac{1}{2}A\sqrt{q}$ ..... $x = -\cot \frac{1}{2}A\sqrt{q}$.

4. For quadratics of the form $x^2 - px = -q$.

Make $\sin A = \frac{2}{p}\sqrt{q}$: then will

$x = +\tan \frac{1}{2}A\sqrt{q}$ ..... $x = +\cot \frac{1}{2}A\sqrt{q}$.

In the last two cases, if $\frac{2}{p}\sqrt{q}$ exceed unity, $\sin A$ is imaginary, and consequently the values of x.

The logarithmic application of these formulæ is very simple. Thus, in case 1st. Find A by making

$$10 + \log 2 + \tfrac{1}{2}\log q - \log p = \log \tan A.$$

$$\text{Then } \log x = \begin{cases} + \log \tan \frac{1}{2}A + \frac{1}{2}\log q - 10. \\ -(\log \cot \frac{1}{2}A + \frac{1}{2}\log q - 10). \end{cases}$$

Note. This method of solving quadratics, is chiefly of use when the quantities p and q are large integers, or complex fractions.

II. *Rules for the Solution of Cubic Equations by tables of Sines, Tangents, and Secants.*

1. For cubics of the form $x^3 + px \pm q = 0$.

Make $\tan B = \frac{\frac{1}{3}p}{q} \cdot 2\sqrt{\frac{1}{3}p}$ ..... $\tan A = \sqrt[3]{\tan \frac{1}{2}B}$.

Then $x = \mp \cot 2A \cdot 2\sqrt{\frac{1}{3}p}$.

2. For cubics of the form $x^3 - px \pm q = 0$.

Make $\sin B = \frac{\frac{1}{3}p}{q} \cdot 2\sqrt{\frac{1}{3}p}$ ..... $\tan A = \sqrt[3]{\tan \frac{1}{2}B}$.

Then $x = \mp \operatorname{cosec} 2A \cdot 2\sqrt{\frac{1}{3}p}$.

Here, if the value of $\sin B$ should exceed unity, B would be imaginary, and the equation would fall in what is called the

the *irreducible case* of cubics. In that case we must make cosec $3A = \frac{\frac{1}{3}p}{q} \cdot 2\sqrt{\frac{1}{3}p}$: and then the three roots would be

$$x = \pm \sin A \,.\, 2\sqrt{\tfrac{1}{3}p}.$$
$$x = \pm \sin(60^\circ - A) \,.\, 2\sqrt{\tfrac{1}{3}p}.$$
$$x = \pm \sin(60^\circ + A) \,.\, 2\sqrt{\tfrac{1}{3}p}.$$

If the value of sin B were 1, we should have $B = 90^\circ$, tan $A = 1$; therefore $A = 45^\circ$, and $x = \mp 2\sqrt{\frac{1}{3}p}$. But this would not be the only root. The second solution would give cosec $3A = 1$: therefore $A = \frac{90^\circ}{3}$; and then

$$x = \pm \sin 30^\circ \,.\, 2\sqrt{\tfrac{1}{3}p} = \pm\sqrt{\tfrac{1}{3}p}.$$
$$x = \pm \sin 30^\circ \,.\, 2\sqrt{\tfrac{1}{3}p} = \pm\sqrt{\tfrac{1}{3}p}.$$
$$x = \mp \sin 90^\circ \,.\, 2\sqrt{\tfrac{1}{3}p} = \mp 2\sqrt{\tfrac{1}{3}p}.$$

Here it is obvious that the first two roots are equal, that their sum is equal to the third with a contrary sign, and that this third is the one which is produced from the first solution*.

In these solutions, the double signs in the value of x, relate to the double signs in the value of q.

N. B. Cardan's Rule for the solution of Cubics is given in the first volume of this course.

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* The tables of sines, tangents, &c, besides their use in trigonometry, and in the solution of the equations, are also very useful in finding the value of algebraic expressions where extraction of roots would be otherwise required. Thus, if a and b be any two quantities, of which a is the greater. Find x, z, &c, so, that $\tan x = \sqrt{\frac{b}{a}}$, $\sin z = \sqrt{\frac{b}{a}}$, $\sec y = \frac{a}{b}$, $\tan u = \frac{b}{a}$, and $\sin t = \frac{b}{a}$: then will

$$\log \sqrt{(a^2 - b^2)} = \log a + \log \sin y = \log b + \log \tan y.$$
$$\log \sqrt{(a^2 - b^2)} = \tfrac{1}{2}[\log(a+b) + \log(a-b)].$$
$$\log \sqrt{(a^2 + b^2)} = \log a + \log \sec u = \log b + \log \operatorname{cosec} u.$$
$$\log \sqrt{(a+b)} = \tfrac{1}{2}\log a + \log \sec x = \tfrac{1}{2}\log a + \tfrac{1}{2}\log 2 + \log\cos \tfrac{1}{2}y.$$
$$\log \sqrt{(a-b)} = \tfrac{1}{2}\log a + \log\cos z = \tfrac{1}{2}\log a + \tfrac{1}{2}\log 2 + \log\sin \tfrac{1}{2}y.$$
$$\log(a \pm b)^{\frac{m}{n}} = \frac{m}{n}[\log a + \log\cos t + \log\tan(45^\circ \pm \tfrac{1}{2}t)].$$

The first three of these formulæ will often be useful, when two sides of a right-angled triangle are given, to find the third.

III. *Solution*

III. *Solution of Biquadratic Equations.*

Let the proposed biquadratic be $x^4 + 2px^3 - qx^2 + rx + s$. Now $(x^2 + px + n)^2 = x^4 + 2px^3 + (p^2 + 2n)x^2 + 2pnx + n^2$: if therefore $(p^2 + 2n)x^2 + 2pnx + n^2$ be added to both sides of the proposed biquadratic, the first will become a complete square $(x^2 + px + n)^2$, and the latter part $(p^2 + 2n + q)x^2 + (2pn + r)x + n^2 + s$, is a complete square if $4(p^2 + 2n + q) \cdot (n^2 + s) = 2pn + r^2$; that is, multiplying and arranging the terms according to the dimensions of n, if $8n^3 + 4qn^2 + (8s - 4rp)n + 4qs + 4p^2s - r^2 = 0$. From this equation let a value of n be obtained, and substituted in the equation $(x^2 + px + n)^2 = (p^2 + 2n + q)x^2 + (2pn + r)x + n^2 + s$; then extracting the square root on both sides

$$x^2 + px + n = \pm \left\{ \sqrt{(p^2 + 2n + q)}x + \sqrt{(n^2 + s)} \right\} \text{ when } 2pn + r \text{ is positive;}$$

$$\text{or } x^2 + px + n = \pm \left\{ \sqrt{(p^2 + 2n + q)}x - \sqrt{(n^2 + s)} \right\} \text{ when } 2pn + r \text{ is negative.}$$

And from these two quadratics, the four roots of the given biquadratic may be determined*.

Note. Whenever, by taking away the second term of a biquadratic, after the manner described in cor. th. 3, the fourth term also vanishes, the roots may immediately be obtained by the solution of a quadratic only.

A biquadratic may also be solved independently of cubics, in the following cases:

1. When the difference between the coefficient of the third term, and the square of half that of the second term, is equal to the coefficient of the fourth term, divided by half that of the second. Then if p be the co-efficient of the second term, the equation will be reduced to a quadratic by dividing it by $x^2 \pm \frac{1}{2}px$.

2. When the last term is negative, and equal to the square of the coefficient of the fourth term divided by 4 times that of the third term, *minus* the square of that of the second: then to complete the square, subtract the terms of the proposed biquadratic from $(x^2 \pm \frac{1}{2}px)^2$, and add the remainder to both its sides.

3. When the co-efficient of the fourth term divided by that of the second term, gives for a quotient the square root of the last term: then to complete the square, add the square of half the coefficient of the second term, to twice the square

*This rule, for solving biquadratics, by conceiving each to be the difference of two squares, is frequently ascribed to Dr. *Waring*; but its original inventor was Mr. *Thomas Simpson*, formerly Professor of Mathematics in the Royal Military Academy.

root

root of the last term, multiply the sum by x^2, from the product take the third term, and add the remainder to both sides of the biquadratics.

4. The fourth term will be made to go out by the usual operation for taking away the second term, when the difference between the cube of half the coefficient of the second term and half the product of the coefficients of the second and third term, is equal to the coefficient of the fourth term.

IV. *Euler's Rule for the Solution of Biquadratics.*

Let $x^4 - ax^2 - bx - c = 0$, be the given biquadratic equation wanting the second term. Take $f = \frac{1}{2}a$, $g = \frac{1}{16}a^2 + \frac{1}{4}c$, and $h = \frac{1}{64}b^2$, or $\sqrt{h} = \frac{1}{8}b$; with which values of f, g, h, form the cubic equation, $z^3 - fz^2 + gz - h = 0$. Find the roots of this cubic equation, and let them be called p, q, r. Then shall the four roots of the proposed biquadratic be these following: viz.

When $\frac{1}{8}b$ is positive.	When $\frac{1}{8}b$ is negative:
1. $x = \sqrt{p} + \sqrt{q} + \sqrt{r}$.	$x = \sqrt{p} + \sqrt{q} - \sqrt{r}$.
2. $x = \sqrt{p} - \sqrt{q} - \sqrt{r}$.	$x = \sqrt{p} - \sqrt{q} + \sqrt{r}$.
3. $x = -\sqrt{p} + \sqrt{q} - \sqrt{r}$.	$x = -\sqrt{p} + \sqrt{q} + \sqrt{r}$.
4. $x = -\sqrt{p} - \sqrt{q} + \sqrt{r}$.	$x = -\sqrt{p} - \sqrt{q} - \sqrt{r}$.

Note 1. In any biquadratic equation having all its terms, if $\frac{3}{8}$ of the square of the coefficient of the 2d term be greater than the product of the coefficients of the 1st and 3d terms, or $\frac{3}{8}$ of the square of the coefficient of the 4th term be greater than the product of the coefficients of the 3d and fifth terms, or $\frac{4}{9}$ of the square of the coefficient of the 3d term greater than the product of the coefficients of the 2d and 4th terms; then all the roots of that equation will be real and unequal: but if either of the said parts of those squares be less than either of those products, the equation will have imaginary roots.

2. In a biquadratic $x^4 + ax^3 + bx^2 + cx + d = 0$, of which two roots are impossible, and d an affirmative quantity, then the two possible roots will be both negative, or both affirmative, according as $a^3 - 4ab + 8c$, is an affirmative or a negative quantity, if the signs of the coefficients a, b, c, d, are neither all affirmative, nor alternately $-$ and $+$*.

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* Various *general* rules for the solution of equations have been given by Demoivre, Bezout, Lagrange, &c; but the most universal in their application are approximating rules, of which a very simple and useful one is given in our first volume.

EXAMPLES.

EXAMPLES.

Ex. 1. Find the roots of the equation $x^2 + \frac{7}{44}x = \frac{1695}{12716}$, by tables of sines and tangents.

Here $p = \frac{7}{44}$, $q = \frac{1695}{12716}$, and the equation agrees with the 1st. form. Also $\tan A = \frac{88}{7}\sqrt{\frac{1695}{12716}}$, and $x = \tan \frac{1}{2}A = \sqrt{\frac{1695}{12716}}$.

In logarithms thus:

$$\begin{array}{rl}
\text{Log } 1695 = & 3\cdot2291697 \\
\text{Arith. com. log } 12716 = & 5\cdot8956495 \\
\hline
\text{sum} + 10 = & 19\cdot1248192 \\
\hline
\text{half sum} = & 9\cdot5624096 \\
\log 88 = & 1\cdot9444827 \\
\text{Arith. com. log } 7 = & 9\cdot1549020 \\
\hline
\text{sum} - 10 = \log\tan A = & 10\cdot6617943 = \log\tan 77^\circ 42' 31''\tfrac{3}{4}; \\
\hline
\log\tan\tfrac{1}{2}A = & 9\cdot9061115 = \log\tan 38^\circ 51' 15''\tfrac{7}{8}; \\
\log\sqrt{q}\text{, as above} = & 9\cdot5624096 \\
\hline
\text{sum} - 10 = \log x = - & 1\cdot4685211 = \log \cdot2941176.
\end{array}$$

This value of x, viz $\cdot2941176$, is nearly equal to $\frac{5}{17}$. To find whether that is the exact root, take the arithmetical compliment of the last logarithm, viz. 0 5314379, and consider it as the logarithm of the denominator of a fraction whose numerator is unity: thus is the fraction found to be $\frac{1}{3\cdot4}$ exactly, and this is manifestly equal to $\frac{5}{17}$. As to the other root of the equation, it is equal to $-\frac{1695}{12716} \div \frac{5}{17} = -\frac{339}{748}$.

Ex. 2. Find the roots of the cubic equation
$x^3 - \frac{403}{441}x + \frac{46}{147} = 0$, by a table of sines.

Here $p = \frac{403}{441}$, $q = \frac{46}{147}$, the second term is negative, and $4p^3 > 27q^2$: so that the example falls under the irreducible case. Hence, $\sin 3A = \frac{3\times46}{147} \times \frac{441}{403} \times \frac{1}{2\sqrt{\frac{403}{3,441}}} = \frac{414}{403} \cdot \frac{1}{\sqrt{\frac{1612}{1323}}}$.

The three values of x therefore, are

$$x = \sin A\sqrt{\frac{1612}{1323}}.$$

$$x = \sin(60^\circ - A)\sqrt{\frac{1612}{1323}}.$$

$$x = -\sin(60^\circ + A)\sqrt{\frac{1612}{1323}}.$$

The

The logarithmic computation is subjoined.

$$
\begin{array}{rcll}
\text{Log } 1612 & = & 3{\cdot}2073650 & \\
\text{Arith. com. log } 1323 & = & 6{\cdot}8784402 & \\
\text{sum} - 10 \ldots\ldots & = & 0{\cdot}0858052 & \\
\text{half sum} & = & 0{\cdot}0429026 & \text{const. log.} \\
\text{Arith. com. const. log.} & = & 9{\cdot}9570974 & \\
\log 414 \ldots & = & 2{\cdot}6170003 & \\
\text{Arith. com. log } 403\ . & = & 7{\cdot}3946950 & \\
\log \sin 3\text{A} \ldots & = & 9{\cdot}9687927 & = \log \sin 68^\circ 32' \, 18''\tfrac{1}{2}. \\
\text{Log} \sin \text{A} & = & 9{\cdot}5891206 & \\
\text{const. log} & & 0{\cdot}0429026 & \\
1.\ \text{sum} - 10 = \log x & = & -\overline{1}{\cdot}6320232 & = \log {\cdot}4285714 = \log \tfrac{3}{7}. \\
\text{Log} \sin (60^\circ - \text{A}) & = & 9{\cdot}7810061 & \\
\text{const. log} \ldots & = & 0{\cdot}0429026 & \\
2.\ \text{sum} - 10 = \log x & = & -\overline{1}{\cdot}8239087 & = \log {\cdot}6666666 = \log \tfrac{2}{3}. \\
\text{Log} \sin (60^\circ + \text{A}) & = & 9{\cdot}9966060 & \\
\text{const. log} \ldots & = & 0{\cdot}0429026 & \\
3.\ \text{sum} - 10 = \log - x & = & 0{\cdot}0395086 & = \log 1{\cdot}095238 = \log \tfrac{23}{21}.
\end{array}
$$

So that the three roots are $\frac{3}{7}$, $\frac{2}{3}$, and $-\frac{23}{21}$; of which the first two are together equal to the third with its sign changed, as they ought to be.

Ex. 3. Find the roots of the biquadratic $x^4 - 25x^2 + 60x - 36 = 0$, by Euler's rule.

Here $a = 25$, $b = -60$, and $c = 36$; therefore

$$f = \frac{25}{2},\ g = \frac{625}{16} + 9 = \frac{769}{16}, \text{ and } h = \frac{225}{4}.$$

Consequently the cubic equation will be

$$z^3 - \frac{25}{2}z^2 + \frac{769}{16}z - \frac{225}{4} = 0.$$

The three roots of which are

$$z = \frac{9}{4} = p, \text{ and } z = 4 = q, \text{ and } z = \frac{25}{4} = r;$$

the square roots of these are $\sqrt{p} = \frac{3}{2}$, $\sqrt{q} = 2$ or $\frac{4}{2}$, $\sqrt{r} = \frac{5}{2}$. Hence, as the value of $\frac{1}{8}b$ is negative, the four roots are

$$
\begin{array}{lrcl}
\text{1st. } x = & \frac{3}{2} + \frac{4}{2} - \frac{5}{2} & = & 1, \\
\text{2d. } x = & \frac{3}{2} - \frac{4}{2} + \frac{5}{2} & = & 2, \\
\text{3d. } x = & -\frac{3}{2} + \frac{4}{2} + \frac{5}{2} & = & 3, \\
\text{4th. } x = & -\frac{3}{2} - \frac{4}{2} - \frac{5}{2} & = & -6.
\end{array}
$$

Ex. 4. Produce a quadratic equation whose roots shall be $\frac{3}{4}$ and $\frac{4}{5}$.

Ans $x^2 - \frac{31}{20}x + \frac{3}{5} = 0$.

Ex. 5. Produce a cubic equation whose roots shall be 2, 5, and -3.

Ans. $x^3 - 4x^2 - 11x + 30 = 0$.

Ex. 6.

Ex. 6. Produce a biquadratic which shall have for the roots 1, 4, -5, and 6 respectively.

Ans. $x^4 - 6x^3 - 21x^2 + 146x - 120 = 0$.

Ex. 7. Find x, when $x^2 + 347x = 22110$.

Ans. $x = 55$, $x = -402$.

Ex. 8. Find the roots of the quadratic $x^2 - \frac{55}{12}x = \frac{325}{6}$.

Ans. $x = 10$, $x = -\frac{65}{12}$.

Ex. 9. Solve the equation $x^2 - \frac{264}{25}x = -\frac{695}{25}$.

Ans. $x = 5$, $x = \frac{139}{25}$.

Ex. 10. Given $x^2 - 24113x = -481860$, to find x.

Ans. $x = 20$, $x = 24093$.

Ex. 11. Find the roots of the equation $x^3 - 3x - 1 = 0$.

Ans. the roots are sin 70°, $-$ sin 50°, and $-$ sin 10°, to a radius $= 2$; or the roots are twice the sines of those arcs as given in the tables.

Ex. 12. Find the real root of $x^3 - x - 6 = 0$.

Ans. $\frac{2}{3}\sqrt{3} \times \sec 54° 44' 20''$.

Ex. 13. Find the real root of $25x^3 + 75x - 46 = 0$.

Ans. $2 \cot 74° 27' 48''$.

Ex. 14. Given $x^4 - 8x^3 - 12x^2 + 84x - 63 = 0$, to find x by quadratics.

Ans. $x = 2 + \sqrt{7} \pm \sqrt{11 + \sqrt{7}}$.

Ex. 15. Given $x^4 + 36x^3 - 400x^2 - 3168x + 7744 = 0$, to find x, by quadratics.

Ans. $x = 11 + \sqrt{209}$.

Ex. 16. Given $x^4 + 24x^3 - 114x^2 - 24x + 1 = 0$ to find x.

Ans. $x = \pm\sqrt{197} - 14$, $x = 2 \pm \sqrt{5}$.

Ex. 17. Find x, when $x^4 - 12x - 5 = 0$.

Ans. $x = 1 \pm \sqrt{2}$, $x = -1 \pm 2\sqrt{-1}$.

Ex. 18. Find x, when $x^4 - 12x^3 + 47x^2 - 72x + 36 = 0$.

Ans. $x = 1$, or 2, or 3, or 6.

Ex. 19. Given $x^5 - 5ax^4 - 80a^2x^3 - 68a^3x^2 + 7a^4x + a^5 = 0$, to find x.

Ans. $x = -a$, $x = 6a \pm a\sqrt{37}$, $x = \pm a\sqrt{10} - 3a$.

CHAPTER

CHAPTER IX.

ON THE NATURE AND PROPERTIES OF CURVES, AND THE CONSTRUCTION OF EQUATIONS.

SECTION I.

Nature and Properties of Curves.

Def. 1. A curve is a line whose several parts proceed in different directions, and are successively posited towards different points in space, which also may be cut by one right line in two or more points.

If all the points in the curve may be included in one plane, the curve is called a *plane* curve; but if they cannot all be comprised in one plane, then is the curve one of *double curvature.*

Since the word direction implies straight lines, and in strictness no part of a curve is a right line, some geometers prefer defining curves otherwise: thus, in a straight line, to be called the line of the abscissas, from a certain point let a line arbitrarily taken be called the abscissa, and denoted (commonly) by x: at the several points corresponding to the different values of x, let straight lines be continually drawn, making a certain angle with the line of the abscissas: these straight lines being regulated in length according to a certain law or equation, are called ordinates; and the line or figure in which their extremities are continually found is, in general, a curve line. This definition however is not free from objection; for a right line may be denoted by an equation between its abscissas and ordinates, such as $y = ax + b$.

Curves are distinguished into algebraical or geometrical, and transcendental or mechanical.

Def. 2. *Algebraical* or geometrical curves, are those in which the relations of the abscissas to the ordinates can be denoted by a common algebraical expression; such, for example, as the equations to the conic sections, given in page 532 &c, of vol. 2.

Def. 3. *Transcendental* or mechanical curves, are such as cannot be so defined or expressed by a pure algebraical equation; or when they are expressed by an equation, having one

of its terms a variable quantity, or a curve line. Thus, $y = \log x$, $y = A . \sin x$, $y = A . \cos x$, $y = A^x$, are equations to transcendental curves; and the latter in particular is an equation to an *exponential* curve

Def. 4. Curves that turn round a fixed point or centre, gradually receding from it, are called *spiral* or *radial* curves.

Def. 5. *Family* or *tribe* of curves, is an assemblage of several curves of different kinds, all defined by the same equation of an indeterminate degree; but differently, according to the diversity of their kind. For example, suppose an equation of an indeterminate degree, $a^{m-1}x = y^m$: if $m = 2$, then will $ax = y^2$; if $m = 3$, then will $a^2x = y^3$; if $m = 4$, then is $a^3x = y^4$; &c: all which curves are said to be of the same family or tribe.

Def. 6. The *axis* of a figure is a right line passing through the centre of a curve, when it has one: if it bisects the ordinates, it is called a *diameter*.

Def. 7. An *asymptote* is a right line which continually approaches towards a curve, but never can touch it, unless the curve could be extended to an infinite distance.

Def. 8. An abscissa and an ordinate, whether right or oblique, are, when spoken of together, frequently termed *co-ordinates*.

Art. 1. The most convenient mode of classing algebraical curves, is according to the orders or dimensions of the equations which express the relation between the co-ordinates. For then the equation for the same curve, remaining always of the same order so long as each of the assumed systems of co-ordinates is supposed to retain constantly the same inclination of ordinate to abscissa, while referred to different points of the curve, however the axis and the origin of the abscissas, or even the inclination of the co-ordinates in different systems, may vary; the same curve will never be ranked under different orders, according to this method. If therefore we take, for a distinctive character, the number of dimensions which the co-ordinates, whether rectangular or oblique, form in the equation, we shall not disturb the order of the classes, by changing the axis and the origin of the abscissas, or by varying the inclination of the co-ordinates.

2. As algebraists call orders of different kinds of equations, those which constitute the greater or less number of dimensions, they distinguish by the same name the different kinds of resulting lines. Consequently the general equation of the first order being $0 = \alpha + \beta x + \gamma y$; we may refer to the first order all the lines which, by taking x and y for the co-ordinates, whether rectangular or oblique, give rise to this

equation.

equation. But this equation comprises the right line alone, which is the most simple of all lines; and since, for this reason, the name of curve does not properly apply to the first order, we do not usually distinguish the different orders by the name of curve lines, but simply by the generic term of lines: hence the first order of lines does not comprehend any curves, but solely the right line.

As for the rest, it is indifferent whether the co-ordinates are perpendicular or not; for if the ordinates make with the axis an angle φ whose sine is μ and cosine ν, we can refer the equation to that of the rectangular co-ordinates, by making $y = \frac{u}{\mu}$, and $x = \frac{\nu u}{\mu} + t$; which will give for an equation between the perpendiculars t and u,

$$0 = \alpha + \beta t + \left(\frac{\beta\nu}{\mu} + \frac{\gamma}{\mu}\right)u.$$

Thus it follows evidently, that the signification of the equation is not limited by supposing the ordinates to be rightly applied: and it will be the same with equations of superior orders, which will not be less general though the co-ordinates are perpendicular. Hence, since the determination of the inclination of the ordinates applied to the axis, takes nothing from the generality of a general equation of any order whatever, we put no restriction on its signification by supposing the co-ordinates rectangular; and the equation will be of the same order whether the co-ordinates be rectangular or oblique.

3. All the lines of the second order will be comprised in the general equation.

$$0 = \alpha + \beta x + \gamma y + \delta x^2 + \epsilon xy + \zeta y^2;$$

that is to say, we may class among lines of the second order all the curve lines which this equation expresses, x and y denoting the rectangular co-ordinates. These curve lines are therefore the most simple of all, since there are no curves in the first order of lines; it is for this reason that some writers call them curves of the first order. But the curves included in this equation are better known under the name of CONIC SECTIONS, because they all result from sections of the cone. The different kinds of these lines are the ellipse, the circle, or ellipse with equal axes, the parabola, and the hyperbola; the properties of all which may be deduced with facility from the preceding general equation. Or this equation may be transformed into the subjoined one:

$$y^2 + \frac{\epsilon x + \gamma}{\zeta}y + \frac{\delta x^2 + \beta x + \alpha}{\zeta} = 0:$$

and

and this again may be reduced to the still more simple form $y^2 = fx^2 + gx + h$.

Here, when the first term fx^2 is *affirmative*, the curve expressed by the equation is a hyperbola; when fx^2 is *negative* the curve is an ellipse; when that term is *absent*, the curve is a parabola. When x is taken upon a *diameter*, the equations reduce to those already given in sect. 4 ch. i.

The mode of effecting these transformations is omitted for the sake of brevity. This section contains a *summary*, not an *investigation* of properties: the latter would require many volumes, instead of a section.

4. Under lines of the third order, or curves of the second, are classed all those which may be expressed by the equation
$0 = \alpha + \beta x + \gamma y + \delta x^2 + \epsilon xy + \zeta y^2 + \eta x^3 + \theta x^2 y + \iota xy^2 + \kappa y^3$.
And in like manner we regard as lines of the fourth order, those curves which are furnished by the general equation

$$0 = \alpha + \beta x + \gamma y + \delta x^2 + \epsilon xy + \zeta y^2 + \eta x^3 + \theta x^2 y + \iota xy^2 + \kappa y^3 + \lambda x^4 + \mu x^3 y + \nu x^2 y^2 + \xi xy^3 + o y^4;$$

taking always x and y for rectangular co-ordinates. In the most general equation of the third order, there are 10 constant quantities, and in that of the fourth order 15, which may be determined at pleasure; whence it results that the kinds of lines of the third order, and, much more, those of the fourth order, are considerably more numerous than those of the second.

5. It will now be easy to conceive, from what has gone before, what are the curve lines that appertain to the fifth, sixth, seventh, or any higher order; but as it is necessary to add to the general equation of the fourth order, the terms

$$x^5,\ x^4 y,\ x^3 y^2,\ x^2 y^3,\ xy^4,\ y^5,$$

with their respective constant co-efficients, to have the general equation comprising all the lines of the fifth order, this latter will be composed of 21 terms: and the general equation comprehending all the lines of the sixth order, will have 28 terms; and so on, conformably to the law of the triangular numbers. Thus, the most general equation for lines of the order n, will contain $\frac{(n+1)\cdot(n+2)}{1\ \cdot\ 2}$ terms, and as many constant letters, which may be determined at pleasure.

6. Since the order of the proposed equation between the co-ordinates, makes known that of the curve line; whenever we have given an algebraic equation between the co-ordinates x and y, or t and u, we know at once to what order it is necessary to refer the curve represented by that equation. If the equation be irrational, it must be freed from radicals, and if

if there be fractions, they must be made to disappear; this done, the greatest number of dimensions formed by the variable quantities x and y, will indicate the order to which the line belongs. Thus the curve which is denoted by this equation $y^2 - ax = 0$, will be of the second order of lines, or of the first order of curves; while the curve represented by the equation $y^2 = x\sqrt{(a^2 - x^2)}$, will be of the third order (that is, the fourth order of lines), because the equation is of the fourth order when freed from radicals; and the line which is indicated by the equation $y = \frac{a^3 - ax^2}{a^2 + x^2}$, will be of the third order, or of the second order of curves, because the equation when the fraction is made to disappear, becomes $a^2y + x^2 y = a^3 - ax^2$, where the term x^2y contains three dimensions.

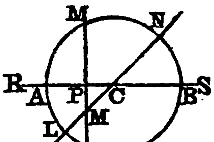
7. It is possible that one and the same equation may give different curves, according as the applicates or ordinates fall upon the axis perpendicularly or under a given obliquity. For instance, this equation, $y^2 = ax - x^2$, gives a circle, when the co-ordinates are supposed perpendicular; but when the co-ordinates are oblique, the curve represented by the same equation will be an ellipse. Yet all these different curves appertain to the same order, because the transformation of rectangular into oblique co-ordinates, and the contrary, does not affect the order of the curve, or of its equation. Hence, though the magnitude of the angles which the ordinates form with the axis, neither augments nor diminishes the generality of the equation, which expresses the lines of each order; yet, a particular equation being given, the curve which it expresses can only be determined when the angle between the co-ordinates is determined also.

8. That a curve line may relate properly to the order indicated by the equation, it is requisite that this equation be not decomposable into rational factors; for if it could be composed of two or of more such factors, it would then comprehend as many equations, each of which would generate a particular line, and the re-union of these lines would be all that the equation proposed could represent. Those equations, then, which may be decomposed into such factors, do not comprise one continued curve, but several at once, each of which may be expressed by a particular equation; and such combinations of separate curves are denoted by the term complex curves.

Thus, the equation $y^2 = ay + xy - ax$, which seems to appertain to a line of the second order, if it be reduced to zero by making $y^2 - ay - xy + ax = 0$, will be composed of the factors $(y - x)(y - a) = 0$; it therefore comprises the

the two equations $y - x = 0$, and $y - a = 0$, both of which belong to the right line: the first forms with the axis at the origin of the abscissas an angle equal to half a right angle; and the second is parallel to the axis, and drawn at a distance $= a$. These two lines, considered together, are comprized in the proposed equation $y^2 = ay + xy - ax$. In like manner we may regard as complex this equation, $y^4 - xy^3 - a^2x^2 - ay^3 + ax^2y + a^2xy = 0$; for its factors being $(y-x)(y-a)(y^2-ax) = 0$, instead of denoting one continued line of the fourth order, it comprizes three distinct lines, viz, two right lines, and one curve denoted by the equa. $y^2 - ax = 0$.

9. We may therefore form at pleasure any complex lines whatever, which shall contain 2 or more right lines or curves. For, if the nature of each line is expressed by an equation referred to the same axis, and to the same origin of the abscissas, and after having reduced each equation to zero, we multiply them one by another, there will result a complex equation which at once comprizes all the lines assumed. For example, if from the centre C, with a radius CA $= a$, a circle be described; and further, if a right line LN be drawn through the centre C; then we may, for any assumed axis, find an equation which will at once include the circle and the right line, as though these two lines formed only one.



Suppose there be taken for an axis the diameter AB, that forms with the right line LN an angle equal to half a right angle: having placed the origin of the abscissas in A, make the abscissa AP $= x$, and the applicate or ordinate PM $= y$; we shall have for the right line, PM $=$ CP $= a - x$; and since the point M of the right line falls on the side of those ordinates which are reckoned negative, we have $y = -a + x$, or $y - x + a = 0$: but, for the circle, we have PM^2 = AP . PB, and BP $= 2a - x$, which gives $y^2 = 2ax - x^2$, or $y^2 + x^2 - 2ax = 0$. Multiplying these two equations together we obtain the complex equation of the third order,

$$y^3 - y^2x + yx^2 - x^3 + ay^2 - 2axy + 3ax^2 - 2a^2x = 0,$$

which represents, at once, the circle and the right line. Hence, we shall find that to the abscissa AP $= x$, corresponds three ordinates, namely, two for the circle, and one for the right line. Let, for example, $x = \frac{1}{2}a$, the equation will become $y^3 + \frac{1}{2}ay^2 - \frac{3}{4}a^2y - \frac{3}{8}a^3 = 0$; whence we first find $y + \frac{1}{2}a = 0$, and by dividing by this root, we obtain $y^2 - \frac{3}{4}a^2 = 0$, the two roots of which being taken and ranked with the former, give the three following values of y:

I. $y =$

$$\text{I. } y = -\tfrac{1}{2}a.$$
$$\text{II. } y = +\tfrac{1}{2}a\sqrt{3}.$$
$$\text{III. } y = -\tfrac{1}{2}a\sqrt{3}.$$

We see, therefore, that the whole is represented by one equation, as if the circle together with the right line formed only one continued curve.

10. This difference between simple and complex curves being once established, it is manifest that the lines of the second order are either continued curves, or complex lines formed of two right lines; for if the general equation have rational factors, they must be of the first order, and consequently will denote right lines. Lines of the third order will be either simple, or complex, formed either of a right line and a line of the second order, or of three right lines. In like manner, lines of the fourth order will be continued and simple, or complex, comprizing a right line and a line of the third order, or two lines of the second order, or lastly, four right lines. Complex lines of the fifth and superior orders will be susceptible of an analogous combination, and of a similar enumeration. Hence it follows, that any order whatever of lines may comprize, at once, all the lines of inferior order, that is to say, that they may contain a complex line of any inferior orders with one or more right lines, or with lines of the second, third, &c, order; so that if we sum the numbers of each order, appertaining to the simple lines, there will result the number indicating the order of the complex line.

Def. 9. That is called an *hyperbolic* leg, or branch of a curve, which approaches constantly to some asymptote; and that a *parabolic* one which has no asymptote.

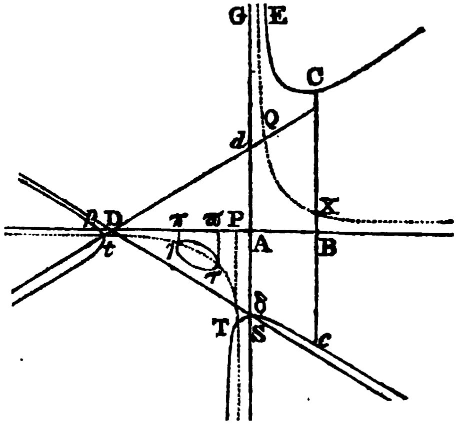
ART. 11. All the legs of curves of the second and higher kinds, as well as of the first, infinitely drawn out, will be of either the hyperbolic or the parabolic kind: and these legs are best known from the tangents. For if the point of contact be at an infinite distance, the tangent of a hyperbolic leg will coincide with the asymptote, and the tangent of a parabolic leg will recede *in infinitum*, will vanish and be no where found. Therefore the asymptote of any leg is found by seeking the tangent to that leg at a point infinitely distant: and the course, or way of an infinite leg, is found by seeking the position of any right line which is parallel to the tangent where the point of contact goes off *in infinitum*: for this right line is directed the same way with the infinite leg.

Sir Isaac Newton's Reduction of all Lines of the Third Order to four Cases of Equations; with the Enumeration of those lines.

CASE I.

CASE I.

12. All the lines of the first, third, fifth, and seventh order, or of any odd order, have at least two legs or sides proceeding on *ad infinitum*, and towards contrary parts. And all lines of the *third* order have two such legs or branches running out contrary ways, and towards which no other of their infinite legs (except in the Cartesian parabola) tend. If the legs are of the *hyperbolic* kind, let GAS be their asymptote; and to it



let the parallel CBc be drawn, terminated (if possible) at both ends at the curve. Let this parallel be bisected in X, and then will the locus of that point X be the conical or common hyperbola XQ, one of whose asymptotes is AS. Let its other asymptote be AB. Then the equation by which the relation between the ordinate BC $= y$, and the abscissa AB $= x$, is determined, will always be of this form: viz.

$$xy^2 + ey = ax^3 + bx^2 + cx + d \ldots \text{(I.)}$$

Here the coefficients e, a, b, c, d, denote given quantities, affected with their signs $+$ and $-$, of which terms any one may be wanting, provided the figure through their defect does not become transformed into a conic section. The conical hyperbola XQ may coincide with its asymptotes, that is, the point X may come to be in the line AB; and then the term $+ ey$ will be wanting.

CASE II.

13. But if the right line CBc cannot be terminated both ways at the curve, but will come to it only in one point; then draw any line in a given position which shall cut the asymptote AS in A; as also any other right line, as BC, parallel to the

the asymptote, and meeting the curve in the point C; then the equation, by which the relation between the ordinate BC and the abscissa AB is determined, will always assume this form: viz. $xy = ax^3 + bx^2 + cx + d$ .... (II.)

CASE III.

14. If the opposite legs be of the parabolic kind, draw the right line CBc, terminated at both ends (if possible) at the curve, and running according to the course of the legs; which line bisect in B: then shall the locus of B be a right line. Let that right line be AB, terminated at any given point, as A: then the equation, by which the relation between the ordinate BC and the abscissa AB is determined, will always be of this form: $y^2 = ax^3 + bx^2 + cx + d$ .... (III.)

CASE IV.

15. If the right line CBc meet the curve only in one point, and therefore cannot be terminated at the curve at both ends; let the point where it comes to the curve be C, and let that right line at the point B, fall on any other right line given in position, as AB, and terminated at any given point, as A. Then will the equation expressing the relation between BC and AB, assume this form:

$$y = ax^3 + bx^2 + cx + d \text{ .... (IV.)}$$

16. In the first case, or that of equation I, if the term ax^3 be affirmative, the figure will be a triple hyperbola with six hyperbolic legs, which will run on infinitely by the three asymptotes, of which none are parallel, two legs towards each asymptote, and towards contrary parts; and these asymptotes, if the term bx^2 be not wanting in the equation, will mutually intersect each other in 3 points, forming thereby the triangle D$d\delta$. But if the term bx^2 be wanting, they will all converge to the same point. This kind of hyperbola is called *redundant*, because it exceeds the conic hyperbola in the number of its hyperbolic legs.

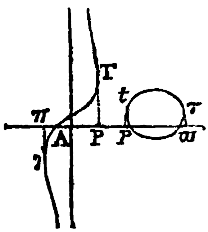
In every redundant hyperbola, if neither the term ey be wanting, nor $b^2 - 4ac = ae\sqrt{a}$, the curve will have *no* diameter; but if either of those occur separately, it will have only *one* diameter; and *three*, if they both happen. Such diameter will always pass through the intersection of two of the asymptotes, and bisect all right lines which are terminated each way by those asymptotes, and which are parallel to the third asymptote.

17. If the redundant hyperbola have no diameter, let the four roots or values of x in the equation $ax^4 + bx^3 + cx^2 + \ldots + \frac{1}{4}e^2 = 0$, be sought; and suppose them to be AP, Aπ.

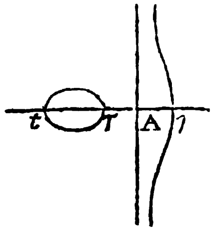
$A\pi$, and Ap (see the preceding figure). Let the ordinates PT, $\varpi\tau$, $\pi 7$, pt, be erected; they shall touch the curve in the points T, τ, 7, t, and by that contact shall give the limits of the curve, by which its species will be discovered.

Thus, if all the roots AP, $A\varpi$, $A\pi$, Ap, be real, and have the same sign, and are unequal, the curve will consist of three hyperbolas and an oval: viz. an *inscribed hyperbola* as EC; a *circumscribed hyperbola*, as T&c; an *ambigeneal hyperbola*, (i. e. lying within one asymptote and beyond another) as *pt*; and an oval $\tau 7$. This is reckoned the *first* species. Other relations of the roots of the equation, give 8 more different species of redundant hyperbolas without diameters; 12 each with but *one* diameter; 2 each with *three* diameters; and 9 each with three asymptotes converging to a common point. Some of these have ovals, some points of decussation, and in some the ovals degenerate into nodes or knots.

18. When the term ax^3 in equa. I, is negative, the figure expressed by that equation, will be a deficient or *defective hyperbola*; that is, it will have fewer legs than the complete conic hyperbola. Such is the marginal figure, representing Newton's 33d species; which is constituted of an *anguineal* or serpentine hyperbola, (both legs approaching a common asymptote by means of a contrary flexure, and a conjugate oval. There are 6 species of defective hyperbolas, each having but one asymptote, and only two hyperbolic legs, running out contrary ways, *ad infinitum*; the asymptote being the first and principal ordinate. When the term ey is not absent, the figure will have no diameter; when it is absent, the figure will have one diameter. Of this latter class there are 7 different species, one of which, namely Newton's 40th species, is exhibited in the margin.

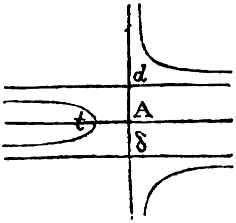


19. If, in equation I, the term ax^3 be wanting, but bx^2 not, the figure expressed by the equation remaining, will be a parabolic hyperbola, having two hyperbolic legs to one asymptote, and two parabolic legs converging one and the same way. When the term ey is not wanting, the figure will have no diameter; if that term be wanting, the figure will have one diameter. There are 7 species appertaining to the former case; and 4 to the latter.



20. When

20. When, in equa, I, the terms ax^3, bx^2, are wanting, or when that equation becomes $xy^2 + ey \mp cx + d$, it expresses a figure consisting of three hyperbolas opposite to one another, one lying between the parallel asymptotes, and the other two without: each of these curves having three asymptotes, one of which is the first and principal ordinate, the other two parallel to the abscissa, and equally distant from it; as in the annexed figure of Newton's 60th species. Otherwise the said equation expresses two opposite circumscribed hyperbolas, and an anguineal hyperbola between the asymptotes. Under this class there are 4 species, called by Newton *Hyperbolismæ of an hyperbola.* By hyperbolismæ of a figure he means to signify when the ordinate comes out, by dividing the rectangle under the ordinate of a given conic section and a given right line, by the common abscissa.

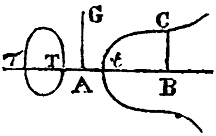


21. When the term cx^2 is negative, the figure expressed by the equation $xy^2 + ey = - cx^2 + d$, is either a serpentine hyperbola, having only one asymptote, being the principal ordinate; or else it is a conchoidal figure. Under this class there are 3 species, called *Hyperbolismæ of an ellipse.*

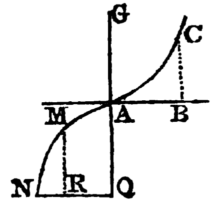
22 When the term cx^2 is absent, the equa. $xy^2 + ey = d$, expresses two hyperbolas, lying, not in the opposite angles of the asymptotes (as in the conic hyperbola), but in the adjacent angles. Here there are only 2 species, one consisting of an inscribed and an ambigeneal hyperbola, the other of two inscribed hyperbolas. These two species are called the *Hyperbolismæ of a parabola.*

23. In the second case of equations, or that of equation II, there is but one figure; which has four infinite legs. Of these, two are hyperbolic about one asymptote, tending towards contrary parts, and two converging parabolic legs, making with the former nearly the figure of a *trident*, the familiar name given to this species. This is the *Cartesian* parabola, by which equations of 6 dimensions are sometimes constructed: it is the 66th species of Newton's enumeration.

24. The third case of equations, or ıa. III, expresses a figure having two ·abolic legs runningout contraryways; these there are 5 different species, led *diverging* or *bell-form* parabolas; which 2 have ovals, 1 is nodate, 1 ·ctate, and 1 cuspidate. The figure shows Newton's 67th species;



species; in which the oval must always be so small that no right line which cuts it twice can cut the parabolic curve ct more than once.



25. In the case to which equa. IV refers, there is but one species. It expresses the *cubical* parabola with contrary legs. This curve may easily be described mechanically by means of a square and an equilateral hyperbola. Its most simple property is, that RM (parallel to AQ) always varies as $QN^3 - QR^3$.

26. Thus according to Newton there are 72 species of lines of the third order. But Mr. Stirling discovered four more species of redundant hyperbolas; and Mr. Stone two more species of deficient hyperbolas, expressed by the equation $xy^2 = bx^2 + cx + d$: i. e. in the case when $bx^2 + cx + d = 0$, has two unequal negative roots, and in that where the equation has two equal negative roots. So that there are *at least* 78 different species of lines of the third order. Indeed Euler, who classes all the varieties of lines of the third order under 16 general species, affirms that they comprehend more than 80 varieties; of which the preceding enumeration necessarily comprizes nearly the whole.

27. Lines of the fourth order are divided by Euler into 146 classes; and these comprize more than 5000 varieties: they all flow from the different relations of the quantities in the 10 general equations subjoined.

$$\left.\begin{array}{ll}
1. & y^4 + fx^2y^2 + gxy^3 + hx^2y + iy^2 + hxy + ly \\
2. & y^4 + fxy^3 + gx^2y + hxy^2 + ixy + ky \;.\;. \\
3. & x^2y^2 + fy^3 + gx^2y + hy^3 + ky \;.\;.\;.\;. \\
4. & x^2y^2 + fy^3 + gy^2 + hxy + iy \;.\;.\;.\;. \\
5. & y^3 + fxy^2 + gx^2y + hy \;.\;.\;.\;.\;.\;. \\
6. & y^3 + fxy^2 + gxy + hy \;.\;.\;.\;.\;.\;.
\end{array}\right\} = ax^4 + bx^3 + cx^2 + dx + e.$$

$$\left.\begin{array}{ll}
7. & y^4 + ex^3y + fxy^3 + gxy^2 + hy^2 + ixy + ky \\
8. & x^3y + exy^3 + fx^2y + gy^2 + hxy + iy \;.\;. \\
9. & x^3y + ey^3 + fxy^2 + gxy + hy \;.\;.\;.\;. \\
10. & x^3y + ey^3 + fy^2 + gxy + hy \;.\;.\;.\;.
\end{array}\right\} = ax^3 + bx^2 + cx + d.$$

28. Lines of the fifth and higher orders, of necessity become still more numerous; and present too many varieties to admit of any classification, at least in moderate compass. Instead, therefore, of dwelling upon these; we shall give a concise sketch of the most curious and important properties of curve lines in general, as they have been deduced from a contemplation of the nature and mutual relation of the roots of the equations representing those curves. Thus a curve being called of n dimensions, or a line of the nth order when its representative equation rises to n dimensions; then since

for

for every different value of x there are n values of y, it will commonly happen that the ordinate will cut the curve in n or in $n - 2$, $n - 4$, &c, points, according as the equation has n, or $n - 2$, $n - 4$, &c, possible roots. It is not however to be inferred, that a right line will cut a curve of n dimensions, in n, $n - 2$, $n - 4$, &c, points, only; for if this were the case, a line of the 2d order, a conic section for instance, could only be cut by a right line in two points;—but this is manifestly incorrect, for though a conic parabola will be cut in two points by a right line oblique to the axis, yet a right line parallel to the axis can only cut the curve in one point.

29. It is true in general, that lines of the n order cannot be cut by a right line in more than n points; but it does not hence follow, that any right line whatever will cut in n points every line of that order; it may happen that the number of intersections is $n - 1$, $n - 2$, $n - 3$, &c, to $n - n$. The number of intersections that any right line whatever makes with a given curve line, cannot therefore determine the order to which a curve line appertains. For, as Euler remarks, if the number of intersections be $= n$, it does not follow that the curve belongs to the n order, but it may be referred to some superior order; indeed it may happen that the curve is not algebraic, but transcendental. This case excepted, however, Euler contends that we may always affirm positively that a curve line which is cut by a right line in n points, cannot belong to an order of lines inferior to n. Thus, when a right line cuts a curve in 4 points, it is certain that the curve does not belong to either the second or third order of lines; but whether it be referred to the fourth, or a superior order, or whether it be transcendental, is not to be decided by analysis.

30. Dr. Waring has carried this enquiry a step further than Euler, and has demonstrated that there are curves of any number of odd orders, that cut a right line in 2, 4, 6, &c, points only; and of any number of even orders that cut a right line in 3, 5, 7, &c, points only; whence this author likewise infers, that the order of the curve cannot be announced from the number of points in which it cuts a right line. See his Proprietates Algebraicarum Curvarum.

31. Every geometrical curve being continued, either returns into itself, or goes on to an infinite distance. And if any plane curve has two infinite branches or legs, they join one another either at a finite, or at an infinite distance.

32. In any curve, if tangents be drawn to all points of the curve; and if they always cut the abscissa at a finite distance from its origin; that curve has an asymptote, otherwise not.

33. A

33. A line of any order may have as many asymptotes as it has dimensions, and no more.

34. An asymptote may intersect the curve in so many points abating two, as the equation of the curve has dimensions. Thus, in a conic section, which is the second order of lines, the asymptote does not cut the curve at all; in the third order it can only cut it in one point; in the fourth order, in two points; and so on.

35. If a curve have as many asymptotes, as it has dimensions, and a right line be drawn to cut them all, the parts of that measured from the asymptotes to the curve, will together be equal to the parts measured in the same direction, from the curve to the asymptotes.

36. If a curve of n dimensions have n asymptotes, then the content of the n abscissas will be to the content of the n ordinates, in the same ratio in the curve and asymptotes; the sum of their n subnormals, to ordinates perpendicular to their abscissas, will be equal to the curve and the asymptotes; and they will have the same central and diametral curves.

37. If two curves of n and m dimensions have a common asymptote; or the terms of the equations to the curves of the greatest dimensions have a common divisor; then the curves cannot intersect each other in $n \times m$ points, possible or impossible. If the two curves have a common general centre, and intersect each other in $n \times m$ points, then the sum of the affirmative abscissas, &c, to those points, will be equal to the sum of the negative; and the sum of the n subnormals to a curve which has a general centre, will be proportional to the distance from that centre.

38. Lines of the third, fifth, seventh, &c order, or any odd number, have, as before remarked, at least two infinite legs or branches, running contrary ways; while in lines of the second, fourth, sixth, or any even number of dimensions, the figure may return into itself, and be contained within certain limits.

39. If the right lines AP, PM, forming a given angle APM, cut a geometrical line of any order in as many points as it has dimensions, the product of the segments of the first terminated by P and the curve, will always be to the product of the segments of the latter, terminated by the same point and the curve, in an invariable ratio.

40. With respect to double, triple, quadruple, and other multiple points, or the points of intersection of 2, 3, 4, or more branches of a curve, their nature and number may be estimated by means of the following principles. 1. A curve of the n order is determinate when it is subjected to pass through the

the number $\frac{(n+1).(n+2)}{2} - 1$ points. 2. A curve of the n order cannot intersect a curve of the m order in more than mn points.

Hence it follows that a curve of the second order, for example, can always pass through 5 given points (not in the same right line), and cannot meet a curve of the m order in more than mn points; and it is impossible that a curve of the m order should have 5 points whose degrees of multiplicity make together more than $2m$ points. Thus, a line of the fourth order cannot have four double points; because the line of the second order which would pass through these four double points, and through a fifth simple point of the curve of the fourth dimension, would meet 9 times; which is impossible, since there can only be an intersection 2×4 or 8 times.

For the same reason, a curve line of the fifth cannot, with one triple point, have more than three double points: and in a similar manner we may reason for curves of higher orders.

Again, for the known proposition, that we can always make a line of the third order pass through nine points, and that a curve of that order cannot meet a curve of the m order in more than $3m$ points, we may conclude that a curve of the m order cannot have nine points, the degrees of multiplicity of which make together a number greater than $3m$. Thus, a line of the fifth order cannot have more than 6 double points; a line of the 6th order, which cannot have more than one quadruple point, cannot have with that quadruple point more than 6 double points; nor with two triple points more than 5 double points; nor even with one triple point more than 7 double points. Analogous conclusions obtain with respect to a line of the fourth order, which we may cause to pass through 14 points, and which can only meet a curve of the m order in $4m$ points, and so on.

41. The properties of curves of a superior order, agree, under certain modifications, with those of all inferior orders. For though some line or lines become evanescent, and others become infinite, some coincide, others become equal; some points coincide, and others are removed to an infinite distance; yet, under these circumstances, the general properties still hold good with regard to the remaining quantities; so that whatever is demonstrated generally of any order, holds true in the inferior orders: and, on the contrary, there is hardly any property of the inferior orders, but there is some similar to it, in the superior ones.

For, as in the conic sections, if two parallel lines are drawn

to

to terminate at the section, the right line that bisects these will bisect all other lines parallel to them; and is therefore called a *diameter* of the figure, and the bisected lines *ordinates*, and the intersections of the diameter with the curve *vertices*; the common intersection of all the diameters the *centre*; and that diameter which is perpendicular to the ordinates, the *vertex*. So likewise in higher curves, if two parallel lines be drawn, each to cut the curve in the number of points that indicate the order of the curve; the right line that cuts these parallels so, that the sum of the parts on one side of the line, estimated to the curve, is equal to the sum of the parts on the other side, it will cut in the same manner all other lines parallel to them that meet the curve in the same number of points; in this case also the divided lines are called *ordinates*, the line so dividing them a *diameter*, the intersection of the diameter and the curve *vertices*; the common intersection of two or more diameters the *centre*; the diameter perpendicular to the ordinates, if there be any such, the *axis*; and when all the diameters concur in one point, that is the *general centre*.

Again, the conic hyperbola, being a line of the second order, has two asymptotes; so likewise, that of the third order may have three; that of the fourth, four; and so on; and they can have no more. And as the parts of any right line between the hyperbola and its asymptotes are equal; so likewise in the third order of lines, if any line be drawn cutting the curve and its asymptotes in three points; the sum of two parts of it falling the same way from the asymptotes to the curve, will be equal to the part falling the contrary way from the third asymptote to the curve; and so of higher curves.

Also, in the conic sections which are not parabolic: as the square of the ordinate, or the rectangle of the parts of it on each side of the diameter, is to the rectangle of the parts of the diameter, terminating at the vertices, in a constant ratio, viz, that of the latus rectum, to the transverse diameter. So in non-parabolic curves of the next superior order, the solid under the three ordinates, is to the solid under the three abscissas, or the distances to the three vertices; in a certain given ratio. In which ratio if there be taken three lines proportional to the three diameters, each to each; then each of these three lines may be called a *latus rectum*, and each of the corresponding diameters a *transverse diameter*. And, in the common, or Apollonian parabola, which has but one vertex for one diameter, the rectangle of the ordinates is equal to the rectangle of the abscissa and latus rectum; so, in those curves of the second kind, or lines of the third kind which

have

have only two vertices to the same diameter, the solid under the three ordinates, is equal to the solid under the two abscissas, and a given line, which may be reckoned the latus rectum.

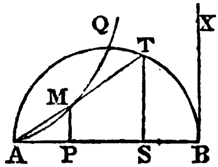
Lastly, since in the conic sections where two parallel lines terminating at the curve both ways, are cut by two other parallels likewise terminated by the curve; we have the rectangle of the parts of one of the first, to the rectangle of the parts of one of the second lines, as the rectangle of the parts of the second of the former, to the rectangle of the parts of the second of the latter pair, passing also through the common point of their division. So, when four such lines are drawn in a curve of the second kind, and each meeting it in three points; the solid under the parts of the first line, will be to that under the parts of the third, as the solid under the parts of the second, to that under the parts of the fourth. And the analogy between curves of different orders may be carried much further: but as enough is given for the objects of this work; we shall now present a few of the most useful problems.

PROBLEM I.

Knowing the Characteristic Property, or the Manner of Description of a Curve, to find its Equation.

This in most cases will be a matter of great simplicity; because the manner of description suggests the relation between the ordinates and their corresponding abscissas; and this relation, when expressed algebraically, is no other than the *equation to the curve*. Examples of this problem have already occurred in sec. 4 of vol. 1: to which the following are now added to exercise the student.

Ex. 1. Find the equation to the cissoid of Diocles; whose manner of description is as below.



From any two points P, S, at equal distances from the extremities A, B, of the diameter of a semicircle, draw ST, PM, perpendicular to AB. From the point T where ST cuts the semicircle, draw a right line AT, it will cut PM in M, a point of the curve required.

Now, by theor. 87 Geom. AS . SB $=$ ST^2; and by the construction, AS . SB $=$ AP . PB. Also the similar triangles APM, AST, give AP : PM :: AS : ST :: PB : ST $= \frac{\text{PM . PB}}{\text{AP}}$. Conse-

quently $ST^2 = \frac{PM^2 \cdot PB^2}{AP^2} = AP.PB$ and lastly $\frac{PM^2.PB^2}{PB} = AP.AP^2$, or $PA^3 = PB \cdot PM^2$. Hence, if the diameter $AB = d$, $AP = x$, $PM = y$; the equation is $x^3 = y^2(d-x)$.

The complete cissoid will have another branch equal and similar to AMQ, but turned contrary ways; being drawn by means of points T′ falling in the other half of the circle. But the same equation will comprehend both branches of the curve; because the square of $-y$, as well as that of $+y$, is positive.

Cor. All cissoids are similar figures; because the abscissæ and ordinates of several cissoids will be in the same ratio, when either of them is in a given ratio to the diameter of its generating circle.

Ex. 2. Find the equation to the logarithmic curve, whose fundamental property is, that when the abscissas increase or decrease in arithmetical progression, the corresponding ordinates increase or decrease in geometrical progression.

Ans. $y = a^x$, a being the number whose logarithm is 1, in the system of logarithms represented by the curve.

Ex. 3. Find the equation to the curve called the *Witch*, whose construction is this: a semicircle whose diameter is AB being given; draw, from any point P in the diameter, a perpendicular ordinate, cutting the semicircle in D, and terminating in M, so that AP : PD : : AB : PM; then is M always a point in the curve.

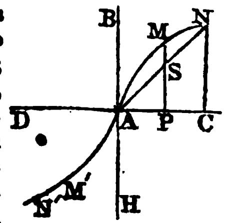
Ans. $y = d\sqrt{\frac{d-x}{x}}$.

PROBLEM II.

Given the Equation to a Curve, to Describe it, and trace its Chief Properties.

The method of effecting this is obvious: for any abscissas being assumed, the corresponding values of the ordinates become known from the equation; and thus the curve may be traced, and its limits and properties developed.

Ex. 1. Let the equation $y^3 = a^2x$, or $y = \sqrt[3]{a^2x}$, to a line of the third order be proposed.



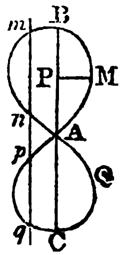
First, drawing the two indefinite lines BH, DC, to make an angle BAC equal to the assumed angle of the co-ordinates; let the values of x be taken upon AC, and those of y upon AB, or upon lines parallel to AB. Then, let it be enquired whether the curve passes through the point A, or not. In order to this, we must ascertain what y will be when $x = 0$:

$x = 0$: and in that case $y = \sqrt[3]{(a^2 \times 0)}$, that is, $y=0$. Therefore the curve passes through A. Let it next be ascertained whether the curve cuts the axis AC in any other point; in order to which, find the value of x when $y = 0$: this will be $\sqrt[3]{a^2 x} = 0$, or $x = 0$. Consequently the curve does not cut the axis in any other point than A. Make $x = \text{AP} = \frac{1}{2}a$, and the given equa. will become $y = \sqrt[3]{\frac{1}{2}a^3} = a\sqrt[3]{\frac{1}{2}}$. Therefore draw PM parallel to AB and equal to $a\sqrt[3]{\frac{1}{2}}$, so will M be a point in the curve. Again, make $x = \text{AC} = a$; then the equation will give $y = \sqrt[3]{a^3} = a$. Hence, drawing CN parallel to AB, and equal to AC or a, N will be another point in the curve. And by assuming other values of y, other ordinates, and consequently other points of the curve, may be obtained. Once more, making x infinite, or $x=\infty$, we shall have $y=\sqrt[3]{(a^2 \times \infty)}$; that is, y is infinite when x is so; and therefore the curve passes on to infinity. And further, since when x is taken $= 0$, it is also $y = 0$, and when $x = \infty$, it is also $y = \infty$; the curve will have no asymptotes that are parallel to the co-ordinates.

Let the right line AN be drawn to cut PM (produced if necessary) in S. Then because CN=AC, it will be PS=AP=$\frac{1}{2}a$. But PM $= a\sqrt[3]{\frac{1}{2}} = \frac{1}{2}a\sqrt[3]{4}$, which is manifestly greater than $\frac{1}{2}a$; so that PM is greater than PS, and consequently the curve is concave to the axis AC.

Now, because in the given equation $y^3 = a^2x$ the exponent of x is *odd*, when x is taken negatively or on the other side of A, its sign should be changed, and the reduced equation will then be $y = \sqrt[3]{-a^2x}$. Here it is evident that, when the values of x are taken in the negative way from A towards D, but equal to those already taken the positive way, there will result as many negative values of y, to fall below AD, and each equal to the corresponding values of y, taken above AC. Hence it follows that the branch AM′N′ will be similar and equal to the branch AMN; but contrarily posited.

Ex. 2. Let the *lemniscate* be proposed, which is a line of the fourth order, denoted by the equation $a^2y^2 = a^2x^2 - x^4$.

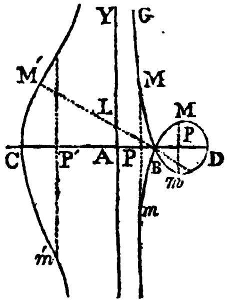


In this equation we have $y = \pm \frac{x}{a}\sqrt{(a^2 - x^2)}$; where, when $x = 0$, $y = 0$, therefore the curve passes through A, the point from which the values of x are measured. When $x = \pm a$, then $y = 0$; therefore the curve passes through B and C, supposing AB and AC each $= \pm a$. If x were assumed greater than a, the value of y would become imaginary; therefore no part of the curve lies beyond B or C. When $x = \frac{1}{2}a$, then

then $y = \frac{1}{2}\sqrt{a^2 - \frac{1}{4}a^2} = \frac{1}{4}a\sqrt{3}$; which is the value of the semi-ordinate PM when AP $= \frac{1}{2}$ AB. And thus, by assuming other values of x, other values of y may be ascertained, and the curve described. It has obviously two equal and similar parts, and a double point at A. A right line may cut this curve in either 2 points, or in 4: even the right line BAC is conceived to cut it in 4 points; because the double point A is that in which two branches of the curve, viz, MAp, and nAQ, are intersected.

Ex. 3. Let there be proposed the *Conchoid* of the ancients, which is a line of the fourth order defined by the equation

$$(a^2 - x^2 \cdot (x-b)^2 = x^2y^2, \text{ or } y = \pm \frac{x-b}{x}\sqrt{(a^2 - x^2)}.$$



Here, if $x = 0$, then y becomes infinite; and therefore the ordinate at A (the origin of the abscissas) is an asymptote to the curve. If AB $= b$, and P be taken between A and B, then shall PM and pm be equal, and lie on different sides of the abscissa AP. If $x = b$, then the two values of y vanish, because $x - b = 0$, and consequently the curve passes through B, having there a double point. If AP be taken greater than AB, then will there be two values of y, as before, having contrary signs; that value which was positive before being now negative, and *vice versa*. But if AD be taken $= a$, and P comes to D, then the two values of y vanish, because in that case $\sqrt{(a^2 - x^2)} = 0$. If AP be taken greater than AD or a, then $a^2 - x^2$ becomes negative, and the value of y impossible: so that the curve does not go beyond D.

Now let x be considered as negative, or as lying on the side of A towards C. Then $y = \pm \frac{x+b}{x}\sqrt{(a^2 - x^2)}$. Here if x vanish, both these values of y become infinite; and consequently the curve has two indefinite arcs on each side the asymptote or directrix AY. If x increase, y manifestly diminishes; and when $x = a$, then y vanishes: that is, if AC $=$ AD, then one branch of the curve passes through C, while the other passes through D. Here also, if x be taken greater than a, y becomes imaginary; so that no part of the curve can be found beyond C.

If $a = b$, the curve will have a cusp in B, the node between B and D vanishing in that case. If a be less than b, then B will become a conjugate point.

In

In the figure, M'cm' represents what is termed the *superior conchoid*, and GBmDMBm the *inferior conchoid*. The point B is called the *pole* of the conchoid; and the curve may be readily constructed by radial lines from this point, by means of the polar equation $z = \frac{b}{\cos. \varphi} \pm a$. It will merely be requisite to set off from any assumed point A, the distance AB $= b$; then to draw through B a right line mLM' making any angle φ with CB, and from L, the point, where this line cuts the directrix AY (drawn perpendicular to CB) set off upon it LM' $=$ Lm $= a$; so shall M' and m be points in the superior and inferior conchoids respectively.

Ex. 4. Let the principal properties of the curve whose equation is $yx^n = a^{n+1}$, be sought; when n is an odd number, and when n is an even number.

Ex. 5. Describe the line which is defined by the equation $xy + ay + cy = bc + bx$.

Ex. 6. Let the Cardioide, whose equation is $y^4 - 6ay^3 + (2x^2 + 12a^2)y^2 - (6ax^2 + 8a^3)y + (x^2 + 3a^2)x^2 = 0$, be proposed.

Ex. 7. Let the Trident, whose equation is $xy = ax^3 + bx^2 + cx + d$, be proposed.

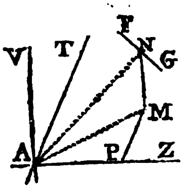
Ex. 8. Ascertain whether the *Cissoid* and the *Witch*, whose equations are found in the preceding problem, have asymptotes.

PROBLEM III.

To determine the Equation to any proposed Curve Surface.

Here the required equation must be deduced from the law or manner of construction of the proposed surface, the reference being to *three* co-ordinates, commonly rectangular ones, the variable quantities being x, y, and z. Of these, two, namely x and y, will be found in one plane, and the third z will always mark the distance from that plane.

Ex. 1. Let the proposed surface be that of a sphere, FNG.



The position of the fixed point A, which is the origin of the co-ordinates PM, MN, being arbitrary; let it be [illegible]posed, for the greater convenience, [illegible]t it is at the centre of the sphere. [illegible] MA, NA, be drawn, of which the [illegible]er is manifestly equal to the radius [illegible] the sphere, and may be denoted by r. Then, if AP $= x$ $= y$, MN $= z$; the right-angled triangle APM will give AM²

$AM^2 = AP^2 + PM^2 = x^2 + y^2$. In like manner, the right-angled triangle AMN, posited in a plane perpendicular to the former, will give $AN^2 = AM^2 + MN^2$, that is, $r^2 = x^2 + y^2 + z^2$; or $z^2 = r^2 - x^2 - y^2$, the equation to the spherical surface, as required.

Scholium. Curve surfaces, as well as plane curves, are arranged in orders according to the dimensions of the equations, by which they are represented. And, in order to determine the properties of curve surfaces, processes must be employed, similar to those adopted when investigating the properties of plane curves. Thus, in like manner as in the theory of curve lines, the supposition that the ordinate y is equal to 0, gives the point or points where the curve cuts its axis; so, with regard to curve surfaces, the supposition of $z = 0$, will give the equation of the curve made by the intersection of the surface and its base, or the plane of the co-ordinates x, y. Hence, in the equation to the spherical surface, when $z = 0$, we have $x^2 + y^2 = r^2$, which is that of a circle whose radius is equal to that of the sphere. See p. 534 vol. 1.

Ex. 2. Let the curve surface proposed be that produced by a parabola turning about its axis.

Here the abscissas x being reckoned from the vertex or summit of the axis, and on a plane passing through that axis; the two other co-ordinates being, as before, y and z; and the parameter of the generating parabola being p: the equation of the parabolic surface will be found to be $z^2 + y^2 - px = 0$.

Now, in this equation, if z be supposed $= 0$, we shall have $y^2 = px$, which (pa. 534 vol. 1) is the equation to the generating parabola, as it ought to be. If we wished to know what would be the curve resulting from a section parallel to that which coincides with the axis, and at the distance a from it, we must put $z = a$; this would give $y^2 = px - a^2$, which is still an equation to a parabola, but in which the origin of the abscissas is distant from the vertex before assumed by the quantity $\frac{a^2}{p}$.

Ex. 3. Suppose the curve surface of a right cone were proposed.

Here we may most conveniently refer the equation of the surface to the plane of the circular base of the cone. In this case, the perpendicular distance of any point in the surface from the base, will be to the axis of the cone, as the distance of the foot of that perpendicular from the circumference (measured

(measured on a radius), to the radius of the base: that is, if the values of x be estimated from the centre of the base, and r be the radius, z will vary as $r - \sqrt{(x^2+y^2)}$. Consequently, the simplest equation of the conic surface, will be $z \backsim r = - \sqrt{(x^2+y^2)}$, or $r^2 - 2rz + z^2 = x^2 + y^2$.

Now from this the nature of curves formed by planes cutting the cone in different directions, may readily be inferred. Let it be supposed, first, that the cutting plane is inclined to the base of a right-angled cone in the angle of 45°, and passes through its centre: then will $z = x$, and this value of z substituted for it in the equation of the surface, will give $r^2 - 2rx = y^2$, which is the equation of the projection of the curve on the plane of the cone's base: and this (art. 3 of this chap.) is manifestly an equation to a *parabola*.

Or, taking the thing more generally, let it be supposed that the cutting plane is so situated, that the ratio of x to z shall be that of 1 to m: then will $mx = z$, and $m^2x^2 = z^2$. These substituted for z and z^2 in the equation of the surface, will give, for the equation of the projection of the section on the plane of the base, $r^2 - 2mx + (m^2 - 1)x^2 = y^2$. Now this equation, if m be greater than unity, or if the cutting plane pass between the vertex of the cone and the parabolic section, will be that of an *hyperbola*: and if, on the contrary, the cutting plane pass between the parabola and the base, i. e. if m be less than unity, the term $(m^2 - 1)x^2$ will be negative, when the equation will obviously designate an *ellipse*.

Schol. It might here be demonstrated, in a nearly similar manner, that every surface formed by the rotation of any conic section on one of its axes, being cut by any plane whatever, will always give a conic section. For the equation of such surface will not contain any power of x, y, or z, greater than the second; and therefore the substitution of any values of z in terms of x or of y, will never produce any powers of x or of y exceeding the square. The section therefore must be a line of the second order. See, on this subject, Hutton's Mensuration, part iii, sect. 4.

Ex. 3. Let the equation to the curve surface be $xyz = a^3$.

Then will the curve surface bear the same relation to the *solid* right angle, which the curve line whose equation is $xy = a^2$ bears to the *plane* right angle. That is, the curve surface will be posited between the three rectangular faces bounding such solid right angle, in the same manner as the equilateral hyperbola is posited between its rectangular asymptotes. And in like manner as there may be 4 equal equilateral

teral hyperbolas comprehended between the same rectangular asymptotes, when produced both ways from the angular point; so there may be 6 equal hyperboloids posited within the 6 solid right angles which meet at the same summit, and all placed between the same three asymptotic planes.

SECTION II.

On the Construction of Equations.

PROBLEM I.

To Construct Simple Equations, Geometrically.

Here the sole art consists in resolving the fractions, to which the unknown quantity is equal, into proportional terms; and then constructing the respective proportions, by means of probs. 8, 9, 10, and 27 Geometry. A few simple examples will render the method obvious.

1. Let $x = \frac{ab}{c}$; then $c : a :: b : x$. Whence x may be found by constructing according to prob. 9 Geometry.

2. Let $x = \frac{abc}{de}$. First construct the proportion $d : a :: b : \frac{ab}{d}$, which 4th term call g; then $x = \frac{gc}{e}$; or $e : c :: g : x$.

3. Let $x = \frac{a^2 - b^2}{c}$. Then, since $a^2 - b^2 = (a+b) \times (a-b)$; it will merely be necessary to construct the proportion $c : a + b :: a - b : x$.

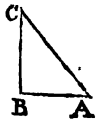
4. Let $x = \frac{a^2b - bc^2}{ad}$. Find, as in the first case, $g = \frac{ab}{d} = \frac{a^2b}{ad}$, and $h = \frac{bc}{d}$, so that $\frac{bc^2}{ad}$ may $= \frac{hc}{a}$. Then find by the first case $i = \frac{hc}{a}$. So shall $x = g - i$, the difference of those lines, found by construction.

5. Let $x = \frac{a^2b - bad}{af + bc}$. First find $\frac{af}{b}$, the fourth proportional to b, a and f, which make $= h$. Then $x = \frac{a(a-d)}{h+c}$; or, by construction it will be $h + c : a - d :: a : x$.

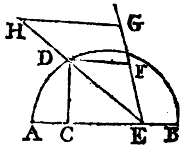
6. Let $x = \frac{a^2 + b^2}{c}$. Make the right-angled triangle ABC such

that

that the leg AB $= a$, BC $= b$; then AC $= \surd(\text{AB}^2 + \text{BC}^2) = \surd(a^2 + b^2)$, by th. 34 Geom. Hence $x = \frac{\text{AC}^2}{c}$. Construct therefore the proportion $c : \text{AC} :: \text{AC} : x$, and the unknown quantity will be found, as required.



7. Let $x = \frac{a^2 + cd}{h + c}$. First, find CD a mean proportional between AC $= c$, and CB $= d$, that is, find CD $= \surd\, cd$. Then make CE $= a$, and join DE, which will evidently be $= \surd(a^2 + cd)$. Next on any line EG set off EF $= h + c$, EG $=$ ED; and draw GH parallel to FD, to meet DE (produced if need be) in H. So shall EH be $= x$, the third proportional to $h + c$, and $\surd(a^2 + cd)$, as required.

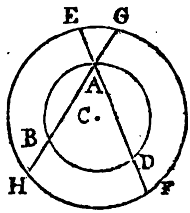


Note. Other methods suitable to different cases which may arise are left to the student's invention. And in all constructions the accuracy of the results, will increase with the size of the diagrams; within convenient limits for operation.

PROBLEM II.

To Find the Roots of Quadratic Equations by Construction.

In most of the methods commonly given for the construction of quadratics, it is required to set off the square root of the last term; an operation which can only be performed accurately when that term is a rational square. We shall here describe a method which, at the same time that it is very simple in practice, has the advantage of showing clearly the relations of the roots, and of dividing the third term into two factors, one of which at least may be a whole number.



In order to this construction, all quadratics may be classed under 4 forms: viz,

1. $x^2 + ax - bc = 0$.
2. $x^2 - ax - bc = 0$.
3. $x^2 + ax + bc = 0$.
4. $x^2 - ax + bc = 0$.

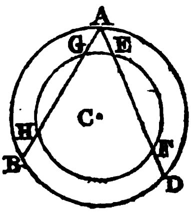
1. One general mode of construction will include the first two of these forms. Let $x^2 \mp ax - bc = 0$, and b greater than c. Describe any circle ABD having its diameter not less than the given quantities a and $b - c$, and within this circle

inscribe

inscribe two chords AB $= a$, AD $= b - c$, both from any common assumed point A. Then, produce AD to F so that DF $= c$, and about the centre C of the former circle, with the radius CF, describe another circle, cutting the chords AD, AB, produced, in F, E, G, H: so shall AG be the *affirmative* and AH the *negative root* of the equation $x^2 + ax - bc = 0$; and contrariwise AG will be the *negative* and AH the *affirmative* root of the equation $x^2 - ax - bc = 0$.

For, AF or AD + DF $= b$, and DF or AE $= c$; and, making AG or BH $= x$, we shall have AH $= a + x$: and by the property of the circle EGFH (theor. 61 Geom.) the rectangle EA . AF = GA . AH, or $bc = (a + x)x$, or again by transposition $x^2 + ax - bc = 0$. Also if AH be $= -x$, we shall have AG or BH or AH — AB $= -x - a$: and conseq. GA . AH $= x^2 + ax$, as before. So that, whether AG be $= x$, or AH $= -x$, we shall always have $x^2 + ax - bc = 0$. And by an exactly similar process it may be proved that AG is the negative, and AH the positive root of $x^2 - ax - bc = 0$.

Cor. In quadratics of the form $x^2 + ax - bc = 0$, the positive root is always *less* than the negative root; and in those of the form $x^2 - ax - bc = 0$, the positive root is always *greater* than the negative one.



2. The third and fourth cases also are comprehended under one method of construction, with two concentric circles. Let $x^2 \mp ax + bc = 0$. Here describe any circle ABD, whose diameter is not less than either of the given quantities a and $b + c$; and within that circle inscribe two chords AB $= a$, AD $= b + c$, both from the same point A. Then in AD assume DF $= c$, and about C the centre of the circle ABD, with the radius CF describe a circle, cutting the chords AD, AB, in the points F, E, G, H: so shall AG, AH, be the two *positive* roots of the equation $x^2 - ax + bc = 0$, and the two *negative* roots of the equation $x^2 + ax + bc = 0$. The demonstration of this also is similar to that of the first case.

Cor. 1. If the circle whose radius is CF just touches the chord AB, the quadratic will have two equal roots; which can only happen when $\frac{1}{4}a^2 = bc$.

Cor. 2. If that circle neither cut nor touch the chord AB, the roots of the equation will be imaginary; and this will always happen, in these two forms, when bc is greater than $\frac{1}{4}a^2$.

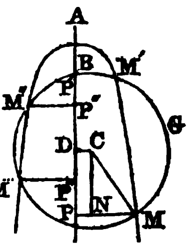
PROBLEM

PROBLEM III.

To Find the Roots of Cubic and Biquadratic Equations, by Construction.

1. In finding the roots of any equation, containing only one unknown quantity, by construction, the contrivance consists chiefly in bringing a new unknown quantity into that equation; so that various equations may be had, each containing the two unknown quantities; and further, such that any two of them contain *together* all the known quantities of the proposed equation. Then from among these equations two of the most simple are selected, and their corresponding loci constructed; the intersection of those loci will give the roots sought.

Thus it will be found that cubics may be constructed by two parabolas, or by a circle and a parabola, or by a circle and an equilateral hyperbola, or by a circle and an ellipse, &c: and biquadratics by a circle and a parabola, or by a circle and an ellipse, or by a circle and an hyperbola, &c. Now, since a parabola of given parameter may be easily constructed by the rule in cor. 2 th. 4 Parabola, we select the circle and the parabola, for the construction of both biquadratic and cubic equations. The general method applicable to both, will be evident from the following description.



2. Let M'' AM'M be a parabola whose axis is AP, M'' M'GM a circle whose centre is C and radius CM, cutting the parabola in the points M, M', M'', M''': from these points draw the ordinates to the axis MP, M'P', M''P'', M'''P'''; and from C let fall CD perpendicularly to the axis; also draw CN parallel to the axis, meeting PM in N. Let AD $= a$, DC $= b$, CM $= n$, the parameter of the parabola $= p$, AP $= x$, PM $= y$. Then (pa. 534 vol. 1) $px = y^2$: also $CM^2 = CN^2 + NM^2$, or $n^2 = (x \mp a)^2 + (y \mp b)^2$; that is, $x^2 \pm 2ax + a^2 + y^2 \pm 2by + b^2 = n^2$. Substituting in this equation for x, its value $\frac{y^2}{p}$, and arranging the terms according to the dimensions of y, there will arise

$$y^4 \pm (2pa + p^2)y^2 \pm 2bp^2y + (a^2 + b^2 - n^2)p^2 = 0,$$

a biquadratic equation, whose roots will be expressed by the ordinates PM, P'M', P''M'', P'''M''', at the points of intersection of the given parabola and circle.

3. To make this coincide with any proposed biquadratic whose second term is taken away (by cor. theor. 3); assume

$y^4 -$

$y^4 - qy^2 + ry - s = 0$. Assume also $p = 1$; then comparing the terms of the two equations, it will be, $2a - 1 = q$, or $a = \frac{q+1}{2}$, $-2b = r$, or $b = \frac{-r}{2}$; $a^2 + b^2 - n^2 = -s$, or $n^2 = a^2 + b^2 + s$, and consequently $n = \sqrt{(a^2 + b^2 + s)}$. Therefore describe a parabola whose parameter is 1, and in the axis take AD $= \frac{q+1}{2}$: at right angles to it draw DC and $= -\frac{1}{2}r$; from the centre C, with the radius $\sqrt{(a^2 + b^2 + s)}$, describe the circle M″M′GM, cutting the parabola in the points M, M′, M″, M‴; then the ordinates PM, P′M′, P″M″, P‴M‴, will be the roots required.

Note. This method, of making $p = 1$, has the obvious advantage of requiring only one parabola for any number of biquadratics, the necessary variation being made in the radius of the circle.

Cor. 1. When DC represents a negative quantity, the ordinates on the same side of the axis with C represent the negative roots of the equation; and the contrary.

Cor. 2. If the circle *touch* the parabola, two roots of the equation are equal; if it cut it only in two points, or touch it in one, two roots are impossible; and if the circle fall wholly within the parabola, all the roots are impossible.

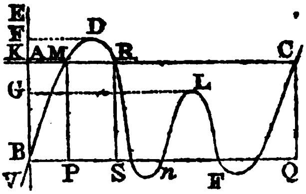
Cor. 3. If $a^2 + b^2 = n^2$, or the circle pass through the point A, the last term of the equation, i.e. $(a^2 + b^2 - n^2)p^2 = 0$; and therefore $y^4 \pm (2pa + p^2)y^2 \pm 2bp^2y = 0$, or $y^3 \pm (2pa + p^2)y \pm 2bp^2 = 0$. This cubic equation may be made to coincide with any proposed cubic, wanting its second term, and the ordinates PM, P″M″, P‴M‴, are its roots.

Thus if the cubic be expressed generally by $y^3 \pm qy \pm s = 0$. By comparing the terms of this and the preceding equation, we shall have $\pm 2pa + p^2 = \pm q$, and $\pm 2bp^2 = \pm s$, or $\mp a = \frac{1}{2}p \mp \frac{q}{2p}$, and $b = \pm \frac{s}{2p^2}$. So that, to construct a cubic equation, with any *given* parabola, whose half parameter is AB (see the preceding figure): from the point B take in the axis, (forward if the equation have $-q$, but backward if q be positive) the line BD $= \frac{q}{2p}$; then raise the perpendicular DC $= \frac{s}{2p^2}$, and from C describe a circle passing through the vertex A of the parabola; the ordinates PM, &c, drawn from the points of intersection of the circle and parabola, will be the roots required.

PROBLEM

PROBLEM IV.

To Construct an Equation of any Order by means of a Locus of the same Degree as the Equation proposed, and a Right Line.



As the general method is the same in all equations, let it be one of the 5th degree, as $x^5 - bx^4 + acx^3 - a^2dx^2 + a^3ex - a^4f = 0$. Let the last term a^4f be transposed; and, taking one of the linear divisors, f, of the last term, make it equal to z, for example, and divide the equation by a^4; then will $z = \frac{x^5 - bx^4 + acx^3 - a^2dx^4 + a^3ex}{a^4}$.

On the indefinite line BQ describe the curve of this equation, BMDRLFC, by the method taught in prob. 2, sect. 1, of this chapter, taking the values of x from the fixed point B. The ordinates PM, SR, &c, will be equal to z; and therefore, from the point B draw the right line BA $= f$, parallel to the ordinates PM, SR, and through the point A draw the indefinite right line KC both ways, and parallel to BQ. From the points in which it cuts the curve, let fall the perpendiculars MP, RS, CQ; they will determine the abscissas BP, BS, BQ, which are the roots of the equation proposed. Those from A towards Q are positive, and those lying the contrary way are negative.

If the right line AC touch the curve in any point, the corresponding abscissa x will denote two equal roots; and if it do not meet the curve at all, all the roots will be imaginary.

If the sign of the last term, a^4f, had been positive, then we must have made $z = -f$, and therefore must have taken BA $= -f$, that is, below the point P, or on the negative side.

EXERCISES.

Ex. 1. Let it be proposed to divide a given arc of a circle into three equal parts.

Suppose the radius of the circle to be represented by r, the sine of the given arc by a, the unknown sine of its third part by x, and let the known arc be $3u$, and of course, the required arc be u. Then, by equa. VIII, IX, chap. iii, we shall have

$$\sin 3u = \sin(2u + u) = \frac{\sin 2u \,.\, \cos u + \cos 2u \,.\, \sin u}{r},$$

$$\sin 2u = \sin(u + u) = \frac{2 \sin u \,.\, \cos u}{r},$$

$$\cos 2u = \cos(u + u) = \frac{\cos^2 u - \sin^2 u}{r}.$$

Putting

Putting, in the first of these equations, for sin $3u$ its given value a, and for sin $2u$, cos $2u$, their values given in the two other equations, there will arise

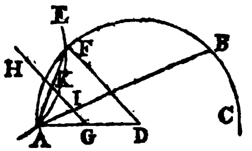
$$a = \frac{3 \sin u \,.\, \cos^2 u \,.\, \sin^3 u}{r}.$$

Then substituting for sin u its value x, and for $\cos^2 u$ its value $r^2 - x^2$, and arranging all the terms according to the powers of x, we shall have

$$x^3 - \tfrac{3}{4}r^2x + \tfrac{1}{4}\,ar^2 = 0,$$

a cubic equation of the form $x^3 - px + q = 0$, with the condition that $\frac{1}{27}p^3 > \frac{1}{4}q^2$; that is to say, it is a cubic equation falling under the irreducible case, and its three roots are represented by the sines of the three arcs u, $u + 120°$, and $u + 240°$.

Now, this cubic may evidently be constructed by the rule in prob. 3 cor. 3. But the trisection of an arc may also be effected by means of an equilateral hyperbola, in the following manner.



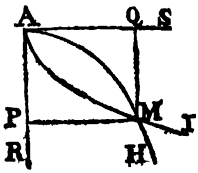
Let the arc to be trisected be AB. In the circle ABC draw the semidiameter AD, and to AD as a diameter, and to the vertex A, draw the equilateral hyperbola AE to which the right line AB (the chord of the arc to be trisected) shall be a tangent in the point A; then the arc AF, included within this hyperbola, is one third of the arc AB.

For, draw the chord of the arc AF, bisect AD at G, so that G will be the centre of the hyperbola, join DF, and draw GH parallel to it, cutting the chords AB, AF, in I and K. Then, the hyperbola being equilateral, or having its transverse and conjugate equal to one another, it follows from Def. 16 Conic Sections, that every diameter is equal to its parameter, and from cor. theor. 2 Hyperbola, that GK . KI $=$ AK^2, or that GK : AK : : AK : KI; therefore the triangles GKA, AKI are similar, and the angle KAI $=$ AGK, which is manifestly $=$ ADF. Now the angle ADF at the centre of the circle being equal to KAI or FAB; and the former angle at the centre being measured by the arc AF, while the latter at the circumference is measured by half FB; it follows that AF $= \frac{1}{2}$FB, or $= \frac{1}{3}$AB, as it ought to be.

Ex. 2. Given the side of a cube, to find the side of another of double capacity.

Let the side of the given cube be a, and that of a double one y, then $2a^3 = y^3$, or, by putting $2a = b$, it will be $a^2b = y^3$; there are therefore to be found two mean proportionals between

tween the side of the cube and twice that side, and the first of those mean proportionals will be the side of the double cube. Now these may be readily found by means of two parabolas; thus:



Let the right lines AR, AS, be joined at right angles; and a parabola AMH be described about the axis AR, with the parameter a; and another parabola AMI about the axis AS, with the parameter b: cutting the former in M. Then AP $= x$, PM $= y$, are the two mean proportionals, of which y is the side of the double cube required.

For, in the parabola AMH the equation is $y^2 = ax$, and in the parabola AMI it is $x^2 = by$. Consequently $a : y :: y : x$, and $y : x :: x : b$. Whence $yx = ab$; or, by substitution, $y \sqrt{by} = ab$, or, by squaring $y^3 b = a^2 b^2$; or lastly, $y^3 = a^2 b = 2a^3$, as it ought to be.

THE

THE DOCTRINE OF FLUXIONS.

DEFINITIONS AND PRINCIPLES.

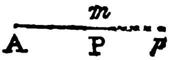
Art. 1. In the Doctrine of Fluxions, magnitudes or quantities of all kinds are considered, not as made up of a number of small parts, but as generated by continued motion, by means of which they increase or decrease. As, a line by the motion of a point; a surface by the motion of a line; and a solid by the motion of a surface. So likewise, time may be considered as represented by a line, increasing uniformly by the motion of a point. And quantities of all kinds whatever, which are capable of increase and decrease, may in like manner be represented by geometrical magnitudes, conceived to be generated by motion.

2. Any quantity thus generated, and variable, is called a Fluent, or a Flowing Quantity. And the rate or proportion according to which any flowing quantity increases, at any position or instant, is the Fluxion of the said quantity, at that position or instant: and it is proportional to the magnitude by which the flowing quantity would be uniformly increased in a given time, with the generating celerity uniformly continued during that time.

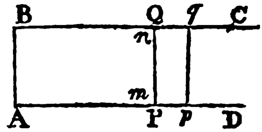
3. The small quantities that are actually generated, produced, or described, in any small given time, and by any continued motion, either uniform or variable, are called Increments.

4. Hence, if the motion of increase be uniform, by which increments are generated, the increments will in that case be proportional, or equal, to the measures of the fluxions: but if the motion of increase be accelerated, the increment so generated, in a given finite time, will exceed the fluxion: and if it be a decreasing motion, the increment, so generated, will be less than the fluxion. But if the time be indefinitely small, so that the motion be considered as uniform for that instant; then these nascent increments will always be proportional, or equal, to the fluxions, and may be substituted instead of them, in any calculation.

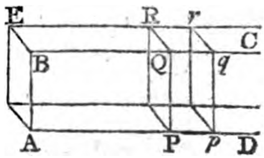
5. To



5. To illustrate these definitions: Suppose a point m be conceived to move from the position A, and to generate a line AP, by a motion any how regulated; and suppose the celerity of the point m, at any position P, to be such, as would, if from thence it should become or continue uniform, be sufficient to cause the point to describe, or pass uniformly over, the distance Pp, in the given time allowed for the fluxion: then will the said line Pp represent the fluxion of the fluent, or flowing line, AP, at that position.



6. Again, suppose the right line mn to move, from the position AB, continually parallel to itself, with any continued motion, so as to generate the fluent or flowing rectangle ABQP, while the point m describes the line AP: also, let the distance Pp be taken, as before, to express the fluxion of the line or base AP; and complete the rectangle PQqp. Then, like as Pp is the fluxion of the line AP, so is Pq the fluxion of the flowing parallelogram AQ; both these fluxions, or increments, being uniformly described in the same time.

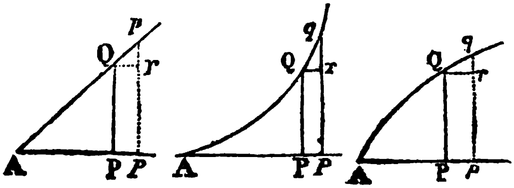


7. In like manner, if the solid AERP be conceived to be generated by the plane PQR, moving from the position ABE, always parallel to itself, along the line AD; and if Pp denote the fluxion of the line AP: Then, like as the rectangle PQqp, or PQ × Pp, denotes the fluxion of the flowing rectangle ABQP, so also shall the fluxion of the variable solid, or prism ABERQP, be denoted by the prism PQRrqp, or the plane PR × Pp. And, in both these last two cases, it appears that the fluxion of the generated rectangle, or prism, is equal to the product of the generating line, or plane, drawn into the fluxion of the line along which it moves.

8. Hitherto the generating line, or plane, has been considered as of a constant and invariable magnitude; in which case the fluent, or quantity generated, is a rectangle, or a prism, the fomer being described by the motion of a line, and the latter by the motion of a plane. So, in like manner are other figures, whether plane or solid, conceived to be de-

 scribed

scribed by the motion of a Variable Magnitude, whether it be a line or a plane. Thus, let a variable line PQ be carried by a parallel motion along AP; or while a point P is carried along, and describes the line AP, suppose another point



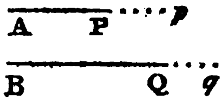
Q to be carried by a motion perpendicular to the former, and to describe the line PQ: let pq be another position of PQ, indefinitely near to the former; and draw Qr parallel to AP. Now in this case there are several fluents, or flowing quantities, with their respective fluxions: namely, the line or fluent AP, the fluxion of which is Pp or Qr; the line or fluent PQ, the fluxion of which is rq; the curve or oblique line AQ, described by the oblique motion of the point Q, the fluxion of which is Qq; and lastly, the surface APQ, described by the variable line PQ, the fluxion of which is the rectangle PQrp, or PQ $\times$ Pp. In the same manner may any solid be conceived to be described, by the motion of a variable plane parallel to itself, substituting the variable plane for the variable line; in which case the fluxion of the solid, at any position, is represented by the variable plane, at that position, drawn into the fluxion of the line along which it is carried.

9. Hence then it follows in general, that the fluxion of any figure, whether plane or solid, at any position, is equal to the section of it, at that position, drawn into the fluxion of the axis, or line along which the variable section is supposed to be perpendicularly carried; that is, the fluxion of the figure AQP, is equal to the plane PQ $\times$ Pp, when that figure is a solid, or to the ordinate PQ $\times$ Pp, when the figure is a surface.

10. It also follows from the same premises, that in any curve, or oblique line AQ, whose absciss is AP, and ordinate is PQ, the fluxions of these three form a small right-angled plane triangle Qqr; for Qr = Pp is the fluxion of the absciss AP, qr the fluxion of the ordinate PQ, and Qq the fluxion of the curve or right line AQ. And consequently that, in any curve, the square of the fluxion of the curve, is equal to the sum

sum of the squares of the fluxions of the absciss and ordinate, when these two are at right angles to each other.

11. From the premises it also appears, that contemporaneous fluents, or quantities that flow or increase together, which are always in a constant ratio to each other, have their fluxions also in the same constant ratio, at every position. For, let AP and BQ be two contemporaneous fluents, described in the same time by the motion of the points P and Q, the contemporaneous positions being P, Q, and p, q; and let AP be to BQ, or Ap to Bq, constantly in the ratio of 1 to n.



Then - - - - - - is $n \times$ AP $=$ BQ,
and $n \times$ Ap $=$ Bq;
therefore, by subtraction, $n \times$ Pp $=$ Qq;
that is, the fluxion - Pp : fluxion Qq : : 1 : n,
the same as the fluent AP : fluent BQ : : 1 : n,
or, the fluxions and fluents are in the same constant ratio.

But if the ratio of the fluents be variable, so will that of the fluxions be also, though not in the same variable ratio with the former, at every position.

NOTATION, &c.

12. To apply the foregoing principles to the determination of the fluxions of algebraic quantities, by means of which those of all other kinds are assigned, it will be necessary first to premise the notation commonly used in this science, with some observations. As, first, that the final letters of the alphabet z, y, x, u, &c, are used to denote variable or flowing quantities; and the initial letters a, b, c, d, &c, to denote constant or invariable ones: Thus, the variable base AP of the flowing rectangular figure ABQP, in art. 6, may be represented by x; and the invariable altitude PQ, by a: also, the variable base or absciss AP, of the figures in art. 8, may be represented by x, the variable ordinate PQ, by y; and the variable curve or line AQ, by z.

Secondly, that the fluxion of a quantity denoted by a single letter, is represented by the same letter with a point over it: Thus, the fluxion of x is expressed by $\dot{x}$, the fluxion of y by $\dot{y}$, and the fluxion of z by $\dot{z}$. As to the fluxions of constant or invariable quantities, as of a, b, c, &c, they are equal to nothing, because they do not flow or change their magnitude.

Thirdly,

Thirdly, that the increments of variable or flowing quantities, are also denoted by the same letters with a small ′ over them: Thus, the increments of x, y, z, are x', y', z'.

13. From these notations, and the foregoing principles, the quantities and their fluxions, there considered, will be denoted as below. Thus, in all the foregoing figures, put

the variable or flowing line - - AP $= x$,
in art 6, the constant line - - PQ $= a$,
in art. 8, the variable ordinate - PQ $= y$,
also, the variable line or curve - AQ $= z$:

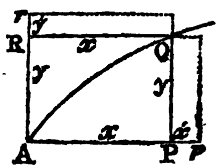
Then shall the several fluxions be thus represented, namely,

$\dot{x} =$ Pp the fluxion of the line AP,
$a\dot{x} =$ PQqp the fluxion of ABQP in art. 6,
$y\dot{x} =$ PQrp the fluxion of APQ in art. 8,
$\dot{z} =$ Qq $= \sqrt{(\dot{x}^2 + \dot{y}^2)}$ the fluxion of AQ; and
$a\dot{x} =$ Pr the fluxion of the solid in art. 7, if a denote the constant generating plane PQR; also,
$nx =$ BQ in the figure to art. 11, and
$n\dot{x} =$ Qq the fluxion of the same.

14. The principles and notation being now laid down, we may proceed to the practice and rules of this doctrine; which consists of two principal parts, called the Direct and Inverse Method of Fluxions; namely, the direct method, which consists in finding the fluxion of any proposed fluent or flowing quantity; and the inverse method, which consists in finding the fluent of any proposed fluxion. As to the former of these two problems, it can always be determined, and that in finite algebraic terms; but the latter, or finding of fluents, can only be effected in some certain cases, except by means of infinite series.—First then, of

THE DIRECT METHOD OF FLUXIONS.

To find the Fluxion of the Product or Rectangle of two Variable Quantities.



15. Let ARQP, $= xy$, be the flowing or variable rectangle, generated by two lines PQ and RQ, moving always perpendicular to each other, from the positions AR and AP; denoting the one by x and the other by y; supposing x and y to be so related, that the curve line AQ may always pass through the intersection Q of those lines, or the opposite angle of the rectangle. Now,

Now, the rectangle consists of the two trilinear spaces APQ, ARQ, of which, the
fluxion of the former is PQ $\times$ Pp, or $y\dot{x}$,
that of the latter is - RQ $\times$ Rr, or $x\dot{y}$, by art. 8;
therefore the sum of the two $\dot{x}y + x\dot{y}$, is the fluxion of the whole rectangle xy or ARQP.

The Same Otherwise.

16. Let the sides of the rectangle x and y, by flowing, become $x + x'$ and $y + y'$: then the product of these two, or $xy + xy' + yx' + x'y'$ will be the new or contemporaneous value of the flowing rectangle PR or xy: subtract the one value from the other, and the remainder, $xy' + yx' + x'y'$, will be the increment generated in the same time as x' or y'; of which the last term $x'y'$ is nothing, or indefinitely small, in respect of the other two terms, because x' and y' are indefinitely small in respect of x and y; which term being therefore omitted, there remains $xy' + yx'$ for the value of the increment; and hence by substituting $\dot{x}$ and $\dot{y}$, for x' and y', to which they are proportional, there arises $x\dot{y} + y\dot{x}$ for the true value of the fluxion of xy; the same as before.

17. Hence may be easily derived the fluxion of the powers and products of any number of flowing or variable quantities whatever; as of xyz, or $uxyz$, or $vuxyz$, &c. And first, for the fluxion of xyz: put $p = xy$, and the whole given fluent $xyz = q$, or $q = xyz = pz$. Then, taking the fluxions of $q = pz$, by the last article, they are $\dot{q} = \dot{p}z + p\dot{z}$; but $p = xy$, and so $\dot{p} = \dot{x}y + x\dot{y}$ by the same article; substituting therefore these values of p and $\dot{p}$ instead of them, in the value of $\dot{q}$, this becomes $\dot{q} = \dot{x}yz + x\dot{y}z + xy\dot{z}$, the fluxion of xyz required; which is therefore equal to the sum of the products, arising from the fluxion of each letter, or quantity, multiplied by the product of the other two.

Again, to determine the fluxion of $uxyz$, the continual product of four variable quantities; put this product, namely $uxyz$, or $qu = r$, where $q = xyz$ as above. Then, taking the fluxions by the last article, $\dot{r} = \dot{q}u + q\dot{u}$; which, by substituting for q and $\dot{q}$ their values as above, becomes - -
$\dot{r} = \dot{u}xyz + u\dot{x}yz + ux\dot{y}z + uxy\dot{z}$, the fluxion of $uxyz$ as required: consisting of the fluxion of each quantity, drawn into the products of the other three.

In

In the very same manner it is found, that the fluxion of $vuxyz$ is $\dot{v}uxyz + v\dot{u}xyz + vu\dot{x}yz + vux\dot{y}z + vuxy\dot{z}$; and so on, for any number of quantities whatever; in which it is always found, that there are as many terms as there are variable quantities in the proposed fluent; and that these terms consist of the fluxion of each variable quantity, multiplied by the product of all the rest of the quantities.

18. Hence is easily derived the fluxion of any power of a variable quantity, as of x^2, or x^3, or x^4, &c. For, in the product or rectangle xy, if $x = y$, then is $xy = xx$ or x^2, and also its fluxion $\dot{x}y + x\dot{y} = \dot{x}x + x\dot{x}$ or $2x\dot{x}$, the fluxion of x^2.

Again, if all the three x, y, z be equal; then is the product of the three $xyz = x^3$; and consequently its fluxion $\dot{x}yz + x\dot{y}z + xy\dot{z} = \dot{x}xx + x\dot{x}x + xx\dot{x}$ or $3x^2\dot{x}$, the fluxion of x^3.

In the same manner, it will appear that
the fluxion of x^4 is $= 4x^3\dot{x}$, and
the fluxion of x^5 is $= 5x^4\dot{x}$ and, in general,
the fluxion of x^n is $= nx^{n-1}\dot{x}$;
where n is any positive whole number whatever.

That is, the fluxion of any positive integral power is equal to the fluxion of the root ($\dot{x}$), multiplied by the exponent of the power (n), and by the power of the same root whose index is less by 1, (x^{n-1}).

And thus, the fluxion of $a + cx$ being $c\dot{x}$,
that of $(a + cx)^2$ is $2c\dot{x} \times (a + cx)$ or $2ac\dot{x} + 2c^2x\dot{x}$,
that of $(a + cx^2)^2$ is $4cx\dot{x} \times (a + cx^2)$ or $4acx\dot{x} + 4c^2x^3\dot{x}$,
that of $(x^2 + y^2)^2$ is $(4x\dot{x} + 4y\dot{y}) \times (x^2 + y^2)$,
that of $(x + cy^2)^3$ is $(3\dot{x} + 6cy\dot{y}) \times (x + cy^2)^2$.

19. From the conclusions in the same article, we may also derive the fluxion of any fraction, or the quotient of one variable quantity divided by another, as of $\frac{x}{y}$. For, put the quotient or fraction $\frac{x}{y} = q$; then, multiplying by the denominator, $x = qy$; and, taking the fluxions,

$$\dot{x} = \dot{q}y + q\dot{y}, \text{ or } \dot{q}y = \dot{x} - q\dot{y}; \text{ and, by division,}$$

$$\dot{q} = \frac{\dot{x}}{y} - \frac{q\dot{y}}{y} = \text{(by substituting the value of } q\text{, or } \tfrac{x}{y}\text{),}$$

$$\frac{\dot{x}}{y} - \frac{x\dot{y}}{y^2} = \frac{\dot{x}y - x\dot{y}}{y^2}, \text{ the fluxion of } \frac{x}{y}, \text{ as required.}$$

That

That is the fluxion of any fraction, is equal to the fluxion of the numerator drawn into the denominator, minus the fluxion of the denominator drawn into the numerator, and the remainder divided by the square of the denominator. So that the fluxion of $\frac{ax}{y}$ is $a \times \frac{\dot{x}y - x\dot{y}}{y^2}$ or $\frac{a\dot{x}y - ax\dot{y}}{y^2}$.

REMARK BY THE EDITOR.

The fluxion of the algebraic quantity xy is properly $y\dot{x} + x\dot{y}$ in all cases of increase or decrease. We should always use the signs of the fluxions of algebraic expressions as those signs arise from the known rules, without considering whether the quantities increase or decrease; but in denoting, algebraically, the simple fluxions of geometrical quantities, we should prefix the sign *minus* to the fluxions of such as decrease: and thus we may, in any case, use the fluxions of algebraic equations, together with the fluxions derived from geometrical figures, without embarrassment or apprehension of error.

20. Hence too is easily derived the fluxion of any negative integer power of a variable quantity, as of x^{-n}, or $\frac{1}{x^n}$, which is the same thing. For here the numerator of the fraction is 1, whose fluxion is nothing; and therefore, by the last article, the fluxion of such a fraction, or negative power, is barely equal to minus the fluxion of the denominator, divided by the square of the said denominator. That is the fluxion of x^{-n}, or $\frac{1}{x^n}$ is $-\frac{nx^{n-1}\dot{x}}{x^{2n}}$ or $-\frac{n\dot{x}}{x^{n+1}}$ or $-nx^{-n-1}\dot{x}$; or the fluxion of any negative integer power of a variable quantity, as x^{-n}, is equal to the fluxion of the root, multiplied by the exponent of the power, and by the next power less by 1; the same rule as for positive powers.

The same thing is otherwise obtained thus: Put the proposed fraction, or quotient $\frac{1}{x^n} = q$; then is $qx^n = 1$; and, taking the fluxions, we have $\dot{q}x^n + qnx^{n-1}\dot{x} = 0$: hence $\dot{q}x^n = -qnx^{n-1}\dot{x}$; divide by x^n, then $\dot{q} = -\frac{qn\dot{x}}{x} =$ (by substituting $\frac{1}{x^n}$ for q), $\frac{-n\dot{x}}{x^{n+1}}$ or $= -nx^{-n-1}\dot{x}$; the same as before.

Hence the fluxion of x^{-1} or $\frac{1}{x}$ is $-x^{-2}\dot{x}$, or $-\frac{\dot{x}}{x^2}$,

that of - x^{-2} or $\frac{1}{x^2}$ is $-2x^{-3}\dot{x}$ or $-\frac{2\dot{x}}{x^3}$,

that of - x^{-3} or $\frac{1}{x^3}$ is $-3x^{-4}\dot{x}$ or $-\frac{3\dot{x}}{x^4}$,

that

that of ax^{-4} or $\frac{a}{x^4}$ is $-4ax^{-5}\dot{x}$ or $-\frac{4a\dot{x}}{x^5}$

that of $(a+x)^{-1}$ or $\frac{1}{a+x}$ is $-(a+x)^{-2}\dot{x}$ or $\frac{\dot{x}}{(a+x)^2}$,

that of $c(a+3x^2)^{-2}$ or $\frac{c}{(a+3x^2)^2}$ is $-12cx\dot{x}\times(a+3x^2)^{-3}$,

or $-\frac{12cx\dot{x}}{(a+3x^2)^3}$.

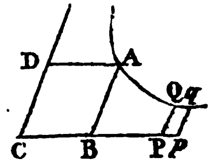
21. Much in the same manner is obtained the fluxion of any fractional power of a fluent quantity, as of $x^{\frac{m}{n}}$, or $\sqrt[n]{x^m}$.

For, put the proposed quantity $x^{\frac{m}{n}} = q$; then, raising each side to the n power, gives $x^m = q^n$;
taking the fluxions, gives $mx^{m-1}\dot{x} = nq^{n-1}\dot{q}$; then

dividing by nq^{n-1}, gives $\dot{q} = \frac{mx^{m-1}\dot{x}}{nq^{n-1}} = \frac{mx^{m-1}\dot{x}}{nx^{m-\frac{m}{n}}} = \frac{m}{n}x^{\frac{m}{n}-1}\dot{x}$.

Which is still the same rule, as before, for finding the fluxion of any power of a fluent quantity, and which therefore is general, whether the exponent be positive or negative, integral or fractional. And hence the fluxion of $ax^{\frac{3}{2}}$ is $\frac{3}{2}ax^{\frac{1}{2}}\dot{x}$, that of $ax^{\frac{1}{2}}$ is $\frac{1}{2}ax^{\frac{1}{2}-1}\dot{x} = \frac{1}{2}ax^{-\frac{1}{2}}\dot{x} = \frac{a\dot{x}}{2x^{\frac{1}{2}}} = \frac{a\dot{x}}{2\sqrt{x}}$; and that of $\sqrt{(a^2-x^2)}$ or $(a^2-x^2)^{\frac{1}{2}}$ is $\frac{1}{2}(a^2-x^2)^{-\frac{1}{2}}\times -2x\dot{x} = \frac{-x\dot{x}}{\sqrt{(a^2-x^2)}}$

22. Having now found out the fluxions of all the ordinary forms of algebraical quantities; it remains to determine those of logarithmic expressions; and also of exponential ones, that is, such powers as have their exponents variable or flowing quantities. And first, for the fluxion of Napier's, or the hyperbolic logarithm.



23. Now, to determine this from the nature of the hyperbolic spaces. Let A be the principal vertex of an hyperbola, having its asymptotes CD, CP, with the ordinates DA, BA, PQ, &c, parallel to them. Then, from the nature of the hyperbola and of logarithms, it is known, that any space ABPQ is the log. of the ratio of CB to CP, to the modulus ABCD. Now, put $1 =$ CB or BA the side of the square or rhombus DB; $m =$ the modulus, or CB $\times$ BA; or area of DB, or sine of the angle C to the radius 1; also the absciss CP $= x$, and the

the ordinate PQ $= y$. Then, by the nature of the hyperbola, CP $\times$ PQ is always equal to DB, that is, $xy = m$; hence $y = \frac{m}{x}$, and the fluxion of the space, $\dot{x}y$ is $\frac{m\dot{x}}{x} =$ PQqp the fluxion of the log. of x, to the modulus m. And, in the hyperbolic logarithms, the modulus m being 1, therefore $\frac{\dot{x}}{x}$ is the fluxion of the hyp. log. of x; which is therefore equal to the fluxion of the quantity, divided by the quantity itself.

Hence the fluxion of the hyp. log.

of $1 + x$ is $\frac{\dot{x}}{1 + x}$,

of $1 - x$ is $\frac{-\dot{x}}{1 - x}$,

of $x + z$ is $\frac{\dot{x} + \dot{z}}{x + z}$,

of $\frac{a + x}{a - x}$ is $\frac{\dot{x}(a - x) + \dot{x}(a + x)}{(a - x)^2} \times \frac{a - x}{a + x} = \frac{2a\dot{x}}{a^2 - x^2}$,

of ax^n is $\frac{nax^{n-1}\dot{x}}{ax^n} = \frac{n\dot{x}}{x}$.

24. By means of the fluxions of logarithms, are usually determined those of exponential quantities, that is, quantities which have their exponent a flowing or variable letter. These exponentials are of two kinds, namely, when the root is a constant quantity, as e^x, and when the root is variable as well as the exponent, as y^x.

25. In the first case put the exponential, whose fluxion is to be found, equal to a single variable quantity z, namely, $z = e^x$; then take the logarithm of each, so shall log. $z = x \times$ log. e; take the fluxions of these, so shall $\frac{\dot{z}}{z} = \dot{x} \times$ log. e, by the last article; hence $\dot{z} = z\dot{x} \times$ log. $e = e^x \dot{x} \times$ log. e, which is the fluxion of the proposed quantity e^x or z; and which therefore is equal to the said given quantity drawn into the fluxion of the exponent, and into the log. of the root.

Hence also, the fluxion of $(a + c)^{nx}$ is $(a + c)^{nx} \times n\dot{x} \times$ log. $(a+c)$.

26. In like manner, in the second case, put the given quantity $y^x = z$; then the logarithms give log. $z = x \times$ log. y, and the fluxions give $\frac{\dot{z}}{z} = \dot{x} \times$ log. $y + x \times \frac{\dot{y}}{y}$; hence $\dot{z} = z\dot{x} \times$ log. $y + \frac{zx\dot{y}}{y} =$ (by substituting y^x for z) $y^x\dot{x} \times$

 log.

log. $y + xy^{x-1}\dot{y}$, which is the fluxion of the proposed quantity y^x; and which therefore consist of two terms, of which the one is the fluxion of the given quantity considering the exponent as constant, and the other the fluxion of the same quantity considering the root as constant.

OF SECOND, THIRD, &c, FLUXIONS.

Having explained the manner of considering and determining the first fluxions of flowing or variable quantities; it remains now to consider those of the higher orders, as second, third, fourth, &c, fluxions.

27. If the rate or celerity with which any flowing quantity changes its magnitude, be constant, or the same at every position; then is the fluxion of it also constantly the same. But if the variation of magnitude be continually changing, either increasing or decreasing; then will there be a certain degree of fluxion peculiar to every point or position; and the rate of variation or change in the fluxion, is called the Fluxion of the Fluxion, or the Second Fluxion of the given fluent quantity. In like manner, the variation or fluxion of this second fluxion, is called the Third Fluxion of the first proposed fluent quantity; and so on.

These orders of fluxions are denoted by the same fluent letter with the corresponding number of points over it: namely, two points for the second fluxion, three points for the third fluxion, four points for the fourth fluxion, and so on. So, the different orders of the fluxion of x, are $\dot{x}$, $\ddot{x}$, $\dddot{x}$, $\ddddot{x}$, &c; where each is the fluxion of the one next before it.

28. This discription of the higher orders of fluxions may be illustrated by the figures exhibited in art. 8, page 306; where, if x denote the absciss AP, and y the ordinate PQ; and if the ordinate PQ or y flow along the absciss AP or x, with a uniform motion; then the fluxion of x, namely, $\dot{x} =$ Pp or Qr, is a constant quantity, or $\ddot{x} = 0$, in all the figures. Also, in fig. 1, in which AQ is a right line, $\dot{y} = rq$, or the fluxion of PQ, is a constant quantity, or $\ddot{y} = 0$; for, the angle Q, = the angle A, being constant, Qr is to rq, or $\dot{x}$ to $\dot{y}$, in a constant ratio. But in the 2d fig. rq, or the fluxion of PQ, continually increases more and more; and in

in fig. 3 it continually decreases more and more, and therefore in both these cases y has a second fluxion, being positive in fig. 2, but negative in fig. 3. And so on, for the other orders of fluxions.

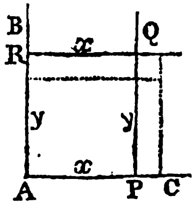
Thus if, for instance, the nature of the curve be such, that x^3 is every where equal to a^2y; then, taking the fluxions, it is $a^2\dot{y} = 3x^2\dot{x}$; and, considering $\dot{x}$ always as a constant quantity, and taking always the fluxions, the equations of the several orders of fluxions will be as below, viz.

the 1st fluxions $a^2\dot{y} = 3x^2\dot{x}$.
the 2d fluxions $a^2\ddot{y} = 6x\dot{x}^2$,
the 3d fluxions $a^2\dddot{y} = 6\dot{x}^3$,
the 4th fluxions $a^2\ddddot{y} = 0$,
and all the higher fluxions also $= 0$, or nothing.

Also, the higher orders of fluxions are found in the same manner as the lower ones. Thus,

the first fluxion of y^3 is - - - - - -	$3y^2\dot{y}$;
its 2d flux. or the flux. of $3y^2\dot{y}$, considered as the rectangle of $3y^2$, and $\dot{y}$, is - - - - - - - -	$3y^2\ddot{y} + 6y\dot{y}^2$:
and the flux. of this again, or the 3d flux. of y^3, is - - - - - - -	$3y^2\dddot{y} + 18y\dot{y}\ddot{y} + 6\dot{y}^3$.

29. In the foregoing articles, it has been supposed that the fluents increase, or that their fluxions are positive; but it often happens that some fluents decrease, and that therefore their fluxions are negative: and whenever this is the case, the sign of the fluxion must be changed, or made contrary to that of the fluent. So, of the rectangle xy, when both x and y increase together, the fluxion is $\dot{x}y + x\dot{y}$; but if one of them, as y, decrease, while the other, x, increases; then, the fluxion of y being $-\dot{y}$, the fluxion of xy will in that case be $\dot{x}y - x\dot{y}$. This may be illustrated by the annexed rectangle, APQR $= xy$, supposed to be generated by the motion of the line PQ from A towards C, and by the motion of the line RQ from B towards A: For, by the motion of PQ, from A towards C, the rectangle is increased, and its fluxion is $+ \dot{x}y$; but, by the motion of RQ, from B towards A, the rectangle is decreased, and the fluxion of the decrease is $x\dot{y}$; therefore,



fore, taking the fluxion of the decrease from that of the increase, the fluxion of the rectangle xy, when x increases and y decreases, is $\dot{x}y - x\dot{y}$.

30. We may now collect all the rules together, which have been demonstrated in the foregoing articles, for finding the fluxions of all sorts of quantities. And hence,

1st, *For the fluxion of any Power of a flowing quantity.*—Multiply all together the exponent of the power, the fluxion of the root, and the power next less by 1 of the same root.

2d, *For the fluxion of the Rectangle of two quantities.*—Multiply each quantity by the fluxion of the other, and connect the two products together by their proper signs.

3d, *For the fluxion of the Continual Product of any number of flowing quantities.*—Multiply the fluxion of each quantity by the product of all the other quantities, and connect all the products together by their proper signs.

4th, *For the fluxion of a Fraction.*—From the fluxion of the numerator drawn into the denominator, subtract the fluxion of the denominator drawn into the numerator, and divide the result by the square of the denominator.

5th, *Or, the 2d, 3d, and 4th cases may be all included under one, and performed thus.*—Take the fluxion of the given expression as often as there are variable quantities in it, supposing first only one of them variable, and the rest constant; then another variable, and the rest constant; and so on, till they have all in their turns been singly supposed variable, and connect all these fluxions together with their own signs.

6th, *For the fluxion of a Logarithm.*—Divide the fluxion of the quantity by the quantity itself, and multiply the result by the modulus of the system of logarithms.

Note. The modulus of the hyperbolic logarithms is 1,
and the modulus of the common logs, is - 0·43429448.

7th, *For the fluxion of an Exponential quantity having the Root Constant.*—Multiply all together, the given quantity the fluxion of its exponent, and the hyp. log. of the root.

8th, *For the fluxion of an Exponential quantity having the Root Variable.*—To the fluxion of the given quantity, found by the 1st rule, as if the root only were variable, and the fluxion of the same quantity found by the 7th rule, as if the exponent only were variable; and the sum will be the fluxion for both of them variable.

Note. When the given quantity consists of several terms, find the fluxion of each term separately, and connect them all together with their proper signs.

31. PRACTICAL

31. PRACTICAL EXAMPLES TO EXERCISE THE FOREGOING RULES.

1. The fluxion of axy is
2. The fluxion of $bxyz$ is
3. The fluxion of $cx \times (ax - cy)$ is
4. The fluxion of $x^m y^n$ is
5. The fluxion of $x^m y^n z^r$ is
6. The fluxion of $(x + y) \times (x - y)$ is
7. The fluxion of $2ax^2$ is
8. The fluxion of $2x^3$ is
9. The fluxion of $3x^4 y$ is
10. The fluxion of $4x^{\frac{2}{3}} y^4$ is
11. The fluxion of $ax^2 y - x^{\frac{1}{2}} y^3$ is
12. The fluxion of $4x^4 - x^2 y + 3byz$ is
13. The fluxion of $\sqrt[n]{x}$ or $x^{\frac{1}{n}}$ is
14. The fluxion of $\sqrt[n]{x^m}$ or $x^{\frac{m}{n}}$ is
15. The fluxion of $\frac{1}{\sqrt[n]{x^m}}$ or $\frac{1}{x^{\frac{m}{n}}}$ or $x^{-\frac{m}{n}}$ is
16. The fluxion of $\sqrt{x}$ or $x^{\frac{1}{2}}$ is
17. The fluxion of $\sqrt[3]{x}$ or $x^{\frac{1}{3}}$ is
18. The fluxion of $\sqrt[3]{x^2}$ or $x^{\frac{2}{3}}$ is
19. The fluxion of $\sqrt{x^3}$ or $x^{\frac{3}{2}}$ is
20. The fluxion of $\sqrt[4]{x^3}$ or $x^{\frac{3}{4}}$ is
21. The fluxion of $\sqrt[3]{x^4}$ or $x^{\frac{4}{3}}$ is
22. The fluxion of $\sqrt{(a^2 + x^2)}$ or $(a^2 + x^2)^{\frac{1}{2}}$ is
23. The fluxion of $\sqrt{(a^2 - x^2)}$ or $(a^2 - x^2)^{\frac{1}{2}}$ is
24. The fluxion of $\sqrt{(2rx - xx)}$ or $(2rx - xx)^{\frac{1}{2}}$ is
25. The fluxion of $\frac{1}{\sqrt{(a^2 - x^2)}}$ or $(a^2 - x^2)^{-\frac{1}{2}}$ is
26. The fluxion of $(ax - xx)^{\frac{1}{3}}$ is

27. The

27. The fluxion of $2x\sqrt{a^2 \pm x^2}$ is

28. The fluxion of $(a^2 - x^2)^{\frac{3}{2}}$ is

29. The fluxion of $\sqrt{xz}$, or $(xz)^{\frac{1}{2}}$ is

30. The fluxion of $\sqrt{xz - zz}$ or $(xz - zz)^{\frac{1}{2}}$ is

31. The fluxion of $-\frac{1}{a\sqrt{x}}$ or $-\frac{1}{a}x^{\frac{1}{2}}$ is

32. The fluxion of $\frac{ax^3}{a + x}$ is

33. The fluxion of $\frac{x^m}{y^n}$ is

34. The fluxion of $\frac{xy}{z}$ is

35. The fluxion of $\frac{c}{xx}$ is

36. The fluxion of $\frac{3x}{a - x}$ is

37. The fluxion of $\frac{z}{x + z}$ is

38. The fluxion of $\frac{x^3}{z^2}$ is

39. The fluxion of $\frac{x^{\frac{2}{3}}}{y^{\frac{3}{2}}}$ is

40. The fluxion of $\frac{axy^2}{z}$ is

41. The fluxion of $\frac{3}{\sqrt{(x^2 - y^2)}}$ is

42. The fluxion of the hyp. log. of ax is

43. The fluxion of the hyp. log. of $1 + x$ is

44. The fluxion of the hyp. log. of $1 - x$ is

45. The fluxion of the hyp. log. of x^2 is

46. The fluxion of the hyp. log. of $\sqrt{x}$ is

47. The fluxion of the hyp. log. of x^m is

48. The

48. The fluxion of the hyp. log. of $\frac{2}{x}$ is

49. The fluxion of the hyp. log. of $\frac{1+x}{1-x}$ is.

50. The fluxion of the hyp. log. of $\frac{1-x}{1+x}$ is

51. The fluxion of c^x is
52. The fluxion of 10^x is
53. The fluxion of $(a+c)^x$ is
54. The fluxion of 100^{xy} is
55. The fluxion of x^z is
56. The fluxion of y^{10x} is
57. The fluxion of x^x is
58. The fluxion of $(xy)^{xz}$ is
59. The fluxion of $x\dot{y}$ is
60. The fluxion of $\dot{x}\dot{y}^2$ is
61. The second fluxion of xy is
62. The second fluxion of xy, when $\dot{x}$ is constant, is
63. The second fluxion of x^n is
64. The third fluxion of x^n, when $\dot{x}$ is constant, is
65. The third fluxion of xy is

THE INVERSE METHOD, OR THE FINDING OF FLUENTS.

32. It has been observed, that a Fluent, or Flowing Quantity, is the variable quantity which is considered as increasing or decreasing. Or, the fluent of a given fluxion, is such a quantity, that its fluxion, found according to the foregoing rules, shall be the same as the fluxion given or proposed.

33. It may further be observed, that Contemporary Fluents, or Contemporary Fluxions, are such as flow together, or for the same time.—When contemporary fluents are always equal, or in any constant ratio; then also are their fluxions respectively either equal, or in that same constant ratio. hat is, if $x = y$, then is $\dot{x} = \dot{y}$; or if $x : y :: n : 1$, then is $: \dot{y} :: n : 1$; or if $x = ny$, then is $\dot{x} = n\dot{y}$.

34. It is easy to find the fluxions to all the given forms of uents; but, on the contrary, it is difficult to find the fluents many given fluxions; and indeed there are numberless cases

cases in which this cannot at all be done, excepting by the quadrature and rectification of curve lines, or by logarithms, or by infinite series. For, it is only in certain particular forms and cases that the fluents of given fluxions can be found; there being no method of performing this universally, *a priori*, by a direct investigation, like finding the fluxion of a given fluent quantity. We can only therefore lay down a few rules for such forms of fluxions as we know, from the direct method, belong to such and such kinds of flowing quantities: and these rules, it is evident, must chiefly consist in performing such operations as are the reverse of those by which the fluxions are found of given fluent quantities. The principal cases of which are as follow.

35. *To find the Fluent of a Simple Fluxion; or of that in which there is no variable quantity, and only one fluxional quantity.*

This is done by barely substituting the variable or flowing quantity instead of its fluxion; being the result or reverse of the notation only.—Thus,

The fluent of $a\dot{x}$ is ax.
The fluent of $a\dot{y} + 2\dot{y}$ is $ay + 2y$.
The fluent of $\overline{\sqrt{a^2 + x^2}}^{\,\cdot}$ is $\sqrt{a^2 + x^2}$.

36. *When any Power of a flowing quantity is Multiplied by the Fluxion of the Root;*

Then, having substituted, as before, the flowing quantity, for its fluxion, divide the result by the new index of the power. Or, which is the same thing, take out, or divide by, the fluxion of the root; add 1 to the index of the power; and divide by the index so increased. Which is the reverse of the 1st rule for finding fluxions.

So, if the fluxion proposed be - -	$3x^5\dot{x}$
Leave out, or divide by, $\dot{x}$, then it is -	$3x^5$;
add 1 to the index, and it is - -	$3x^6$;
divide by the index 6, and it is - -	$\frac{3}{6}x^6$ or $\frac{1}{2}x^6$,
which is the fluent of the proposed fluxion	$3x^5\dot{x}$.

In like manner,

The fluent of $2ax\dot{x}$ is ax^2.
The fluent of $3x^2\dot{x}$ is x^3.

The

The fluent of $4x^{\frac{1}{2}}\dot{x}$ is $\frac{8}{3}x^{\frac{3}{2}}$.

The fluent of $2y^{\frac{3}{4}}\dot{y}$ is $\frac{8}{7}y^{\frac{7}{4}}$.

The fluent of $az^{\frac{5}{6}}\dot{z}$ is $\frac{6}{11}az^{\frac{11}{6}}$.

The fluent of $x^{\frac{1}{2}}\dot{x} + 3y^{\frac{2}{3}}\dot{y}$ is $\frac{2}{3}x^{\frac{3}{2}} + \frac{9}{5}y^{\frac{5}{3}}$.

The fluent of $x^{n-1}\dot{x}$ is $\frac{1}{n}x^{n}$.

The fluent of $ny^{n-1}\dot{y}$ is

The fluent of $\frac{\dot{z}}{z^2}$, or $z^{-2}\dot{z}$ is

The fluent of $\frac{a\dot{y}}{y^n}$ is

The fluent of $(a + x)^4\dot{x}$ is

The fluent of $(a^4 + y^4)y^3\dot{y}$ is

The fluent of $(a^3 + z^3)^4 z^2\dot{z}$ is

The fluent of $(a^n + x^n)^m x^{n-1}\dot{x}$ is

The fluent of $(a^2 + y^2)^3 y\dot{y}$ is

The fluent of $\frac{z\dot{z}}{\sqrt{(a^2+z^2)}}$ is

The fluent of $\frac{\dot{x}}{\sqrt{(a - x)}}$ is

27. *When the Root under a Vinculum is a Compound Quantity; and the Index of the part or factor Without the Vinculum, increased by 1, is some Multiple of that under the Vinculum:*

Put a single variable letter for the compound root; and substitute its powers and fluxion instead of those of the same value, in the given quantity; so will it be reduced to a simpler form, to which the preceding rule can then be applied.

Thus, if the given fluxion be $\dot{F} = (a^2 + x^2)^{\frac{2}{3}}x^3\dot{x}$, where 3, the index of the quantity without the vinculum, increased by 1, making 4, which is just the double of 2, the exponent of x^2 within the vinculum: therefore, putting $z = a^2 + x^2$, thence $x^2 = z - a^2$, the fluxion of which is $2x\dot{x} = \dot{z}$; hence then $x^3\dot{x} = \frac{1}{2}x^2\dot{z} = \frac{1}{2}\dot{z}(z - a^2)$, and the given fluxion $\dot{F}$, or $(a^2 + x^2)^{\frac{2}{3}}x^3\dot{x}$, is $= \frac{1}{2}z^{\frac{2}{3}}\dot{z}(z - a^2)$, or $= \frac{1}{2}z^{\frac{5}{3}}\dot{z} - \frac{1}{2}a^2z^{\frac{2}{3}}\dot{z}$; and hence the fluent F is $= \frac{3}{16}z^{\frac{8}{3}} - \frac{3}{10}a^2z^{\frac{5}{3}} = 3z^{\frac{5}{3}}(\frac{1}{16}z - \frac{1}{10}a^2)$. Or, by substituting the value of z instead of it, the same fluent is $3(a^2+x^2)^{\frac{5}{3}} \times (\frac{1}{16}x^2 - \frac{3}{80}a^2)$, or $\frac{3}{16}(a^2 + x^2) \times (x^2 - \frac{3}{5}a^2)$.

In

In like manner for the following examples.

To find the fluent of $\sqrt{a + cx} \times x^3\dot{x}$.

To find the fluent of $(a + cx)^{\frac{3}{2}}x^2\dot{x}$.

To find the fluent of $(a + cx^2)^{\frac{1}{3}} \times dx^3\dot{x}$.

To find the fluent of $\dfrac{cz\dot{z}}{\sqrt{a + z}}$ or $(a + z)^{-\frac{1}{2}}cz\dot{z}$.

To find the fluent of $\dfrac{cz^{3n-1}\dot{z}}{\sqrt{a+z^n}}$ or $(a+z^n)^{-\frac{1}{2}}cz^{3n-1}\dot{z}$.

To find the fluent of $\dfrac{\dot{x}\sqrt{a^2+z^2}}{z^6}$ or $(a^2 + z^2)^{\frac{1}{2}}z^{-6}\dot{z}$.

To find the fluent of $\dfrac{\dot{x}\sqrt{a-x^n}}{x^{\frac{7}{2}n-1}}$ or $(a - x^n)^{\frac{1}{2}}x^{\frac{7}{2}n-1}\dot{x}$.

38. *When there are several Terms, involving Two or more Variable Quantities, having the Fluxion of each Multiplied by the other Quantity or Quantities:*

Take the fluent of each term, as if there were only one variable quantity in it, namely, that whose fluxion is contained in it, supposing all the others to be constant in that term; then if the fluents of all the terms, so found, be the very same quantity in all of them, that quantity will be the fluent of the whole. Which is the reverse of the 5th rule for finding fluxions: Thus, if the given fluxion be $\dot{x}y + x\dot{y}$, then the fluent of $\dot{x}y$ is xy, supposing y constant: and the fluent of $x\dot{y}$ is also xy, supposing x constant: therefore xy is the required fluent of the given fluxion $\dot{x}y + x\dot{y}$.

In like manner,

The fluent of $\dot{x}yz + x\dot{y}z + xy\dot{z}$ is xyz.
The fluent of $2xy\dot{x} + x^2\dot{y}$ is x^2y.

The fluent of $\frac{1}{2}x^{-\frac{1}{2}}\dot{x}y^2 + 2x^{\frac{1}{2}}y\dot{y}$ is

The fluent of $\dfrac{\dot{x}y - x\dot{y}}{y^2}$ or $\dfrac{\dot{x}}{y} - \dfrac{x\dot{y}}{y^2}$ is

The fluent of $\dfrac{2ax\dot{x}y^{\frac{1}{2}} - \frac{1}{2}ax^2y^{-\frac{1}{2}}\dot{y}}{y}$ or $\dfrac{2ax\dot{x}}{\sqrt{y}} - \dfrac{ax^2\dot{y}}{2y\sqrt{y}}$ is

39. *When*

39. *When the given Fluxional Expression is in this Form* $\frac{\dot{x}y-x\dot{y}}{y^2}$ *namely, a Fraction, including Two Quantities, being the Fluxion of the former of them drawn into the latter, minus the Fluxion of the latter drawn into the former, and divided by the Square of the latter:*

Then, the fluent is the fraction $\frac{x}{y}$, or the former quantity divided by the latter. That is,

The fluent of $\frac{\dot{x}y-x\dot{y}}{y^2}$ is $\frac{x}{y}$. And, in like manner,

The fluent of $\frac{2x\dot{x}y^2-2x^2y\dot{y}}{y^4}$ is $\frac{x^2}{y^2}$.

Though, indeed, the examples of this case may be performed by the foregoing one. Thus, the given fluxion $\frac{\dot{x}y-x\dot{y}}{y^2}$ reduces to $\frac{\dot{x}}{y}-\frac{x\dot{y}}{y^2}$, or $\frac{\dot{x}}{y}-x\dot{y}y^{-2}$; of which,

the fluent of $\frac{\dot{x}}{y}$ is $\frac{x}{y}$ supposing y constant; and

the fluent of $-x\dot{y}y^{-2}$ is also xy^{-1} or $\frac{x}{y}$, when x is constant;

therefore, by that case, $\frac{x}{y}$ is the fluent of the whole $\frac{\dot{x}y-x\dot{y}}{y^2}$.

40. *When the Fluxion of a Quantity is Divided by the Quantity itself:*

Then the fluent is equal to the hyperbolic logarithm of that quantity; or, which is the same thing, the fluent is equal to 2·30258509 multiplied by the common logarithm of the same quantity.

So, the fluent of $\frac{\dot{x}}{x}$ or $x^{-1}\dot{x}$, is the hyp. log. of x.

The fluent of $\frac{2\dot{x}}{x}$ is $2 \times$ hyp. log. of x, or = hyp. log. x^2.

The fluent of $\frac{a\dot{x}}{x}$, is $a \times$ hyp. log. x, or = hyp. log of x^a.

The fluent of $\frac{\dot{x}}{a+x}$, is

The fluent of $\frac{3x^2\dot{x}}{a+x^3}$, is

41. *Many*

41. *Many fluents may be found by the Direct Method thus:*

Take the fluxion again of the given fluxion, or the second fluxion of the fluent sought; into which substitute $\frac{\dot{x}^2}{x}$ for $\ddot{x}$, $\frac{\dot{y}^2}{y}$ for $\ddot{y}$, &c; that is, make x, $\dot{x}$, $\ddot{x}$, as also y, $\dot{y}$, $\ddot{y}$, &c, to be in continual proportion, or so that $x : \dot{x} :: \dot{x} : \ddot{x}$, and $y : \dot{y} :: \dot{y} : \ddot{y}$, &c; then divide the square of the given fluxional expression by the second fluxion, just found, and the quotient will be the fluent required in many cases.

Or the same rule may be otherwise delivered thus:

In the given fluxion $\dot{F}$, write x for $\dot{x}$, y for $\dot{y}$, &c, and call the result G, taking also the fluxion of this quantity $\dot{G}$; then make $\dot{G} : \dot{F} :: G : F$; so shall the fourth proportional F be the fluent sought in many cases.

It may be proved if this be the true fluent, by taking the fluxion of it again, which, if it agree with the proposed fluxion, will show that the fluent is right; otherwise, it is wrong.

EXAMPLES.

EXAM. 1. Let it be required to find the fluent of $nx^{n-1}\dot{x}$.

Here $\dot{F} = nx^{n-1}\dot{x}$. Write x, for $\dot{x}$, then $nx^{n-1}x$ or $nx^n = G$; the fluxion of this is $\dot{G} = n^2x^{n-1}\dot{x}$; therefore $\dot{G} : \dot{F} :: G : F$, becomes $n^2x^{n-1}\dot{x} : nx^{n-1}\dot{x} :: nx^n : x^n = F$, the fluent sought.

EXAM. 2. To find the fluent of $\dot{x}y + x\dot{y}$.

Here $\dot{F} = \dot{x}y + x\dot{y}$; then, writing x for $\dot{x}$ and y for $\dot{y}$, it is $xy + xy$ or $2xy = G$; hence $\dot{G} = 2\dot{x}y + 2x\dot{y}$; then $\dot{G} : \dot{F} :: G : F$, becomes $2\dot{x}y + 2x\dot{y} : \dot{x}y + x\dot{y} :: 2xy : xy = F$, the fluent sought.

42. *To find Fluents by means of a Table of Forms of Fluxions and Fluents.*

In the following Table are contained the most usual forms of fluxions that occur in the practical solution of problems, with their corresponding fluents set opposite to them; by means of which, namely, by comparing any proposed fluxion with the corresponding form in the table, the fluent of it will be found.

Forms.

Forms.	Fluxions.	Fluents.
I	$x^{n-1}\dot{x}$	$\frac{x^n}{n}$ or $\frac{1}{n}x^n$
II	$(a \pm x^n)^{m-1}x^{n-1}\dot{x}$	$\pm \frac{1}{mn}(a \pm x^n)^m$
III	$\frac{x^{mn-1}\dot{x}}{(a \pm x^n)^{m+1}}$	$\frac{1}{mna} \times \frac{x^{mn}}{(a \pm x^n)^m}$
IV	$\frac{(a \pm x^n)^{m-1}\dot{x}}{x^{mn+1}}$	$\frac{-1}{mna} \times \frac{(a \pm x^n)^m}{x^{mn}}$
V	$(my\dot{x} + nx\dot{y}) \times x^{m-1}y^{n-1}$, or $(\frac{m\dot{x}}{x} + \frac{n\dot{y}}{y})x^my^n$	x^my^n
VI	$mx^{m-1}\dot{x}y^nz^r + nx^my^{n-1}\dot{y}z^r + rx^my^nz^{r-1}\dot{z}$, or $(m\dot{x}yz + nx\dot{y}z + rxy\dot{z})x^{m-1}y^{n-1}z^{r-1}$, or $(\frac{m\dot{x}}{x} + \frac{n\dot{y}}{y} + \frac{r\dot{z}}{z})x^my^nz^r$	$x^my^nz^r$
VII	$\frac{\dot{x}}{x}$ or $x^{-1}\dot{x}$	log. of x.
VIII	$\frac{x^{n-1}\dot{x}}{a \pm x^n}$	$\pm \frac{1}{n}$ log. of $a \pm x^n$
IX	$\frac{x^{-1}\dot{x}}{a \pm x^n}$	$\frac{1}{na}$ log. of $\frac{x^n}{a \pm x^n}$
X	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{a - x^n}$	$\frac{1}{n\sqrt{a}}$ log of $\frac{\sqrt{a} + \sqrt{x^n}}{\sqrt{a} - \sqrt{x^n}}$
XI	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{a + x^n}$	$\frac{2}{n\sqrt{a}} \times$ arc to tan $\sqrt{\frac{x}{a}}$, or $\frac{1}{n\sqrt{a}} \times$ arc to cosine $\frac{a - x^n}{a + x^n}$
XII	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{\sqrt{\pm a + x^n}}$	$\frac{2}{n}$ log of $\sqrt{x^n} + \sqrt{\pm a + x^n}$

Forms.	*Fluxions.*	*Fluents.*
XIII	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{\sqrt{a-x^n}}$	$\frac{2}{n}\times$ arc to sin. $\sqrt{\frac{x^n}{a}}$, or $\frac{1}{n}\times$ arc to vers. $\frac{2x^n}{a}$
XIV	$\frac{x^{-1}\dot{x}}{\sqrt{a\pm x^n}}$	$\frac{1}{n\sqrt{a}}$ log. of $\frac{\pm\sqrt{a\pm x^n}\mp\sqrt{a}}{\sqrt{a\pm x^n}+\sqrt{a}}$
XV	$\frac{x^{-1}\dot{x}}{\sqrt{-a+x^n}}$	$\frac{2}{n\sqrt{a}}\times$ arc to secant $\sqrt{\frac{x^n}{a}}$, or $\frac{1}{n\sqrt{a}}\times$ arc to cosin. $\frac{2a-x^n}{x^n}$
XVI	$\dot{x}\sqrt{dx-x^2}$	$\frac{1}{2}$ circ. seg. to diam. d & vers. x
XVII	$c^{nx}\dot{x}$	$\frac{c^{nx}}{n\ \text{log.}\ c}$
XVIII	$\dot{x}y^x\ \text{log.}\ y+xy^{x-1}\dot{y}$	y^x

Note. The logarithms, in the above forms, are the hyperbolic ones, which are found by multiplying the common logarithms by 2·302585092994. And the arcs, whose sine, or tangent, &c, are mentioned, have the radius 1, and are those in the common tables of sines, tangents and secants. Also, the numbers m, n, &c, are to be some real quantities, as the forms fail when $m=0$, or $n=0$, &c.

The Use of the foregoing Table of Forms of Fluxions and Fluents.

43. In using the foregoing table, it is to be observed, that the first column serves only to show the number of the forms; in the second column are the several forms of fluxions, which are of different kinds or classes; and in the third or last column, are the corresponding fluents.

The method of using the table, is this. Having any fluxion given, to find its fluent: First, Compare the given fluxion with the several forms of fluxions in the second column of the table, till one of the forms be found that agrees with it; which is done by comparing the terms of the given fluxion with the like parts of the tabular fluxion, namely, the radical quantity of the one, with that of the other; and the

the exponents of the variable quantities of each, both within and without the vinculum; all which, being found to agree or correspond, will give the particular values of the general quantities in the tabular form: then substitute these particular values in the general or tabular form of the fluent, and the result will be the particular fluent of the given fluxion; after it is multiplied by any coefficient the proposed fluxion may have.

EXAMPLES.

EXAM. 1. To find the fluent of the fluxion $3x^{\frac{5}{3}}\dot{x}$.

This is found to agree with the first form. And, by comparing the fluxions, it appears that $x = x$, and $n - 1 = \frac{5}{3}$, or $n = \frac{8}{3}$; which being substituted in the tabular fluent, or $\frac{1}{n}x^n$, gives, after multiplying by 3, the co-efficient, $3 \times \frac{3}{8}x^{\frac{8}{3}}$, or $\frac{9}{8}x^{\frac{8}{3}}$, for the fluent sought.

EXAM. 2. To find the fluent of $5x^2\dot{x}\sqrt{c^3-x^3}$, or $5x^2\dot{x}(c^3-x^3)^{\frac{1}{2}}$.

This fluxion, it appears, belongs to the 2d tabular form: for $a = c^3$, and $-x^n = -x^3$, and $n = 3$ under the vinculum, also $m - 1 = \frac{1}{2}$, or $m = \frac{3}{2}$, and the exponent $^{n-1}$ of x^{n-1} without the vinculum, by using 3 for n, is $n - 1 = 2$, which agrees with x^2 in the given fluxion: so that all the parts of the form are found to correspond. Then, substituting these values into the general fluent, $-\frac{1}{mn}(a - x^n)^m$,

it becomes $-\frac{5}{3} \times \frac{2}{3}(c^3 - x^3)^{\frac{3}{2}} = -\frac{10}{9}(c^3 - x^3)^{\frac{3}{2}}$.

EXAM. 3. To find the fluent of $\frac{x^2\dot{x}}{1+x^3}$.

This is found to agree with the 8th form; where . . . $\pm x^n = + x^3$ in the denominator, or $n = 3$; and the numerator x^{n-1} then becomes x^2, which agrees with the numerator in the given fluxion; also $a = 1$. Hence then, by substituting in the general or tabular fluent, $\frac{1}{n}$ log. of $a + x^n$, it becomes $\frac{1}{3}$ log. $1 + x^3$.

EXAM. 4. To find the fluent of $ax^4\dot{x}$.

EXAM. 5. To find the fluent of $2(10 + x^2)^{\frac{2}{3}}x\dot{x}$.

EXAM. 6. To find the fluent of $\frac{a\dot{x}}{(c^2 + x^2)^{\frac{3}{2}}}$.

EXAM. 7. To find the fluent of $\frac{3x^2\dot{x}}{(a - x)^4}$.

EXAM. 8.

Exam. 8. To find the fluent of $\frac{a^2 - x^2}{x^5}\dot{x}$.

Exam. 9. To find the fluent of $\frac{1 + 3x}{2x^4}\dot{x}$.

Exam. 10. To find the fluent of $(\frac{3\dot{x}}{x} + \frac{2\dot{y}}{y})\, x^3y^2$.

Exam. 11. To find the fluent of $(\frac{\dot{x}}{x} + \frac{\dot{y}}{3y})xy^{\frac{1}{3}}$.

Exam. 12. To find the fluent of $\frac{3\dot{x}}{ax}$ or $\frac{3}{a}x^{-1}\dot{x}$.

Exam. 13. To find the fluent of $\frac{a\dot{x}}{3-2x}$.

Exam. 14. To find the fluent of $\frac{3\dot{x}}{2x - x^2}$ or $\frac{3x^{-1}\dot{x}}{2 - x}$.

Exam. 15. To find the fluent of $\frac{2\dot{x}}{x - 3x^3}$ or $\frac{2x^{-1}\dot{x}}{1 - 3x^2}$.

Exam. 16. To find the fluent of $\frac{3x\dot{x}}{1-x^4}$.

Exam. 17. To find the fluent of $\frac{ax^{\frac{3}{2}}\dot{x}}{2-x^5}$.

Exam. 18. To find the fluent of $\frac{2x\dot{x}}{1+x^4}$.

Exam. 19. To find the fluent of $\frac{ax^{\frac{3}{2}}\dot{x}}{2+x^5}$.

Exam. 20. To find the fluent of $\frac{3x\dot{x}}{\sqrt{1+x^4}}$.

Exam. 21. To find the fluent of $\frac{a\dot{x}}{\sqrt{x^2 - 4}}$.

Exam. 22. To find the fluent of $\frac{3x\dot{x}}{\sqrt{1 - x^4}}$.

Exam. 23. To find the fluent of $\frac{a\dot{x}}{\sqrt{4-x^2}}$.

Exam. 24. To find the fluent of $\frac{2x^{-1}\dot{x}}{\sqrt{1- x^2}}$.

Exam. 25. To find the fluent of $\frac{a\dot{x}}{\sqrt{ax^2 + x^{\frac{1}{2}}}}$.

Exam. 26. To find the fluent of $\frac{2x^{-1}\dot{x}}{\sqrt{x^2 - 1}}$.

EXAM. 27. To find the fluent of $\frac{a\dot{x}}{\sqrt{x^{\frac{1}{2}} - ax^2}}$.

EXAM. 28. To find the fluent of $2\dot{x}\sqrt{2x - x^2}$.

EXAM. 29. To find the fluent of $a^x\dot{x}$.

EXAM. 30. To find the fluent of $3a^{3x}\dot{x}$.

EXAM. 31. To find the fluent of $3z^x\dot{x}\log.z + 3xz^{x-1}\dot{z}$.

EXAM. 32. To find the fluent of $(1 + x^3)x\dot{x}$.

EXAM. 33. To find the fluent of $(2 + x^4)x^{\frac{3}{2}}\dot{x}$.

EXAM. 34. To find the fluent of $x^2\dot{x}\sqrt{a^2 + x^2}$.

To find Fluents by Infinite Series.

44. When a given fluxion, whose fluent is required, is so complex, that it cannot be made to agree with any of the forms in the foregoing table of cases, nor made out from the general rules before given; recourse may then be had to the method of infinite series; which is thus performed:

Expand the radical or fraction, in the given fluxion, into an infinite series of simple terms, by the methods given for that purpose in books of algebra; viz. either by division or extraction of roots, or by the binomial theorem, &c; and multiply every term by the fluxional letter, and by such simple variable factor as the given fluxional expression may contain. Then take the fluent of each term separately, by the foregoing rules, connecting them all together by their proper signs; and the series will be the fluent sought, after it is multiplied by any constant factor or co-efficient which may be contained in the given fluxional expression.

45. It is to be noted however, that the quantities must be so arranged, as that the series produced may be a converging one, rather than diverging: and this is effected by placing the greater terms foremost in the given fluxion. When these are known or constant quantities, the infinite series will be an ascending one; that is, the powers of the variable quantity will ascend or increase; but if the variable quantity be set foremost, the infinite series produced will be a descending one, or the powers of that quantity will decrease always more and more in the succeeding terms, or increase in the denominators of them, which is the same thing.

For example, to find the fluent of $\frac{1-x}{1+x-x^2}\dot{x}$.

Here, by dividing the numerator by the denominator, the proposed fluxion becomes $\dot{x}-2x\dot{x}+3x^2\dot{x}-5x^3\dot{x}+8x^4\dot{x}-$ &c; then the fluents of all the terms being taken, give $x-x^2+x^3-\frac{5}{4}x^4+\frac{8}{5}x^5-$ &c, for the fluent sought.

Again, to find the fluent of $\dot{x}\sqrt{1-x^2}$.

Here, by extracting the root, or expanding the radical quantity $\sqrt{1-x^2}$, the given fluxion becomes $\dot{x}-\frac{1}{2}x^2\dot{x}-\frac{1}{8}x^4\dot{x}-\frac{1}{16}x^6\dot{x}-$ &c. Then the fluents of all the terms, being taken, give $x-\frac{1}{6}x^3-\frac{1}{40}x^5-\frac{1}{112}x^7-$ &c, for the fluent sought.

OTHER EXAMPLES.

EXAM. 1. To find the fluent of $\frac{bx\dot{x}}{a-x}$ both in an ascending and descending series.

EXAM. 2. To find the fluent of $\frac{b\dot{x}}{a+x}$ in both series.

EXAM. 3. To find the fluent of $\frac{2\dot{x}}{(a+x)^2}$.

EXAM. 4. To find the fluent of $\frac{1-x^2+2x^4}{1+x-x^2}\dot{x}$.

EXAM. 5. Given $\dot{z}=\frac{b\dot{x}}{a^2+x^2}$, to find z.

EXAM. 6. Given $\dot{z}=\frac{a^2+x^2}{a+x}\dot{x}$ to find z.

EXAM. 7. Given $\dot{z}=3\dot{x}\sqrt{a+x}$, to find z.

EXAM. 8. Given $\dot{z}=2\dot{x}\sqrt{a^2+x^2}$, to find z.

EXAM. 9. Given $\dot{z}=4\dot{x}\sqrt{a^2-x^2}$, to find z.

EXAM. 10. Given $\dot{z}=\frac{5a\dot{x}}{\sqrt{x^2-a^2}}$, to find z.

EXAM. 11. Given $\dot{z}=2\dot{x}\sqrt[3]{a^3-x^3}$, to find z.

EXAM. 12. Given $\dot{z}=\frac{3a\dot{x}}{\sqrt{ax-xx}}$, to find z.

EXAM. 13. Given $\dot{z}=2\dot{x}\sqrt[3]{x^3+x^4+x^5}$, to find z.

EXAM. 14. Given $\dot{z}=5\dot{x}\sqrt{ax-xx}$, to find z.

To Correct the Fluent of any Given Fluxion.

46. The fluxion found from a given fluent, is always perfect and complete; but the fluent found from a given fluxion is not always so; as it often wants a correction, to make it contemporaneous with that required by the problem under consideration, &c: for, the fluent of any given fluxion, as $\dot{x}$ may be either x, which is found by the rule, or it may be $x + c$, or $x - c$, that is x plus or minus some constant quantity c; because both x and $x \pm c$ have the same fluxion $\dot{x}$, and the finding of the constant quantity c, to be added or subtracted with the fluent as found by the foregoing rules, is called *correcting* the fluent.

Now this correction is to be determined from the nature of the problem in hand, by which we come to know the relation which the fluent quantities have to each other at some certain point or time. Reduce, therefore, the general fluential equation, supposed to be found by the foregoing rules, to that point or time; then if the equation be true, it is correct; but if not, it wants a correction; and the quantity of the correction, is the difference between the two general sides of the equation when reduced to that particular point. Hence the general rule for the correction is this:

Connect the constant, but indeterminate, quantity c, with one side of the fluential equation, as determined by the foregoing rules; then, in this equation, substitute for the variable quantities, such values as they are known to have at any particular state, place, or time; and then, from that particular state of the equation, find the value of c, the constant quantity of the correction.

EXAMPLES.

47. Exam. 1. To find the correct fluent of $\dot{z} = ax^3\dot{x}$.

The general fluent is $z = ax^4$, or $z = ax^4 + c$, taking in the correction c.

Now, if it be known that z and x begin together, or that z is $= 0$, when $x = 0$; then writing 0 for both x and z, the general equation becomes $0 = 0 + c$, or $= c$; so that, the value of c being 0, the correct fluents are $z = ax^4$.

But

But if z be $= 0$, when x is $= b$, any known quantity; then substituting 0 for z, and b for x, in the general equation, it becomes $0 = ab^4 + c$, and hence we find $c = -ab^4$; which being written for c in the general fluential equation, it becomes $z = ax^4 - ab^4$, for the correct fluents.

Or, if it be known that z is $=$ some quantity d, when x is $=$ some other quantity as b; then substituting d for z, and b for x, in the general fluential equation $z = ax^4 + c$, it becomes $d = ab^4 + c$; and hence is deduced the value of the correction, namely, $c = d - ab^4$; consequently, writing this value for c in the general equation, it becomes - - - $z = ax^4 - ab^4 + d$, for the correct equation of the fluents in this case.

48. And hence arises another easy and general way of correcting the fluents, which is this: In the general equation of the fluents, write the particular values of the quantities which they are known to have at any certain time or position; then subtract the sides of the resulting particular equation from the corresponding sides of the general one, and the remainders will give the correct equation of the fluents sought.

So, the general equation being $z = ax^4$;
write d for z, and b for x, then $d = ab^4$;
hence, by subtraction, - $z - d = ax^4 - ab^4$,
or $z = ax^4 - ab^4 + d$, the correct fluents as before.

Exam. 2. To find the correct fluents of $\dot{z} = 5x\dot{x}$; z being $= 0$ when x is $= a$.

Exam. 3. To find the correct fluents of $\dot{z} = 3\dot{x}\sqrt{a + x}$; z and x being $= 0$ at the same time.

Exam. 4. To find the correct fluent of $\dot{z} = \frac{2a\dot{x}}{a + x}$; supposing z and x to begin to flow together, or to be each $= 0$ at the same time.

Exam. 5. To find the correct fluents of $\dot{z} = \frac{2\dot{x}}{a^2 + x^2}$; supposing z and x to begin together.

Art. 49.

OF FLUXIONS AND FLUENTS.

ART. 49. In art 42, &c. is given a compendious table of various forms of fluxions and fluents, the truth of which it may be proper here in the first place to prove.

50. As to most of those forms indeed, they will be easily proved, by only taking the fluxions of the forms of fluents, in the last column, by means of the rules before given in art. 30 of the direct method; by which they will be found to produce the corresponding fluxions in the 2d column of the table. Thus, the 1st and 2d forms of fluents will be proved by the 1st of the said rules for fluxions: the 3d and 4th forms of fluents by the 4th rule for fluxions; the 5th and 6th forms, by the 3d rule of fluxions: the 7th, 8th, 9th, 10th, 12th, 14th forms, by the 6th rule of fluxions: the 17th form, by the 7th rule of fluxions: the 18th form, by the 8th rule of fluxions. So that there remains only to prove the 11th, 13th, 15th, and 16th forms.

51. Now, as to the 16th form, that is proved by the 2d example in art. 98, where it appears that $\dot{x}\sqrt{(dx-x^2)}$ is the fluxion of the circular segment, whose diameter is d, and versed sine x. And the remaining three forms, viz, the 11th, 13th, and 15th, will be proved by means of the rectifications of circular arcs, in art. 96.

52. Thus, for the 11th form, it appears by that art. that the fluxion of the circular arc z, whose radius is r and tangent t, is $\dot{z} = \frac{r^2\dot{t}}{r^2+t}$. Now put $t = x^{\frac{1}{2}n}$, or $t^2 = x^n$, and $a = r^2$: then is $\dot{t} = \frac{1}{2}nx^{\frac{1}{2}n-1}\dot{x}$, and $r^2 + t^2 = a + x^n$, and $\dot{z} = \frac{r^2\dot{t}}{r^2+t^2}$ $= \frac{\frac{1}{2}anx^{\frac{1}{2}n-1}\dot{x}}{a+x^n}$; hence $\frac{x^{\frac{1}{2}n-1}\dot{x}}{a+x^n} = \frac{\dot{z}}{\frac{1}{2}an} = \frac{2}{an}\dot{z}$, and the fluent is $\frac{2z}{an} = \frac{2}{na} \times$ arc to radius $\sqrt{a}$ and tang. $x^{\frac{1}{2}n}$ or $= \frac{2}{n\sqrt{a}} \times$ arc to radius 1 and tang. $\sqrt{\frac{x^n}{a}}$, which is the first form of the fluent in n°. XI.

53. And, for the latter form of the fluent in the same n°; because the coefficient of the former of these, viz, $\frac{2}{n\sqrt{a}}$, is double of $\frac{1}{n\sqrt{a}}$ the coefficient of the latter, therefore the arc in the latter case, must be double the arc in the former. But the cosine of double an arc, to radius 1 and tangent t, is $\frac{1-t^2}{1+t^2}$;

$\frac{1-t^2}{1+t^2}$; and because $t^2 = \frac{x^n}{a}$ by the former case, this substituted for t^2 in the cosine $\frac{1-t^2}{1+t^2}$, it becomes $\frac{a-x^n}{a+x^n}$, the cosine as in the latter case of the 11th form.

54. Again, for the first case of the fluent in the 13th form. By art. 61, the fluxion of the circular arc z, to radius r and sine y, is $\dot{z} = \frac{r\dot{y}}{\sqrt{(r^2-y^2)}}$, or $= \frac{\dot{y}}{\sqrt{(1-y^2)}}$ to the radius 1. Now put $y = \sqrt{\frac{x^n}{a}}$, or $y^2 = \frac{x^n}{a}$; hence $\sqrt{(1-y^2)} = \sqrt{(1-\frac{x^n}{a})} = \sqrt{\frac{1}{a}} \times \sqrt{(a-x^n)}$, and $\dot{y} = \sqrt{\frac{1}{a}} \times \frac{1}{2}nx^{\frac{1}{2}n-1}\dot{x}$; then these two being substituted in the value of $\dot{z}$, give $\dot{z}$ or $\frac{\dot{y}}{\sqrt{(1-x^2)}} = \frac{n}{2} \times \frac{x^{\frac{1}{2}n-1}\dot{x}}{\sqrt{(a-x^n)}}$; consequently the given fluxion $\frac{x^{\frac{1}{2}n-1}\dot{x}}{\sqrt{(a-x^n)}}$ is $= \frac{2}{n}\dot{z}$, and therefore its fluent is $\frac{2}{n}z$, that is $\frac{2}{n} \times$ arc to sine $\sqrt{\frac{x^n}{a}}$, as in the table of forms, for the first case of form XIII.

55. And, as the coefficient $\frac{1}{n}$, in the latter case of the said form, is the half of $\frac{2}{n}$ the coefficient in the former case, therefore the arc in the latter case must be double of the arc in the former. But, by trigonometry, the versed sine of double an arc, to sine y and radius 1, is $2y^2$; and, by the former case, $2y^2 = \frac{2x^n}{a}$; therefore $\frac{1}{n} \times$ arc to the versed sine $\frac{2x^n}{a}$ is the fluent, as in the 2d case of form XIII.

56. Again, for the first case of fluent in the 15th form. By art. 61, the fluxion of the circular arc z, to radius r and secant s, is $\dot{z} = \frac{r^2\dot{s}}{s\sqrt{(s^2-r^2)}}$, or $= \frac{\dot{s}}{s\sqrt{(s^2-1)}}$ to radius 1. Now, put $s = \sqrt{\frac{x^n}{a}} = \frac{x^{\frac{1}{2}n}}{\sqrt{a}}$, or $s^2 = \frac{x^n}{a}$; hence $s\sqrt{(s^2-1)} = \frac{x^{\frac{1}{2}n}}{\sqrt{a}}\sqrt{(\frac{x^n}{a}-1)} = \frac{x^{\frac{1}{2}n}}{a}\sqrt{(x^n-a)}$, and $\dot{s} = \sqrt{\frac{1}{a}} \times \frac{1}{2}nx^{\frac{1}{2}n-1}\dot{x}$; then these two being substituted in the value of $\dot{z}$, give $\dot{z}$ or $\frac{\dot{s}}{s\sqrt{(s^2-1)}} = \frac{n\sqrt{a}}{2} \times \frac{x^{-1}\dot{x}}{\sqrt{(x^n-a)}}$; consequently the given fluxion $\frac{x^{-1}\dot{x}}{\sqrt{(x^n-a)}} = \frac{2}{n\sqrt{a}}\dot{z}$, and theref. its fluent is $\frac{2}{n\sqrt{a}}z$, that is $\frac{2}{n\sqrt{a}}$ $\times$ arc

$\times$ arc to secant. $\sqrt{\frac{x^n}{a}}$, as in the table of forms, for the first case of form xv.

57. And, as the coefficient $\frac{1}{n\sqrt{a}}$, in the latter case of the said form, is the half of $\frac{2}{n\sqrt{a}}$ the coefficient of the former case, therefore the arc in the latter case must be double the arc in the former. But, by trigonometry, the cosine of the double arc, to secant s and radius 1, is $\frac{2}{s^2} - 1$; and, by the former case, $\frac{2}{s^2} - 1 = \frac{2a}{x^n} - 1 = \frac{2a - x^n}{x^n}$; therefore $\frac{1}{n\sqrt{a}} \times$ arc to cosine $\frac{2a - x^n}{x^n}$ is the fluent, as in the 2d case of form xv.

Or, the same fluent will be $\frac{2}{n\sqrt{a}} \times$ arc to cosine $\sqrt{\frac{a}{x^n}}$, because the cosine of an arc, is the reciprocal of its secant.

58. It has been just above remarked, that several of the tabular forms of fluents are easily shown to be true, by taking the fluxions of those forms, and finding they come out the same as the given fluxions. But they may also be determined in a more direct manner, by the transformation of the given fluxions to another form. Thus, omitting the first form, as too evident to need any explanation, the 2d form is $\dot{z} = (a + x^n)^{m-1} x^{n-1} \dot{x}$, where the exponent $(n - 1)$ of the unknown quantity without the vinculum, is 1 less than (n) that under the same. Here, putting $y =$ the compound quantity $a + x^n$: then is $\dot{y} = nx^{n-1}\dot{x}$, and $\dot{z} = \frac{y^{m-1}\dot{y}}{n}$; hence, by art. 36, $z = \frac{y^m}{mn} = \frac{(a + x^n)^m}{mn}$ as in the table.

59. By the above example it appears, that such form of fluxion admits of a fluent in finite terms, when the index $(n - 1)$ of the variable quantity (x) without the vinculum, is less by 1 than n, the index of the same quantity under the vinculum. But it will also be found, by a like process, that the same thing takes place in such forms as $(a + x^n)^m x^{cn-1} \dot{x}$, where the exponent $(cn - 1)$ without the vinculum, is 1 less than any multiple (c) of that (n) under the vinculum. And further, that the fluent, in each case, will consist of as many terms as are denoted by the integer number c; viz, of one term when $c = 1$, of two terms when $c = 2$, of three terms when $c = 3$, and so on.

60. Thus, in the general form, $\dot{z} = (a + x^n)^m x^{cn-1} \dot{x}$, putting as before, $a + x^n = y$; then is $x^n = y - a$, and its

fluxion

fluxion $nx^{n-1}\dot{x} = \dot{y}$, or $x^{n-1}\dot{x} = \frac{\dot{y}}{n}$, and $x^{cn-1}\dot{x}$ or x^{cn-n} $x^{n-1}\dot{x} = \frac{1}{n}(y-a)^{c-1}\dot{y}$; also $(a + x^n)^m = y^m$: these values being now substituted in the general form proposed, give $\dot{z} = \frac{1}{n}(y-a)^{c-1}y^m\dot{y}$. Now, if the compound quantity $(y-a)^{c-1}$ be expanded by the binomial theorem, and each term multiplied by $y^m\dot{y}$, that fluxion becomes

$\dot{z} = \frac{1}{n}(y^{m+c-1}\dot{y} - \frac{c-1}{1}ay^{m+c-2}\dot{y} + \frac{c-1}{1}\cdot\frac{c-2}{2}a^2y^{m+c-3}\dot{y} -$ &c); then the fluent of every term, being taken by art. 36, it is

$$z = \frac{1}{n}\left(\frac{y^{m+c}}{m+c} - \frac{c-1}{1}\cdot\frac{ay^{m+c-1}}{m+c-1} + \frac{c-1}{1}\cdot\frac{c-2}{2}\cdot\frac{a^2y^{m+c-2}}{m+c-2} - \&c\right),$$

$$= \frac{y^d}{n}\left(\frac{1}{d} - \frac{c-1}{d-1}\cdot\frac{a}{y} + \frac{c-1\,.\,c-2}{d-2}\cdot\frac{a^2}{2y^2} - \frac{c-1\,.\,c-2\,.\,c-3}{d-3}\cdot\frac{a^3}{2.3y^3}\ \&c\right),$$

putting $d = m + c$, for the general form of the fluent; where, c being a whole number, the multipliers $c-1$, $c-2$, $c-3$, &c, will become equal to nothing, after the first c terms, and therefore the series will then terminate, and exhibit the fluent in that number of terms; viz, there will be only the first term when $c = 1$, but the first two terms when $c = 2$, and the first three terms when $c = 3$, and so on.— Except however the cases in which m is some negative number equal to or less than c; in which cases the divisors, $m + c$, $m + c - 1$, $m + c - 2$, &c, becoming equal to nothing, before the multipliers $c-1$, $c-2$, &c, the corresponding terms of the series, being divided by 0, will be infinite: and then the fluent is said to fail, as in such case nothing can be determined from it.

61. Besides this form of the fluent, there are other methods of proceeding, by which other forms of fluents are derived, of the given fluxion $\dot{z} = (a + x^n)^m x^{cn-1}\dot{x}$, which are of use when the foregoing form fails, or runs into an infinite series; some results of which are given both by Mr. Simpson and Mr. Landen. The two following processes are after the manner of the former author.

62. The given fluxion being $(a + x^n)^m x^{cn-1}\dot{x}$; its fluent may be assumed equal to $(a + x^n)^{m+1}$ multiplied by a general series, in terms of the powers of x combined with assumed unknown co-efficients, which series may be either ascending or descending, that is, having the indices either increasing or decreasing;

viz, $(a+x^n)^{m+1} \times (Ax^r + Bx^{r-s} + Cx^{r-2s} + Dx^{r-3s} + \&c)$,
or $(a+x^n)^{m+1} \times (Ax^r + Bx^{r+s} + Cx^{r+2s} + Dx^{r+3s} + \&c)$.

And

And first, for the former of these, take its fluxion in the usual way, which put equal to the given fluxion $(a+x^n)^m x^{cn-1}\dot{x}$, then divide the whole equation by the factors that may be common to all the terms; after which, by comparing the like indices and the coefficients of the like terms, the values of the assumed indices and coefficients will be determined, and consequently the whole fluent. Thus, the former assumed series in fluxions is,

$$n(m+1)x^{n-1}\dot{x}(a+x^n)^m \times (\mathrm{A}x^r + \mathrm{B}x^{r-s} + \mathrm{C}x^{r-2s}\ \&\text{c.}) + (a+x^n)^{m+1}\dot{x} \times (r\mathrm{A}x^{r-1} + (r-s)\,\mathrm{B}x^{r-s-1} + (r-2s)\,\mathrm{C}x^{r-2s-1}$$

&c); this being put equal to the given fluxion $(a+x^n)^m x^{cn-1}\dot{x}$, and the whole equation divided by $(a+x^n)^m x^{r-1}\dot{x}$, there results

$$\left.\begin{array}{l} n(m+1)x^n \times (\mathrm{A}x^r + \mathrm{B}x^{r-s} + \mathrm{C}x^{r-2s} + \mathrm{D}x^{r-3s} + \&\text{c}) \\ +(a+x^n) \times (r\mathrm{A}x^r + (r-s)\mathrm{B}x^{r-s} + (r-2s)\mathrm{C}x^{r-2s}\&\text{c}) \end{array}\right\} = x^{cn}.$$

Hence, by actually multiplying, and collecting the coefficients of the like powers of x, there results

$$\left.\begin{array}{l} \left.\begin{array}{r} z(m+1) \\ +r \end{array}\right\} \mathrm{A}x^{r+n} \left.\begin{array}{r} +n(m+1) \\ +r-s \end{array}\right\} \mathrm{B}x^{r+n-s+n(m+1)} \left.\begin{array}{r} \\ +r-2s \end{array}\right\} \mathrm{C}x^{r+2n-2s}\&\text{c} \\ \quad -x^{cn} \ldots + \ldots\ ra\mathrm{A}x^r \ldots + (r-s)a\mathrm{B}x^{r-s}\ \&\text{c} \end{array}\right\} = 0.$$

Here, by comparing the greatest indices of x, in the first and second terms, it gives $r+n=cn$, and $r+n-s=r$; which give $r=(c-1)n$, and $n=s$. Then these values being substituted in the last series, it becomes

$$\left.\begin{array}{l} (c+m)n\mathrm{A}x^{cn} + (c+m-1)n\mathrm{B}x^{cn-n} + (c+m-2)n\mathrm{C}x^{cn-2n}\&\text{c} \\ \quad -x^{cn} + (c-1)na\mathrm{A}x^{cn-n} + (c-2)na\mathrm{B}x^{cn-2n}\ \&\text{c} \end{array}\right\} = 0.$$

Now, comparing the coefficients of the like terms, and putting $c+m=d$, there result these equalities:

$$\mathrm{A} = \frac{1}{dn};\quad \mathrm{B} = -\frac{c-1\,.\,a\mathrm{A}}{d-1} = -\frac{c-1\,.\,a}{d-1\,.\,dn};\quad \mathrm{C} = -\frac{c-2\,.\,a\mathrm{B}}{d-2} = +\frac{c-1\,.\,c-2\,.\,a^2}{d-1\,.\,d-2\,.\,dn};$$

&c; which values of A, B, C, &c, with those of r and s, being now substituted in the first assumed fluent, it becomes

$$\frac{(a+x^n)^{m+1}x^{cn-n}}{dn} \times \left(\frac{1}{1} - \frac{c-1\,.\,a}{d-1\,.\,x^n} + \frac{c-1\,.\,c-2\,.\,a^2}{d-1\,.\,d-2\,.\,x^{2n}} - \frac{c-1\,.\,c-2\,.\,c-3\,.\,a^3}{d-1\,.\,d-2\,.\,d-3\,.\,x^{3n}}\right.$$

$+$ &c, the true fluent of $(a+x^n)^m x^{cn-1}\dot{x}$, exactly agreeing with the first value of the 19th form in the table of fluents in my Dictionary. Which fluent therefore, when c is a whole positive number, will always terminate in that number of terms; subject to the same exception as in the former case. Thus, if $c=2$, or the given fluxion be $(a+x^n)^m x^{2n-1}\dot{x}$; then, $c+m$ or d being $= m+2$, the fluent becomes

$$\frac{(a+x^n)^{m+1}x^n}{(m+2)n} \times \left(1 - \frac{ax^{-n}}{m+1}\right) = \frac{(a+x^n)^{m+1}}{n} \times \frac{(m+1)x^n - a}{m+1\,.\,m+2}.$$

And if $c=3$, or the given fluxion be $(a+x^n)^m x^{3n-1}x$; then $m+c$ or d being $= m+3$, the fluent becomes

$$\frac{(a+x^n)^{m+1}x^{2n}}{(m+3)n} \times \left(1 - \frac{2ax^{-n}}{m+2} + \frac{2a^2x^{-2n}}{m+2\,.\,m+1}\right) = \frac{(a+x^n)^{m+1}}{n} \times \left(\frac{x^{2n}}{m+3} - \right.$$

$\frac{2ax^n}{m+3.m+2}+\frac{2a^2}{m+3.m+2.m+1}$). And so on, when c is other whole numbers: but, when c denotes either a fraction or a negative number, the series will then be an infinite one, as none of the multipliers $c-1$, $c-2$, $c-3$, can then be equal to nothing.

63. Again, for the latter or ascending form, $(a+x^n)^{m+1}\times(\text{A}x^r+\text{B}x^{r+s}+\text{C}x^{r+2s}+\text{D}x^{r+3s}+\&c)$, by making its fluxion equal to the proposed one, and dividing, &c, as before, equating the two least indices, &c, the fluent will be obtained in a different form, which will be useful in many cases, when the foregoing one fails, or runs into an infinite series. Thus, if $r+s$, $r+2s$, &c, be written instead of $r-s$, $r-2s$, &c, respectively, in the general equation in the last case, and taking the first term of the 2d line into the first line, there results

$$\left.\begin{array}{l}\left.\begin{array}{c}-x^{cn}+n(m+1)\\+r\end{array}\right\}\text{A}x^{r+n}\left.\begin{array}{c}+n(m+1)\\+r+s\end{array}\right\}\text{B}x^{r+n+s}\ \&c\\+ra\text{A}x^r+(r+s)a\text{B}x^{r+s}+(r+2s)a\text{C}x^{r+2s}\ \&c\end{array}\right\}=0.$$

Here, comparing the two least pairs of exponents, and the coefficients, we have $r=cn$, and $s=n$; then $\text{A}=\frac{1}{ra}=\frac{1}{cna}$; $\text{B}=-\frac{r+n(m+1)}{a(r+s)}$; $\text{A}=-\frac{c+m+1}{c+1}\cdot\frac{\text{A}}{a}=-\frac{c+m+1}{(c+1)cna^2}$; $\text{C}=-\frac{c+m+2}{(c+2)a}\text{B}=+\frac{c+m+1.c+m+2}{c.c+1.c+2.na^3}$ &c. Therefore, denoting $c+m$ by d, as before, the fluent of the same fluxion $(a+x^n)^m x^{cn-1}\dot{x}$, will also be truly expressed by

$$\frac{(a+x^n)^{m+1}x^{cn}}{cna}\times\left(\frac{1}{1}-\frac{d+1.x^n}{c+1.a}+\frac{d+1.d+2.x^{2n}}{c+1.c+2.a^2}-\&c\right);$$

agreeing with the 2d value of the fluent of the 19th form in my Dictionary. Which series will terminate when d or $c+m$ is a negative integer; except when c is also a negative integer less than d; for then the fluent fails, or will be infinite, the divisor in that case first becoming equal to nothing.

To show now the use of the foregoing series, in some example of finding fluents, take first,

64. *Example* 1. To find the fluent of $\frac{6x\dot{x}}{\surd(a+x)}$ or $6x\dot{x}(a+x)^{-\frac{1}{2}}$.

This example being compared with the general form $x^{cn-1}\dot{x}(a+x^n)^m$, in the several corresponding parts of the first series, gives these following equalities: viz, $a=a$, $n=1$, $cn-1=1$, or $c-1=1$, or $c=2$; $m=-\frac{1}{2}$; $y=a+x$,

$d=m$

$d = m + c = 2 - \frac{1}{2} = \frac{3}{2}$, $\frac{1}{n}y^d = (a + x)^{\frac{3}{2}}$, $\frac{1}{d} = \frac{2}{3}$, $\frac{c-1}{d-1} \cdot \frac{a}{y} = \frac{2a}{a+x}$; here the series ends, as all the terms after this become equal to nothing, because the following terms contain the factor $c - 2 = 0$. These values then being substituted in $\frac{y^d}{n}\left(\frac{1}{d} - \frac{c-1}{d-1} \cdot \frac{a}{y}\right)$, it becomes $(a + x)^{\frac{3}{2}} \times \left(\frac{2}{3} - \frac{2a}{a+x}\right) = \left(\frac{2a+2x}{3} - 2a\right) \times (a + x)^{\frac{1}{2}} = \frac{2x-4a}{3}\sqrt{(a+x)}$; which multiplied by 6, the given coefficient in the proposed example, there results $(4x - 8a) . \sqrt{(a + x)}$, for the fluent required.

17. *Exam.* 2. To find the fluent of

$$\frac{3\dot{x}\sqrt{(a^2+x^2)}}{x^6} = (a^2 + x^2)^{\frac{1}{2}} \times 3x^{-6}\dot{x}.$$

The several parts of this quantity being compared with the corresponding ones of the general form, give $a = a^2$, $n = 2$, $m = \frac{1}{2}$, $cn - 1$ or $2c - 1 = -6$, whence $c = \frac{1-6}{2} = -\frac{5}{2}$, and $d = m + c = \frac{1}{2} - \frac{5}{2} = -\frac{4}{2} = -2$, which being a negative integer, the fluent will be obtained by the 3d or last form of series; which on substituting these values of the letters, gives $\frac{3(a^2+x^2)^{\frac{3}{2}}x^{-5}}{-5a^2} \times \left(\frac{1}{1} - \frac{-1 . x^2}{-\frac{3}{2}a^2}\right) = \frac{3(a^2+x^2)^{\frac{3}{2}}}{-5a^2x^5} \times \left(1 - \frac{2x^2}{3a^2}\right) = \frac{(a^2+x^2)^{\frac{3}{2}}}{x^5} \times \frac{2x^2-3a^2}{5a^4}$ for the required fluent of the proposed fluxion.

66. *Exam.* 3. Let the fluxion proposed be

$$\frac{5x^{3n-1}\dot{x}}{\sqrt{(b+x^n)}} = 5(b + x^n)^{-\frac{1}{2}}x^{3n-1}\dot{x}.$$

Here, by proceeding as before, we have $a = b$, $n = n$, $m = -\frac{1}{2}$, $c = 3$, and $d = c + m = \frac{5}{2}$; where c being a positive integer, this case belongs to the 2d series; into which therefore the above values being substituted, it becomes $\frac{5(b+x^n)^{\frac{5}{2}}x^{2n}}{\frac{5}{2}n} \times \left(\frac{1}{1} - \frac{2b}{\frac{3}{2}x^n} + \frac{2 . 1 . b^2}{\frac{3}{2} . \frac{1}{2}x^{2n}}\right) = 2\sqrt{(b+x^n)} \times \frac{3x^{2n}-4bx^n+8b^2}{3n}$.

67. *Exam.* 4. Let the proposed fluxion be $5\left(\frac{1}{3}-z^2\right)^{\frac{1}{2}}z^{-8}\dot{z}$.

Here, proceeding as above, we have $a = \frac{1}{3}$, $n = 2$, $m = \frac{1}{2}$, $cn - 1$ or $2c - 1 = -8$, and $c = -\frac{7}{2}$, $x = -z$, $d = c + m = -3$, which being a negative integer, the case belongs to the 3d or last series; which therefore, by substituting

these

these values, becomes $\frac{5(\frac{1}{3}-z^2)^{\frac{5}{2}}}{-7\,.\,\frac{1}{3}z^7}\times(\frac{1}{1}+\frac{-2z^2}{-\frac{5}{2}.\frac{1}{3}}+\frac{-2\,.\,-1\,.\,z^4}{-\frac{5}{2}.-\frac{3}{2}\,.\,\frac{1}{9}}=$

$\frac{15(\frac{1}{3}-z^2)^{\frac{5}{2}}}{-7z^7}\times(1+\frac{12z^2}{5}+\frac{24z^4}{5})=\frac{-3(\frac{1}{3}-z^2)^{\frac{5}{2}}}{7z^7}\times(5+12z^2+24z^4)$,

the true fluent of the proposed fluxion. And thus may many other similar fluents be exhibited in finite terms, as in these following examples for practice.

Ex. 5. To find the fluent of $-3x^2\dot{x}\sqrt{(a^2-x^2)}$.

Ex. 6. To find the fluent of $-6x^5\dot{x}\,.\,(a^2-x^2)^{-\frac{3}{2}}$

Ex. 7. To find the flu. of $\frac{\dot{x}\sqrt{(a-x^n)}}{x^{\frac{1}{2}n-1}}$ or $(a-x^n)^{\frac{1}{2}}x^{-\frac{1}{2}n+1}\dot{x}$.

68. The case mentioned in art. 37, viz, of compound quantities under the vinculum, the fluxion of which is in a given ratio to the fluxion without the vinculum, with only one variable letter, will equally apply when the compound quantities consist of several variables. Thus,

Example 1. The given fluxion being $(4x\dot{x}+8y\dot{y})\times\sqrt{(x^2+2y^2)}$, or $(4x\dot{x}+8y\dot{y})\times(x^2+2y^2)^{\frac{1}{2}}$, the root being x^2+2y^2, the fluxion of which is $2x\dot{x}+4y\dot{y}$. Dividing the former fluxional part by this fluxion, gives the quotient 2: next, the exponent $\frac{1}{2}$ increased by 1, gives $\frac{3}{2}$: lastly, dividing by this $\frac{3}{2}$, there then results $\frac{4}{3}(x^2+2y^2)^{\frac{3}{2}}$, for the required fluent of the proposed fluxion.

Exam. 2. In like manner, the fluent of

$$(x^2+y^4+z^6)^{\frac{1}{3}}\times(6x\dot{x}+12y^3\dot{y}+18z^5\dot{z})\text{ is}$$

$$\frac{(x^2+y^4+z^6)^{\frac{1}{3}+1}\times(6x\dot{x}+12y^3\dot{y}+18z^5\dot{z})}{(2x\dot{x}+4y^3\dot{y}+6z^5\dot{z})\times\frac{4}{3}}=\frac{9}{4}(x^2+y^4+z^6)^{\frac{4}{3}}.$$

Exam. 3. In like manner, the fluent of

$2x^2(\dot{x}y^2+xy\dot{y}+x^2\dot{x})\sqrt{(x^2+2y^2)}$, is $\frac{1}{5}(x^4+2x^2y^2)^{\frac{3}{2}}$.

69. The fluents of fluxions of the forms $\frac{x^n\dot{x}}{x\pm a}$, $\frac{x^n\dot{x}}{x^2\pm a^2}$, &c, or $\frac{x^{cn-1}\dot{x}}{x^n\pm a^n}$, &c, where c and n are whole numbers, will be found in finite terms, by dividing the numerator by the denominator, using the variable letter x as the first term in the divisor, continuing the division till the powers of x are exhausted; after which, the last remainder will be the fluxion of a logarithm, or of a circular arc, &c.

Exam. 1. To find the fluent of $\frac{x\dot{x}}{a+x}$ or $\frac{x\dot{x}}{x+a}$.

By

By division, $\frac{x\dot{x}}{x+a} = \dot{x} - \frac{a\dot{x}}{x+a}$, where the remainder $\frac{a\dot{x}}{x+a}$ is evidently $= a \times$ the fluxion of the hyperbolic logarithm of $a + x$: therefore the whole fluent of the proposed fluxion is $x - a \times$ hyp. log. of $(a + x)$. In like manner it will be found that,

Ex. 2. The fluent of $\frac{x\dot{x}}{x-a}$, is $x + a \times$ hyp. log. of $(x - a)$.

Ex. 3. The fluent of $\frac{x\dot{x}}{a-x}$, is $-x - a \times$ hyp. log. of $(a - x)$.

Ex. 4. The fluent of $\frac{x^2\dot{x}}{a+x}$, is $\frac{1}{2}x^2 - ax + a^2 \times$ hyp. log. of $(a + x)$.

Ex. 5. The fluent of $\frac{x^2\dot{x}}{a-x}$, is $-\frac{1}{2}x^2 - ax - a^2 \times$ hyp. log. of $(a - x)$.

Ex. 6. The fluent of $\frac{x^2\dot{x}}{x-a}$, is $\frac{1}{2}x^2 + ax + a^2 \times$ hyp. log. of $(x - a)$.

Ex. 7. The fluent of $\frac{x^3\dot{x}}{x+a}$, is $\frac{1}{3}x^3 - \frac{1}{2}ax^2 + a^2x - a^3 \times$ hyp. log. of $(x + a)$.

Ex. 8. The fluent of $\frac{x^3\dot{x}}{x-a}$, is $\frac{1}{3}x^3 + \frac{1}{2}ax^2 + a^2x + a^3 \times$ hyp. log. of $(x - a)$.

Ex. 9. The fluent of $\frac{x^3\dot{x}}{a-x}$, is $-\frac{1}{3}x^3 - \frac{1}{2}ax^2 - a^2x + a^3 \times$ hyp. log. of $(a - x)$.

Ex. 10. The fluent of $\frac{x^4\dot{x}}{a+x}$, is $\frac{1}{4}x^4 - \frac{1}{3}ax^3 + \frac{1}{2}a^2x^2 - a^3x + a^4 \times$ hyp. log. $(a + x)$.

Ex. 11. The fluent of $\frac{x^n\dot{x}}{a+x}$, is $\frac{x^n}{n} - \frac{ax^{n-1}}{n-1} + \frac{a^2x^{n-2}}{n-2} - \frac{a^3x^{n-3}}{n-3} +$ &c $\pm a^n \times$ h. l. $(a+x)$.

Ex. 12. The fluent of $\frac{x^n\dot{x}}{a-x}$, is $-\frac{x^n}{n} - \frac{ax^{n-1}}{n-1} - \frac{a^2x^{n-2}}{n-2} - \frac{a^3x^{n-3}}{n-3}$ &c $- a^n \times$ h. l. $(a - x)$.

Ex. 13. The fluent of $\frac{x^n\dot{x}}{x-a}$, is $\frac{x^n}{n} + \frac{ax^{n-1}}{n-1} + \frac{a^2x^{n-2}}{n-2} + \frac{a^3x^{n-3}}{n-3}$ &c $+ a^n \times$ h. l. $(x - a)$.

Ex. 14. The fluent of $\frac{x^2\dot{x}}{x^2+a^2}$ = (by division) $\dot{x} - \frac{a^2\dot{x}}{x^2+a^2}$,

is

is, (by form 11 this vol.) x — cir. arc of radius a and tang. x or $x - \frac{1}{2}a \times$ cir. arc of rad. 1 and cosine $\frac{a^2-x^2}{a^2+x^2}$. In like manner,

Ex. 15. The fluent of $\frac{x^2\dot{x}}{a^2-x^2}$, or of $-\dot{x}+\frac{a^2\dot{x}}{a^2-x^2}$, is $-x+\frac{1}{2}a \times$ h. l. $\frac{a+x}{a-x}$, by form 10. And

Ex. 16. The fluent of $\frac{x^2\dot{x}}{x^2-a^2} = \dot{x} + \frac{a^2\dot{x}}{x^2-a^2}$, is $x+\frac{1}{2}a \times$ hyp. log. $\frac{x-a}{x+a}$, by the same form.

70. In like manner for the fluents of $\frac{x^4\dot{x}}{a^2 \pm x^2}$. Thus,

Ex. 17. The fluent of $\frac{x^4\dot{x}}{a^2+x^2} = x^2\dot{x} - a^2\dot{x} + \frac{a^4\dot{x}}{a^2+x^2}$, is by form, $\frac{1}{3}x^3 - a^2x + a^2 \times$ cir. arc to rad. a and tang. x, or $\frac{1}{3}x^3 - a^2x + \frac{1}{2}a^3 \times$ cir. arc to rad 1 and cosine $\frac{a^2-x^2}{a^2+x^2}$. And

Ex. 18. The fluent of $\frac{x^4\dot{x}}{a^2-x^2}, = -x^2\dot{x} - a^2\dot{x} + \frac{a^4\dot{x}}{a^2-x^2}$, is $-\frac{1}{3}x^3 - a^2x + \frac{1}{2}a^3 \times$ hyp. log. $\frac{a+x}{a-x}$, by form 10. Also

Ex. 19. The fluent of $\frac{x^4\dot{x}}{x^2-a^2} = x^2\dot{x} + a^2\dot{x} + \frac{a^4\dot{x}}{x^2-a^2}$, is $\frac{1}{3}x^3 + a^2x + \frac{1}{2}a^3 \times$ hyp. log. $\frac{x-a}{x+a}$, by form 10.

71. And in general for the fluent of $\frac{x^n\dot{x}}{x^2 \pm a^2}$, where n is any even positive number, by dividing till the powers of x in the numerator are exhausted, the fluents will be found as before. And first for the denominator x^2+a^2, as in

Ex. 20. For the fluent of $\frac{x^n\dot{x}}{x^2+a^2} =$ (by actual division) $x^{n-2}\dot{x} - a^2x^{n-4}\dot{x} + a^4x^{n-6} - \&c \pm a^{n-2}\dot{x} \mp \frac{a^n\dot{x}}{x^2+a^2}$; the number of terms in the quotient being $\frac{1}{2}n$, and the remainder $\mp \frac{a^4\dot{x}}{x^2+a^2}$, viz, — or + according as that number of terms is odd or even. Hence, as before, the fluent is $\frac{x^{n-1}}{n-1} - \frac{a^2x^{n-3}}{n-3} + \&c \ldots \pm a^{n-2}x \mp a^{n-2} \times$ arc to rad. a and tan x, or $\frac{x^{n-1}}{n-1} - \frac{a^2x^{n-3}}{n-3} + \&c \ldots \pm a^{n-2}x \mp \frac{1}{2}a^{n-1} \times$ arc to rad. 1 and cos. $\frac{a^2-x^2}{a^2+x^2}$.

Ex. 21.

Ex. 21. In like manner, the fluent of $\frac{x^n\dot{x}}{a^2-x^2}$, is $-\frac{x^{n-1}}{n-1}-\frac{a^2x^{n-3}}{n-3}-\frac{a^4x^{n-5}}{n-5}-$ &c $+\frac{1}{2}a^{n-1}\times$ hyp. log. $\frac{a+x}{a-x}$.

Ex. 22. And of $\frac{x^n\dot{x}}{x^2-a^2}$, is $\frac{x^{n-1}}{n-1}+\frac{a^2x^{n-3}}{n-3}+$ &c $+\frac{1}{2}a^{n-1}\times$ hyp. log. $\frac{x-a}{x+a}$.

72. In a similar manner we are to proceed for the fluents of $\frac{x^n\dot{x}}{a^2\pm x^2}$, when n is any odd number, by dividing by the denominator inverted, till the first power of x only be found in the remainder, and when of course there will be one term less in the quotient than in the foregoing case, when n was an even number; but in the present case the log. fluent of the remainder will be found by the 8th form in the table of fluents.

Ex. 22. Thus, for the fluent of $\frac{x^n\dot{x}}{x^2+a^2}$, where n is an odd number, the quotient by division as before, is $x^{n-2}\dot{x}-a^2x^{n-4}\dot{x}+a^4x^{n-6}\dot{x}-$ &c $\pm a^{n-3}x\dot{x}$, the number of terms being $\frac{m-1}{2}$, and the remainder $\mp\frac{a^{n-1}x\dot{x}}{x^2+a^2}$. Therefore the fluent is $\frac{x^{n-1}}{n-1}-\frac{a^2x^{n-3}}{n-3}+$ &c ... $\pm\frac{a^{n-3}x^2}{2}\mp\frac{1}{2}a^{n-1}\times$ h. l. x^2+a^2.

Ex. 23. The fluent of $\frac{x^n\dot{x}}{x^2-a^2}$ is obtained in the same manner, and has the same terms, but the signs are all positive, and the remainder is $+\frac{1}{2}a^{n-1}\times$ hyp. log. x^2-a^2.

Ex. 24. Also the fluent of $\frac{x^n\dot{x}}{a^2-x^2}$ is still the same, but the signs are all negative, and the remainder is $-\frac{1}{2}a^{n-1}\times$ hyp. log. a^2-x^2. Hence also,

Ex. 25. The fluent of $\frac{x^3\dot{x}}{x^2+a^2}$, is $\frac{1}{2}x^2-\frac{1}{2}a^2\times$ hyp. log. of x^2+a^2.

Ex. 26. The fluent of $\frac{x^3\dot{x}}{x^2-a^2}$, is $\frac{1}{2}x^2+\frac{1}{2}a^2\times$ hyp. log. of x^2-a^2.

Ex. 27. The fluent of $\frac{x^3\dot{x}}{a^2-x^2}$, is $-\frac{1}{2}x^2-\frac{1}{2}a^2\times$ hyp. log. of a^2-x^2.

Ex. 28. The fluent of $\frac{x^5\dot{x}}{x^2+a^2}$, is $\frac{1}{4}x^4-\frac{1}{2}a^2x^2+\frac{1}{2}a^4\times$ hyp. log. x^2+a^2.

Ex 29.

Ex. 29. The fluent of $\frac{x^5\dot{x}}{x^2-a^2}$, is $\frac{1}{4}x^4 + \frac{1}{2}a^2x^2 + \frac{1}{2}a^4 \times$ hyp. log. $x^2 - a^2$.

Ex. 30. The fluent of $\frac{x^5\dot{x}}{a^2-x^2}$, is $-\frac{1}{4}x^4 - \frac{1}{2}a^2x^2 - \frac{1}{2}a^4 \times$ hyp. log $a^2 - x^2$.

73. *Ex.* 31. In a similar manner may be found the fluents of $\frac{x^{cn-1}\dot{x}}{x^n \pm a^n}$, where c is any whole positive number, by dividing till the remainder be $\frac{a^{(c-1)n}x^{n-1}\dot{x}}{x^n \pm a^n}$, which can always be done, and the fluent of that remainder will be had by the 8th form in this vol. Thus, by dividing first by $x^n + a^n$, the terms are, $x^{cn-n-1}\dot{x} - a^n x^{cn-2n-1}\dot{x} + a^{2n}x^{cn-3n-1}\dot{x} - +$ &c till the last term be $a^{(d-1)n}x^{(c-d)n-1}$, and the remainder $\frac{a^{dn}x^{(c-d)n-1}\dot{x}}{x^n+a^n} = \frac{a^{(c-1)n}x^{n-1}\dot{x}}{x^n+a^n}$ when d is $= c - 1$, or 1 less than c, which is also the number of the terms in the quotient; and therefore the fluent is

$$\frac{x^{cn-n}}{cn-n} - \frac{a^n x^{cn-2n}}{cn-2n} + \frac{a^{2n}x^{cn-3n}}{cn-3n} \ldots \pm \frac{a^{(c-2)n}x^n}{n} \mp \frac{1}{n}a^{(c-1)n} \times$$

hyp. log. of $x^n + a^n$. In like manner,

Ex. 32. The fluent of $\frac{x^{cn-1}\dot{x}}{x^n - a^n}$ has all the same terms as the former, but their signs all $+$ or positive, and the remainder $\frac{1}{n}a^{(c-1)n} \times$ hyp. log. of $x^n - a^n$. Also in like manner

Ex. 33. The fluent of $\frac{x^{cn-1}\dot{x}}{a^n - x^n}$ has all the very same terms; but all negative, and the remainder $-\frac{1}{n}a^{(c-1)n} \times$ hyp. log. of $a^n - x^n$

Ex. 34. The fluent of $\frac{x^{cn-1}\dot{x}}{b \pm ex^n} = \frac{1}{e} \times \frac{x^{cn-1}\dot{x}}{\frac{b}{e} \pm x^n}$ is also the same with the preceding, by substitut. $\frac{b}{e}$ for a^n, and multiplying the whole series by the fraction $\frac{1}{e}$.

74. When the numerator is compound, as well as the denominator, the expression may, in a similar manner by division, be reduced to like terms admitting of finite fluents. Thus, for

Ex. 35. To find the fluent of $\frac{a-bx}{c+dx^2} \times x\dot{x} = \frac{ax\dot{x} - bx^2\dot{x}}{c+dx^2}$. By

By division this becomes $-\frac{b}{d}x\dot{x}+\frac{ad+bc}{dd}\times\frac{x\dot{x}}{\frac{c}{d}+x^2}$; and its fluent $-\frac{b}{2d}x^2+\frac{ad+bc}{2d^2}\times$ hyp. log. of $\frac{c}{d}+x^2$.

75. There are certain methods of finding fluents one from another, or of deducing the fluent of a proposed fluxion from another fluent previously known or found. There are hardly any general rules however that will suit all cases; but they mostly consist in assuming some quantity y in the form of a rectangle or product of two factors, which are such, that the one of them drawn into the fluxion of the other may be of the form of the proposed fluxion; then taking the fluxion of the assumed rectangle, there will thence be deduced a value of the proposed fluxion in terms that will often admit of finite fluents. The manner in such cases will better appear from the following examples.

Ex. 1. To find the fluent of $\frac{x^2\dot{x}}{\sqrt{(x^2+a^2)}}$.

Here it is obvious that if y be assumed $= x\sqrt{(x^2+a^2)}$, then one part of the fluxion of this product, viz, $x\times$ flux. of $\sqrt{(x^2+a^2)}$, will be of the same form as the fluxion proposed. Putting theref. the assumed rectangle $y=x\sqrt{(x^2+a^2)}$ into fluxions, it is $\dot{y}=\dot{x}\sqrt{(x^2+a^2)}+\frac{x^2\dot{x}}{\sqrt{(x^2+a^2)}}$. But as the former part, viz, $\dot{x}\sqrt{(x^2+a^2)}$, does not agree with any of our preceding forms, which have been integrated, multiply it by $\sqrt{(x^2+a^2)}$, and subscribe the same as a denominator to the product, by which that part becomes $\frac{x^2+a^2}{\sqrt{(x^2+a^2)}}\dot{x}=\frac{x^2\dot{x}+a^2\dot{x}}{\sqrt{(x^2+a^2)}}$; this united with the former part, makes the whole $\dot{y}=\frac{2x^2\dot{x}}{\sqrt{(x^2+a^2)}}+\frac{a^2\dot{x}}{\sqrt{(x^2+a^2)}}$; hence the given fluxion $\frac{x^2\dot{x}}{\sqrt{(x^2+a^2)}}=\frac{1}{2}\dot{y}-\frac{1}{2}a^2\times\frac{\dot{x}}{\sqrt{(x^2+a^2)}}$, and its fluent is therefore $\frac{1}{2}y-\frac{1}{2}a^2\times f\frac{\dot{x}}{\sqrt{(x^2+a^2)}}=\frac{1}{2}x\sqrt{(x^2+a^2)}-\frac{1}{2}a^2\times$ hyp. log. of $x+\sqrt{(x^2+a^2)}$, by the 12th form of fluents.

Ex. 2. In like manner the fluent of $\frac{x^2\dot{x}}{\sqrt{(x^2-a^2)}}$ will be found from that of $\frac{\dot{x}}{\sqrt{(x^2-a^2)}}$ by the same 12th form, and is $=\frac{1}{2}x\sqrt{(x^2-a^2)}+\frac{1}{2}a^2\times$ hyp. log $x+\sqrt{(x^2-a^2)}$.

Ex. 3. Also in a similar manner, by the 13th form, the fluent

fluent of $\frac{x^2\dot{x}}{\sqrt{(a^2-x^2)}}$ will be found from that of $\frac{\dot{x}}{\sqrt{(a^2-x^2)}}$, and comes out $-\frac{1}{2}x\sqrt{(a^2-x^2)}+\frac{1}{2}a\times$ cir. arc to radius a and sine x.

Ex. 4. In like manner, the fluent of $\frac{x^4\dot{x}}{\sqrt{(x^2+a^2)}}$ will be found from that of $\frac{x^2\dot{x}}{\sqrt{(x^2+a^2)}}$. Here it is manifest that y must be assumed $=x^3\sqrt{(x^2+a^2)}$, in order that one part of its fluxion, viz, $\dot{x}\times$ flux. of $\sqrt{(x^2+a^2)}$ may agree with the proposed fluxion. Thus, by taking the fluxion, and reducing as before, the fluent of $\frac{x^4\dot{x}}{\sqrt{(x^2+a^2)}}$ will be found $=$ $\frac{1}{4}x^3\sqrt{(x^2+a^2)}-\frac{3}{4}a^2\times f\frac{x^2\dot{x}}{\sqrt{(x^2+a^2)}}$.

Ex. 5. Thus also the fluent of $\frac{x^4\dot{x}}{\sqrt{(x^2-a^2)}}$ is $\frac{1}{4}x^3\sqrt{(x^2-a^2)}$ $+\frac{3}{4}a^2\times f\frac{x^2\dot{x}}{\sqrt{(x^2-a^2)}}$.

Ex. 6. And the $f\frac{x^4\dot{x}}{\sqrt{(a^2-x^2)}}$, is $-\frac{1}{4}x^3\sqrt{(a^2-x^2)}+$ $\frac{3}{4}a^2\times f\frac{x^2\dot{x}}{\sqrt{(a^2-x^2)}}$.

In like manner the student may find the fluents of $\frac{x^6\dot{x}}{\sqrt{(x^2\pm a^2)}}$, $\frac{x^8\dot{x}}{\sqrt{(x^2\pm a^2)}}$, &c, to $\frac{x^n\dot{x}}{\sqrt{(x^2\pm a^2)}}$, where n is any even number, each from the fluent of that which immediately precedes it in the series, by substituting for y as before. Thus the fluent of $\frac{x^n\dot{x}}{\sqrt{(x^2+a^2)}}=\frac{1}{n}x^{n-1}\sqrt{(x^2+a^2)}-\frac{n-1}{n}$ $a^2\times f\frac{x^{n-2}\dot{x}}{\sqrt{(x^2+a^2)}}$.

76. In like manner we may proceed for the series of similar expressions where the index of the power of x in the numerator is some odd number.

Ex. 1 To find the fluent of $\frac{x^3\dot{x}}{\sqrt{(x^2+a^2)}}$. Here assuming $y=x^2\sqrt{(x^2+a^2)}$, and taking the fluxion, one part of it will be similar to the fluxion proposed. Thus, $\dot{y}=2x\dot{x}$ $\sqrt{(x^2+a^2)}+\frac{x^3\dot{x}}{\sqrt{(x^2+a^2)}}$; hence at once the given fluxion $\frac{x^3\dot{x}}{\sqrt{(x^2+a^2)}}=\dot{y}-2x\dot{x}\sqrt{(x^2+a^2)}$; theref. the required fluent is $y-f.2x\dot{x}\sqrt{(x^2+a^2)}=x^2\sqrt{(x^2+a^2)}-\frac{2}{3}(x^2+a^2)^{\frac{3}{2}}$, by the 2d form of fluents.

Ex. 2. In like manner the fluent of $\frac{x^3\dot{x}}{\sqrt{(x^2+a^2)}}$, is $x^2\sqrt{(x^2-a^2)}-\frac{2}{3}(x^2-a^2)^{\frac{3}{2}}$.

Ex. 3. And the fluent of $\frac{x^3\dot{x}}{\sqrt{(a^2-x^2)}} = -x^2\sqrt{(a^2-x^2)} - \frac{2}{3}(a^2-x^2)^{\frac{3}{2}}$.

Ex. 4. To find the flu. of $\frac{x^5\dot{x}}{\sqrt{(x^2+a^2)}}$, from that of $\frac{x^3\dot{x}}{\sqrt{(x^2+a^2)}}$.

Here it is manifest we must assume $y = x^4\sqrt{(x^2+a^2)}$. This in fluxions and reduced gives $\dot{y} = \frac{5x^5\dot{x}}{\sqrt{(x^2+a^2)}} + \frac{4a^2x^3\dot{x}}{\sqrt{(x^2+a^2)}}$, and hence $\frac{x^5\dot{x}}{\sqrt{(x^2+a^2)}} = \frac{1}{5}\dot{y} - \frac{4a^2}{5} - \frac{x^3\dot{x}}{\sqrt{(x^2+a^2)}}$; and the flu. is $\frac{1}{5}y - \frac{4}{5}a^2 \times f\frac{x^3\dot{x}}{\sqrt{(x^2+a^2)}} = \frac{1}{5}x_4\sqrt{(x^2+a^2)} - \frac{4}{5}a^2 \times f\frac{x^3\dot{x}}{\sqrt{(x^2+a^2)}}$, the fluent of the latter part being as in ex. 1, above.

In like manner the student may find the fluents of $\frac{x^5\dot{x}}{\sqrt{(x^2-a_2)}}$ and $\frac{x^5\dot{x}}{\sqrt{(a^2-x^2)}}$. He may then proceed in a similar way for the fluents of $\frac{x^7\dot{x}}{\sqrt{(x^2\pm a^2)}}$, $\frac{x^9\dot{x}}{\sqrt{(x^2\pm a^2)}}$, &c, $\frac{x^n\dot{x}}{\sqrt{(x^2\pm a^2)}}$, where n is any odd number, viz, always by means of the fluent of each preceding term in the series.

77. In a similar manner may the process be for the fluents of the series of fluxions,

$$\frac{\dot{x}}{\sqrt{(a\pm x)}},\ \frac{x\dot{x}}{\sqrt{(a\pm x)}},\ \frac{x^2\dot{x}}{\sqrt{(a\pm x)}},\ \&c,\ \ldots\ \frac{x^n\dot{x}}{\sqrt{(a\pm x)}},$$

using the fluent of each preceding term in the series, as a part of the next term, and knowing that the fluent of the first term $\frac{\dot{x}}{\sqrt{a\pm x}}$ is given, by the 2d form of fluents, $= 2\sqrt{(a\pm x)}$, of the same sign as x.

Ex. 1. To find the fluent of $\frac{x\dot{x}}{\sqrt{(x+a)}}$, having given that of $\frac{\dot{x}}{\sqrt{(x+a)}} = 2\sqrt{(x+a)} = \text{A}$ suppose. Here it is evident we must assume $y = x\sqrt{(x+a)}$, for then its flux. $\dot{y} = \frac{\frac{1}{2}x\dot{x}}{\sqrt{(x+a)}} + \dot{x}\sqrt{(x+a)} = \frac{\frac{1}{2}x\dot{x}}{\sqrt{(x+a)}} + \frac{x\dot{x}}{\sqrt{(x+a)}} + \frac{a\dot{x}}{\sqrt{(x+a)}} = \frac{\frac{3}{2}x\dot{x}}{\sqrt{(x+a)}} + a\dot{\text{A}}$; hence $\frac{x\dot{x}}{\sqrt{(x+a)}} = \frac{2}{3}\dot{y} - \frac{2}{3}a\dot{\text{A}}$; and the required fluent is $\frac{2}{3}y - \frac{2}{3}a\text{A} = \frac{2}{3}x\sqrt{(x+a)} - \frac{4}{3}a\sqrt{(x+a)} = (x-2a)\times\frac{2}{3}\sqrt{(x+a)}$.

In like manner the student will find the fluents of $\frac{x\dot{x}}{\sqrt{(x-a)}}$ and $\frac{x\dot{x}}{\sqrt{(a-x)}}$.

Ex. 2.

Ex. 2. To find the fluent of $\frac{x^2\dot{x}}{\sqrt{(x+a)}}$, having given that of $\frac{x\dot{x}}{\sqrt{(x+a)}} = \text{B}$. Here y must be assumed $= x^2\sqrt{(x+a)}$; for then taking the flu. and reducing, there is found $\frac{x^2\dot{x}}{\sqrt{(x+a)}} = \frac{2}{5}\dot{y} - \frac{4}{5}a\dot{\text{B}}$; theref. $\int \frac{x^2\dot{x}}{\sqrt{(x+a)}} = \frac{2}{5}y - \frac{4}{5}a\text{B} = \frac{2}{5}x^2\sqrt{(x+a)} - \frac{4}{5}a\text{B} = \frac{2}{5}x^2\sqrt{(x+a)} - \frac{4}{5}a(x-2a) \times \frac{2}{3}\sqrt{(x+a)} = (3x^2 - 4ax + 8a^2) \times \frac{2}{15}\sqrt{(x+a)}$.

In the same manner the student will find the fluents of $\frac{x^2\dot{x}}{\sqrt{(x-a)}}$ and of $\frac{x^2\dot{x}}{\sqrt{(a-x)}}$. And in general, the fluent of $\frac{x^{n-1}\dot{x}}{\sqrt{(x+a)}}$ being given $= \text{C}$, he will find the fluent of $\frac{x^n\dot{x}}{\sqrt{(x+a)}} = \frac{2}{2n+1}x^n\sqrt{(x+a)} - \frac{2n}{2n+1}a\text{C}$.

78. In a similar way we might proceed to find the fluents of other classes of fluxions by means of other fluents in the table of forms; as, for instance, such as $x\dot{x}\sqrt{(dx-x^2)}$, $x^2\dot{x}\sqrt{(dx-x^2)}$, $x^3\dot{x}\sqrt{(dx-x^2)}$, &c, depending on the fluent of $\dot{x}\sqrt{(dx-x^2)}$, the fluent of which, by the 16th tabular form, is the circular semisegment to diameter d and versed sine x, or the half or trilineal segment contained by an arc with its right sine and versed sine, the diameter being d.

Ex. 1. Putting then said semiseg. or fluent of $\dot{x}\sqrt{(dx-x^2)} = \text{A}$, to find the fluent of $x\dot{x}\sqrt{(dx-x^2)}$. Here assuming $y = (dx-x^2)^{\frac{3}{2}}$, and taking the fluxions, they are $\dot{y} = \frac{3}{2}(d\dot{x} - 2x\dot{x})\sqrt{(dx-x^2)}$; hence $x\dot{x}\sqrt{(dx-x^2)} = \frac{1}{2}d\dot{x}\sqrt{(dx-x^2)} - \frac{1}{3}\dot{y} = \frac{1}{2}d\dot{\text{A}} - \frac{1}{3}\dot{y}$; therefore the required fluent, $\int x\dot{x}\sqrt{(dx-x^2)}$, is $\frac{1}{2}d\text{A} - \frac{1}{3}y = \frac{1}{2}d\text{A} - \frac{1}{3}(dx-x^2)^{\frac{3}{2}} = \text{B}$ suppose.

Ex. 2. To find the fluent of $x^2\dot{x}\sqrt{(dx-x^2)}$, having that of $x\dot{x}\sqrt{(dx-x^2)}$ given $= \text{B}$. Here assuming $y = x(dx-x^2)^{\frac{3}{2}}$, then taking the fluxions, and reducing, there results $\dot{y} = (\frac{5}{2}dx\dot{x} - 4x^2\dot{x})\sqrt{(dx-x^2)}$; hence $x^2\dot{x}\sqrt{(dx-x^2)} = \frac{5}{8}dx\dot{x}\sqrt{(dx-x^2)} - \frac{1}{4}\dot{y} = \frac{5}{8}d\dot{\text{B}} - \frac{1}{4}\dot{y}$, the flu. theref. of $x^2\dot{x}\sqrt{(dx-x^2)}$ is $\frac{5}{8}d\text{B} - \frac{1}{4}y = \frac{5}{8}d\text{B} - \frac{1}{4}x(dx-x^2)^{\frac{3}{2}}$.

Ex. 3. In the same manner the series may be continued to any extent; so that in general, the flu. of $x^{n-1}\dot{x}\sqrt{(dx-x^2)}$ being given $= \text{C}$, then the next, or the flu. of $x^n\dot{x}\sqrt{(dx-x^2)}$ will be $\frac{2n+1}{n+2} - \frac{1}{2}d\text{C} - \frac{1}{n+2}x^{n-1}(dx-x^2)^{\frac{3}{2}}$.

79. To find the fluent of such expressions as $\frac{\dot{x}}{\sqrt{(x^2 \pm 2ax)}}$, a case not included in the table of forms. Put

Put the proposed radical $\sqrt{(x^2 \pm 2ax)} = z$, or $x^2 \pm 2ax = z^2$; then, completing the square, $x^2 \pm 2ax + a^2 = z^2 + a^2$, and the root is $x \pm a = \sqrt{(z^2 + a^2)}$. The fluxion of this is $\dot{x} = \frac{z\dot{z}}{\sqrt{(z^2 + a^2)}}$; theref. $\frac{\dot{x}}{\sqrt{(x^2 \pm 2ax)}} = \frac{\dot{z}}{\sqrt{(z^2 + a^2)}}$; the fluent of which, by the 12th form, is the hyp. log. of $z + \sqrt{(z^2 + a^2)}$ = hyp. log. of $x \pm a + \sqrt{(x^2 \pm 2ax)}$, the fluent required.

Ex. 2. To find now the fluent of $\frac{x\dot{x}}{\sqrt{(x^2 + 2ax)}}$, having given, by the above example, the fluent of $\frac{\dot{x}}{\sqrt{(x^2 + 2ax)}} = A$ suppose. Assume $\sqrt{(x^2 + 2ax)} = y$; then its fluxion is $\frac{x\dot{x} + a\dot{x}}{\sqrt{(x^2 + 2ax)}} = \dot{y}$; theref. $\frac{x\dot{x}}{\sqrt{(x^2 + 2ax)}} = \dot{y} - \frac{a\dot{x}}{\sqrt{(x^2 + 2ax)}} = \dot{y} - a\dot{A}$; the fluent of which is $y - aA = \sqrt{(x^2 + 2ax)} - aA$, the fluent sought.

Ex. 3. Thus also, this fluent of $\frac{x\dot{x}}{\sqrt{(x^2 + 2ax)}}$ being given, the flu. of the next in the series, or $\frac{x^2\dot{x}}{\sqrt{(x^2 + 2ax)}}$ will be found, by assuming $x\sqrt{(x^2 + 2ax)} = y$; and so on for any other of the same form. As, if the fluent of $\frac{x^{n-1}\dot{x}}{\sqrt{(x^2 + 2ax)}}$ be given $= C$; then, by assuming $x^{n-1}\sqrt{(x^2 + 2ax)} = y$, the fluent of $\frac{x^n\dot{x}}{\sqrt{(x^2 + 2ax)}} = \frac{1}{n}x^{n-1}\sqrt{(x^2 + 2ax)} - \frac{2n-1}{n}aC$.

Ex. 4. In like manner, the fluent of $\frac{\dot{x}}{\sqrt{(x^2 - 2ax)}}$ being given, as in the first example, that of $\frac{x\dot{x}}{\sqrt{(x^2 - 2ax)}}$ may be found; and thus the series may be continued exactly as in the 3d ex. only taking $- 2ax$ for $+ 2ax$.

80. Again, having given the fluent of $\frac{\dot{x}}{\sqrt{(2ax - x^2)}}$, which, is $\frac{1}{a} \times$ circular arc to radius a and versed sine x, the fluents of $\frac{x\dot{x}}{\sqrt{(2ax - x^2)}}$, $\frac{x^2\dot{x}}{\sqrt{(2ax - x^2)}}$, &c. . . $\frac{x^n\dot{x}}{\sqrt{(2ax - x^2)}}$, may be assigned by the same method of continuation. Thus,

Ex. 1. For the fluent of $\frac{x\dot{x}}{\sqrt{(2ax - x^2)}}$, assume $\sqrt{(2ax - x^2)} = y$; the required fluent will be found $= -\sqrt{(2ax - x^2)} + A$ or arc to radius a and vers. x.

Ex. 2. In like manner the fluent of $\frac{x^2\dot{x}}{\sqrt{(2ax - x^2)}}$ is

f

$f\dfrac{\frac{3}{2}ax\dot{x}}{\sqrt{(2ax-x^2)}}-\frac{1}{2}x\sqrt{(2ax-x^2)}=\frac{3}{2}a\text{A}-\dfrac{3a+x}{2}\sqrt{(2ax-x^2)}$,
where A denotes the arc mentioned in the last example.

Ex. 3. And in general the fluent of $\dfrac{x^n\dot{x}}{\sqrt{(2ax-x^2)}}$ is $\dfrac{2n-1}{n}a\text{C}-\dfrac{1}{n}x^{n-1}\sqrt{(2ax-x^2)}$, where C is the fluent of $\dfrac{x^{n-1}\dot{x}}{\sqrt{(2ax-x^2)}}$, the next preceding term in the series.

81. Thus also, the fluent of $\dot{x}\sqrt{(x-a)}$ being given, $=\frac{2}{3}(x-a)^{\frac{3}{2}}$, by the 2d form, the fluents of $x\dot{x}\sqrt{(x-a)}$, $x^2\dot{x}\sqrt{(x-a)}$, &c . . . $x^n\dot{x}\sqrt{(x-a)}$, may be found. And in general, if the fluent of $x^{n-1}\dot{x}\sqrt{(x-a)}=\text{C}$ be given; then by assuming $x^n(x-a)^{\frac{3}{2}}=y$, the fluent of $x^n\dot{x}\sqrt{(x-a)}$ is found $=\dfrac{2}{2n+3}x^n(x-a)^{\frac{3}{2}}+\dfrac{2na}{2n+3}\text{C}$.

82. Also, given the fluent of $(x-a)^m\dot{x}$ which is $\dfrac{1}{m+1}(x-a)^{m+1}$ by the 2d form, the fluents of the series $(x-a)^m x\dot{x}$, $(x-a)^m x^2\dot{x}$. &c . . . $(x-a)^m x^n\dot{x}$ can be found. And in general, the fluent of $(x-a)^m x^{n-1}\dot{x}$ being given $=\text{C}$; then by assuming $(x-a)^{m+1}x^n=y$, the fluent of $(x-a)^m x^n\dot{x}$ is found $=\dfrac{x^n(x-a)^{m+1}+na\text{C}}{m+n+1}$.

Also, by the same way of continuation, the fluents of $x^n\dot{x}\sqrt{(a\pm x)}$ and of $x^n\dot{x}(a\pm x)^m$ may be found.

83. When the fluxional expression contains a trinomial quantity, as $\sqrt{(b+cx+x^2)}$, this may be reduced to a binomial, by substituting another letter for the unknown one x, connected with half the coefficient of the middle term with its sign. Thus, put $z=x+\frac{1}{2}c$: then $z^2=x^2+cx+\frac{1}{4}c^2$; theref. $z^2-\frac{1}{4}c^2=x^2+cx$, and $z^2+b-\frac{1}{4}c^2=x^2+cx+b$ the given trinomial, which is $=z^2+a^2$, by putting $a^2=b-\frac{1}{4}c^2$.

Ex. 1. To find the fluent of $\dfrac{3\dot{x}}{\sqrt{(5+4x+x^2)}}$.

Here $z=x+2$; then $z^2=x^2+4x+4$, and $z^2+1=5+4x+x^2$, also $\dot{x}=\dot{z}$; theref. the proposed fluxion reduces to $\dfrac{3\dot{z}}{\sqrt{(1+z^2)}}$; the fluent of which, by the 12th form in this vol. is 3 hyp. log. of $z+\sqrt{(1+z)}=3$ hyp. log. $x+2+\sqrt{(5+4x+x^2)}$.

Ex. 2.

Ex. 2. To find the fluent of $\dot{x}\sqrt{}(b+cx+dx^2)=\dot{x}\sqrt{d}\times\sqrt{}(\frac{b}{d}+\frac{c}{d}x+x^2)$.

Here assuming $x+\frac{c}{2d}=z$; then $\dot{x}=\dot{z}$, and the proposed flux. reduces to $\dot{z}\sqrt{d}\times\sqrt{}(z^2+\frac{b}{d}-\frac{c^2}{4d^2})=\dot{z}\sqrt{d}\times\sqrt{}(z^2+a^2)$, putting a^2 for $\frac{b}{d}-\frac{c^2}{4d^2}$; and the fluent will be found by a similar process to that employed in ex. 1 art. 75.

Ex. 3. In like manner, for the flu. of $x^{n-1}\dot{x}\sqrt{}(b+cx^n+dx^{2n})$, assuming $x^n+\frac{c}{2d}=z$, $nx^{n-1}\dot{x}=\dot{z}$, and $x^{n-1}\dot{x}=\frac{1}{n}\dot{z}$; hence $x^{2n}+\frac{c}{d}x^n+\frac{c^2}{4d^2}=z^2$, and $\sqrt{}(dx^{2n}+cx^n+b)=\sqrt{d}\times\sqrt{}(x^{2n}+\frac{c}{d}x+\frac{b}{d})=\sqrt{d}\times\sqrt{}(z^2+\frac{b}{d}-\frac{c^2}{4d^2})=\sqrt{d}\times\sqrt{}(z^2\pm a^2)$, putting $\pm a^2=\frac{b}{d}-\frac{c^2}{4d^2}$; hence the given fluxion becomes $\frac{1}{n}\dot{z}\sqrt{d}\times\sqrt{}(z^2\pm a^2)$, and its fluent as in the last example.

Ex. 4. Also, for the fluent of $\frac{x^{n-1}\dot{x}}{b+cx+dx^2}$; assume $x^n+\frac{c}{2d}=z$, then the fluxion may be reduced to the form $\frac{1}{dn}\times\frac{\dot{x}}{x^2\pm a^2}$, and the fluent found as before.

So far on this subject may suffice on the present occasion. But the student who may wish to see more on this branch, may profitably consult Mr. Dealtry's very methodical and ingenious treatise on Fluxions, lately published, from which several of the foregoing cases and examples have been taken or imitated.

OF MAXIMA AND MINIMA; OR, THE GREATEST AND LEAST MAGNITUDE OF VARIABLE OR FLOWING QUANTITIES.

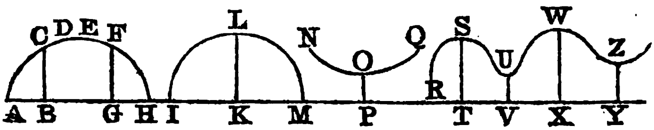
84. Maximum, denotes the greatest state or quantity attainable in any given case, or the greatest value of a variable quantity: by which it stands opposed to Minimum, which is the least possible quantity in any case.

Thus,

Thus, the expression or sum $a^2 + bx$, evidently increases as x, or the term bx, increases; therefore the given expression will be the greatest, or a maximum, when x is the greatest, or infinite: and the same expression will be a minimum, or the least, when x is the least, or nothing.

Again, in the algebraic expression $a^2 - bx$, where a and b denote constant or invariable quantities, and x a flowing or variable one. Now, it is evident that the value of this remainder or difference, $a^2 - bx$, will increase, as the term bx, or as x, decreases; therefore the former will be the greatest, when the latter is the smallest; that is $a^2 - bx$ is a maximum, when x is the least, or nothing at all; and the difference is the least, when x is the greatest.

85. Some variable quantities increase continually; and so have no maximum, but what is infinite. Others again decrease continually; and so have no minimum, but what is of no magnitude, or nothing. But, on the other hand, some variable quantities increase only to a certain finite magnitude, called their Maximum, or greatest state, and after that they decrease again. While others decrease to a certain finite magnitude, called their Minimum, or least state, and afterwards increase again. And lastly, some quantities have several maxima and minima.



Thus, for example, the ordinate BC of the parabola, or such-like curve, flowing along the axis AB from the vertex A, continually increases, and has no limit or maximum. And the ordinate GF of the curve EFH, flowing from E towards H, continually decreases to nothing when it arrives at the point H. But in the circle ILM, the ordinate only increases to a certain magnitude, namely, the radius, when it arrives at the middle as at KL, which is its maximum; and after that it decreases again to nothing, at the point M. And in the curve NOQ, the ordinate decreases only to the position OP, where it is least, or a minimum; and after that it continually increases towards Q. But in the curve RSU &c, the ordinates have several maxima, as ST, WX, and several minima, as VU, YZ, &c.

51. Now

86. Now, because the fluxion of a variable quantity, is the rate of its increase or decrease: and because the maximum or minimum of a quantity neither increases nor decreases, at those points or states; therefore such maximum or minimum has no fluxion, or the fluxion is then equal to nothing. From which we have the following rule.

To find the Maximum or Minimum.

87. From the nature of the question or problem, find an algebraical expression for the value, or general state, of the quantity whose maximum or minimum is required; then take the fluxion of that expression, and put it equal to nothing; from which equation, by dividing by, or leaving out, the fluxional letter and other common quantities, and performing other proper reductions, as in common algebra, the value of the unknown quantity will be obtained, determining the point of the maximum or minimum.

So, if it be required to find the maximum state of the compound expression $100x - 5x^2 \pm c$, or the value of x when $100x - 5x^2 \pm c$ is a maximum. The fluxion of this expression is $100\dot{x} - 10x\dot{x} = 0$: which being made $= 0$, and divided by $10\dot{x}$, the equation is $10 - x = 0$; and hence $x = 10$. That is, the value of x is 10, when the expression $100x - 5x^2 \pm c$ is the greatest. As is easily tried: for if 10 be substituted for x in that expression, it becomes $\pm c + 500$: but if, for x, there be substituted any other number, whether greater or less than 10, that expression will always be found to be less than $\pm c + 500$, which is therefore its greatest possible value, or its maximum.

88. It is evident, that if a maximum or minimum be any way compounded with, or operated on, by a given constant quantity, the result will still be a maximum or minimum. That is, if a maximum or minimum be increased, or decreased, or multiplied, or divided, by a given quantity, or any given power or root of it be taken; the result will still be a maximum or minimum. Thus, if x be a maximum or minimum, then also is $x + a$, or $x - a$, or ax, or $\frac{x}{a}$, or x^2, or $\sqrt[3]{x}$, still a maximum or minimum. Also, the logarithm of the same will be a maximum or a minimum. And therefore, if any proposed maximum or minimum can be made simpler by performing any of these operations, it is better to do so, before the expression is put into fluxions.

89. When the expression for a maximum or minimum contains several variable letters or quantities; take the fluxion of it as often as there are variable letters; supposing first one of them only to flow, and the rest to be constant; then another only to flow, and the rest constant; and so on for all of them: then putting each of these fluxions $= 0$, there will be as many equations as unknown letters, from which these may be all determined. For the fluxion of the expression must be equal to nothing in each of these cases; otherwise the expression might become greater or less, without altering the values of the other letters, which are considered as constant.

So, if it be required to find the values of x and y when $4x^2 - xy + 2y$ is a minimum. Then we have,

First, - $8x\dot{x} - \dot{x}y = 0$, and $8x - y = 0$, or $y = 8x$.

Secondly, $2\dot{y} - x\dot{y} = 0$, and $2 - x = 0$, or $x = 2$.

And hence y or $8x = 16$.

90. *To find whether a proposed quantity admits of a Maximum or a Minimum.*

Every algebraic expression does not admit of a maximum or minimum, properly so called; for it may either increase continually to infinity, or decrease continually to nothing; and in both these cases there is neither a proper maximum nor minimum; for the true maximum is that finite value to which an expression increases, and after which it decreases again: and the minimum is that finite value to which the expression decreases, and after that it increases again. Therefore, when the expression admits of a maximum, its fluxion is positive before the point, and negative after it; but when it admits of a minimum, its fluxion is negative before, and positive after it. Hence then, taking the fluxion of the expression a little before the fluxion is equal to nothing, and again a little after the same; if the former fluxion be positive, and the latter negative, the middle state is a maximum; but if the former fluxion be negative, and the latter positive, the middle state is minimum.

So, if we would find the quantity $ax - x^2$ a maximum or minimum; make its fluxion equal to nothing, that is, - - $a\dot{x} - 2x\dot{x} = 0$, or $(a - 2x)\dot{x} = 0$; dividing by $\dot{x}$, gives $a - 2x = 0$, or $x = \frac{1}{2}a$ at that state. Now, if in the fluxion $(a - 2x)\,\dot{x}$, the value of x be taken rather less than its true value, $\frac{1}{2}a$, that fluxion will evidently be positive; but if x be taken somewhat greater than $\frac{1}{2}a$ the value of $a - 2x$, and consequently of the fluxion, is as evidently negative. Therefore, the fluxion of $ax - x^2$ being positive before, and negative

gative after the state when its fluxion is $=0$, it follows that at this state the expression is not a minimum, but a maximum.

Again, taking the expression $x^3 - ax^2$, its fluxion $3x^2\dot{x} - 2ax\dot{x} = (3x-2a)x\dot{x} = 0$; this divided by $x\dot{x}$ gives $3x - 2a = 0$, and $x = \frac{2}{3}a$, its true value when the fluxion of $x^3 - ax^2$ is equal to nothing. But now to know whether the given expression be a maximum or a minimum at that time, take x a little less than $\frac{2}{3}a$ in the value of the fluxion $(3x - 2a)x\dot{x}$, and this will evidently be negative; and again, taking x a little more than $\frac{2}{3}a$, the value of $3x - 2a$, or of the fluxion, is as evidently positive. Therefore the fluxion of $x^3 - ax^2$ being negative before that fluxion is $= 0$, and positive after it, it follows that in this state the quantity $x^3 - ax^2$ admits of a minimum, but not of a maximum.

91. SOME EXAMPLES FOR PRACTICE.

EXAM. 1. To divide a line, or any other given quantity a, into two parts, so that their rectangle or product may be the greatest possible.

EXAM. 2. To divide the given quantity a into two parts such, that the product of the m power of one, by the n power of the other, may be a maximum.

EXAM. 3. To divide the given quantity a into three parts such, that the continual product of them all may be a maximum.

EXAM. 4. To divide the given quantity a into three parts such, that the continual product of the 1st, the square of the 2d, and the cube of the 3d, may be a maximum.

EXAM. 5. To determine a fraction such, that the difference between its m power and n power shall be the greatest possible.

EXAM. 6. To divide the number 80 into two such parts, x and y, that $2x^2 + xy + 3y^2$ may be a minimum.

EXAM. 7. To find the greatest rectangle that can be inscribed in a given right-angled triangle.

EXAM. 8. To find the greatest rectangle that can be inscribed in the quadrant of a given circle.

EXAM. 9. To find the least right-angled triangle that can circumscribe the quadrant of a given circle.

EXAM. 10. To find the greatest rectangle inscribed in, and the least isosceles triangle circumscribed about, a given semi-ellipse.

EXAM. 11.

Exam. 11. To determine the same for a given parabola.

Exam. 12. To determine the same for a given hyperbola.

Exam. 13. To inscribe the greatest cylinder in a given cone; or to cut the greatest cylinder out of a given cone.

Exam. 14. To determine the dimensions of a rectangular cistern, capable of containing a given quantity a of water, so as to be lined with lead at the least possible expense.

Exam. 15. Required the dimensions of a cylindrical tankard, to hold one quart of ale measure, that can be made of the least possible quantity of silver, of a given thickness.

Exam. 16. To cut the greatest parabola from a given cone.

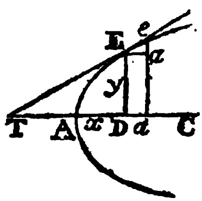
Exam. 17. To cut the greatest ellipse from a given cone.

Exam. 18. To find the value of x when x^x is a minimum.

THE METHOD OF TANGENTS; OR, TO DRAW TANGENTS TO CURVES.

92. The Method of Tangents, is a method of determining the quantity of the tangent and subtangent of any algebraic curve; the equation of the curve being given. Or, *vice versa*, the nature of the curve, from the tangent given.

If AE be any curve, and E be any point in it, to which it is required to draw a tangent TE. Draw the ordinate ED: then if we can determine the subtangent TD, limited between the ordinate and tangent, in the axis produced, by joining the points, T, E, the line TE will be the tangent sought.



93. Let dae be another ordinate, indefinitely near to DE, meeting the curve, or tangent produced in e; and let Ee be parallel to the axis AD. Then is the elementary triangle Eea similar to the triangle TDE; and

therefore

therefore - $ea : aE :: ED : DT$.
But - - $ea : aE ::$ flux. ED : flux. AD.
Therefore - flux. ED : flux. AD :: DE : DT.
That is - - $\dot{y} : \dot{x} :: y : \frac{y\dot{x}}{\dot{y}} = DT$.

which is therefore the general value of the subtangent sought; where x is the absciss AD, and y the ordinate DE.

Hence we have this general rule.

GENERAL RULE.

94. By means of the given equation of the curve, when put into fluxions, find the value of either $\dot{x}$ or $\dot{y}$ or of $\frac{\dot{x}}{\dot{y}}$; which value substitute for it in the expression $DT = \frac{y\dot{x}}{\dot{y}}$, and, when reduced to its simplest terms, it will be the value of the subtangent sought.

EXAMPLES.

EXAM. 1. Let the proposed curve be that which is defined, or expressed by the equation $ax^2 + xy^2 - y^3 = 0$.

Here the fluxion of the equation of the curve is
$2ax\dot{x}+y^2\dot{x} + 2xy\dot{y} - 3y^2\dot{y} = 0$; then, by transposition,
$2ax\dot{x}+y^2\dot{x}= 3y^2\dot{y} - 2xy\dot{y}$; and hence, by division,
$\frac{\dot{x}}{\dot{y}} = \frac{3y^2-2xy}{2ax+y^2}$; consequently $\frac{y\dot{x}}{\dot{y}} = \frac{3y_3-2xy^2}{2ax+y^2}$.
which is the value of the subtangent TD sought.

EXAM. 2. To draw a tangent to a circle; the equation of which is $ax-x^2 = y^2$; where x is the absciss, y the ordinate, and a the diameter.

EXAM. 3. To draw a tangent to a parabola; its equation being $ax = y^2$; where a denotes the parameter of the axis.

EXAM. 4. To draw a tangent to an ellipse; its equation being $c^2(ax - x^2) = a^2y^2$; where a and c are the two axes.

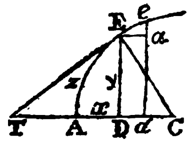
EXAM. 5. To draw a tangent to an hyperbola; its equation being $c^2(ax + x_2) = a^2y^2$; where a and c are the two axes.

EXAM. 6. To draw a tangent to the hyperbola referred to the asymptote as an axis; its equation being $xy = a^2$; where a^2 denotes the rectangle of the absciss and ordinate answering to the vertex of the curve.

OF

OF RECTIFICATIONS; OR, TO FIND THE LENGTHS OF CURVE LINES.

95. RECTIFICATION, is the finding the length of a curve line, or finding a right line equal to a proposed curve.



By art. 10 it appears, that the elementary triangle Eae, formed by the increments of the absciss, ordinate, and curve, is a right-angled triangle, of which the increment of the curve is the hypothenuse; and therefore the square of the latter is equal to the sum of the squares of the two former; that is, $Ee^2 = Ea^2 + ae^2$. Or, substituting, for the increments, their proportional fluxions, it is $\dot{z}\dot{z} = \dot{x}\dot{x} + \dot{y}\dot{y}$, or $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$; where z denotes any curve line AE, x its absciss AD, and y its ordinate DE. Hence this rule.

RULE.

96. From the given equation of the curve put into fluxions, find the value of $\dot{x}^2$ or $\dot{y}^2$, which value substitute instead of it in the equation $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$; then the fluents, being taken, will give the value of z, or the length of the curve, in terms of the absciss or ordinate.

EXAMPLES.

EXAM. 1. To find the length of the arc of a circle, in terms both of the sine, versed sine, tangent, and secant.

The equation of the circle may be expressed in terms of the radius, and either the sine, or the versed sine, or tangent, or secant, &c, of an arc. Let therefore the radius of the circle be CA or CE $= r$, the versed sine AD (of the arc AE) $= x$, the right sine DE $= y$, the tangent TE $= t$, and the secant CT $= s$, then, by the nature of the circle, there arise these equations, viz.

$$y^2 = 2rx - x^2 = \frac{r^2t^2}{r^2 + t^2} = \frac{s^2 - r^2}{s^2}r^2.$$

Then, by means of the fluxions of these equations, with the general fluxional equation $\dot{z}^2 = \dot{x}^2 + \dot{y}^2$, are obtained the following fluxional forms, for the fluxion of the curve; the fluent of any one of which will be the curve itself; viz.

$$\dot{z} = \frac{r\dot{x}}{\sqrt{2rx - xx}} = \frac{r\dot{y}}{\sqrt{r^2 - y^2}} = \frac{r^2\dot{t}}{r^2 + t^2} = \frac{r\dot{s}}{s\sqrt{s^2 - r^2}}.$$

Hence

Hence the value of the curve, from the fluent of each of these, gives the four following forms, in series, viz. putting $d = 2r$ the diameter, the curve is

$$z = (1 + \frac{x}{2.3d} + \frac{3x^2}{2.4.5d^2} + \frac{3.5x^3}{2.4.6.7d^3} + \&c)\sqrt{dx},$$
$$= (1 + \frac{y^2}{2.3r^2} + \frac{3y^4}{2.4.5r^4} + \frac{3.5y^6}{2.4.6.7r^6} + \&c)\ y,$$
$$= (1 - \frac{t^2}{3r^2} + \frac{t^4}{5r^4} - \frac{t^6}{7r^6} + \frac{t^8}{9r^8} - \&c)\ t,$$
$$= (\frac{s - r}{s} + \frac{s^3 - r^3}{2.3s^3} + \frac{3(s^5 - r^5)}{2.4.5s^5} + \&c)\ r.$$

Now, it is evident that the simplest of these series, is the third in order, or that which is expressed in terms of the tangent. That form will therefore be the fittest to calculate an example by in numbers. And for this purpose it will be convenient to assume some arc whose tangent, or at least the square of it, is known to be some small simple number. Now, the arc of 45 degrees, it is known, has its tangent equal to the radius; and therefore, taking the radius $r = 1$, and consequently the tangent of 45°, or t, $= 1$ also, in this case the arc of 45° to the radius 1, or the arc of the quadrant to the diameter 1, will be equal to the infinite series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} -$ &c.

But as this series converges very slowly, it will be proper to take some smaller arc, that the series may converge faster; such as the arc of 30 degrees, the tangent of which is $= \sqrt{\frac{1}{3}}$, or its square $t^2 = \frac{1}{3}$: which being substituted in the series, the length of the arc of 30° comes out - - - $(1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \frac{1}{9.3^4} -$ &c) $\sqrt{\frac{1}{3}}$. Hence, to compute these terms in decimal numbers, after the first, the succeeding terms will be found by dividing, always by 3, and these quotients again by the absolute numbers 3, 5, 7, 9, &c; and lastly, adding every other term together, into two sums, the one the sum of the positive terms, and the other the sum of the negative ones; then lastly, the one sum taken from the other, leaves the length of the arc of 30 degrees; which being the 12th part of the whole circumference when the radius is 1, or the 6th part when the diameter is 1, consequently 6 times that arc will be the length of the whole circumference to the diameter 1. Therefore, multiplying the first term $\sqrt{\frac{1}{3}}$ by 6, the product is $\sqrt{12} = 3{\cdot}4641016$; and hence the operation will be conveniently made as follows:

+ Terms.

		+ Terms.	— Terms.
1)	3·4641016 (	3·4641016	
3)	1·1547005 (		0·3849002
5)	3849002 (	769800	
7)	1283001 (		183286
9)	427667 (	47519	
11)	142556 (		12960
13)	47519 (	3655	
15)	15840 (		1056
17)	5280 (	311	
19)	1760 (		93
21)	587 (	28	
23)	196 (		8
25)	65 (	3	
27)	22 (		1
		+3·5462332	—0·4046406
		—0·4046406	

So that at last 3·1415926 is the whole circumference to the diameter 1.

Exam. 2. To find the length of a parabola.

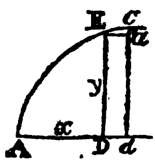
Exam. 3. To find the length of the semicubical parabola, whose equation is $ax^2 = y^3$.

Exam. 4. To find the length of an elliptical curve.

Exam. 5. To find the length of an hyperbolic curve.

OF QUADRATURES; OR, FINDING THE AREAS OF CURVES.

97. The Quadrature of Curves, is the measuring their areas, or finding a square, or other right-lined space, equal to a proposed curvilineal one.



By art. 9 it appears, that any flowing quantity being drawn into the fluxion of the line along which it flows, or in the direction of its motion, there is produced the fluxion of the quantity generated by the flowing. That is, $\text{D}d \times \text{DE}$ or $y\dot{x}$ is the fluxion of the area ADE. Hence this rule.

RULE.

RULE.

98. From the given equation of the curve, find the value either of $\dot{x}$ or of $\dot{y}$; which value substitute instead of it in the expression $y\dot{x}$; then the fluent of that expression, being taken, will be the area of the curve sought.

EXAMPLES.

EXAM. 1. To find the area of the common parabola.

The equation of the parabola being $ax = y^2$; where a is the parameter, x the absciss AD, or part of the axis, and y the ordinate DE.

From the equation of the curve is found $y = \sqrt{ax}$. This substituted in the general fluxion of the area $y\dot{x}$ gives $\dot{x}\sqrt{ax}$ or $a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}$ the fluxion of the parabolic area; and the fluent of this, or $\frac{2}{3}a^{\frac{1}{2}}x^{\frac{3}{2}} = \frac{2}{3}x\sqrt{ax} = \frac{2}{3}xy$, is the area of the parabola ADE, and which is therefore equal to $\frac{2}{3}$ of its circumscribing rectangle.

EXAM. 2. To square the circle, or find its area.

The equation of the circle being $y^2 = ax - x^2$, or $y = \sqrt{ax - x^2}$, where a is the diameter; by substitution, the general fluxion of the area $y\dot{x}$, becomes $\dot{x}\sqrt{ax-x^2}$, for the fluxion of the circular area. But as the fluent of this cannot be found in finite terms, the quantity $\sqrt{ax-x^2}$ is thrown into a series, by extracting the root, and then the fluxion of the area becomes

$$\dot{x}\sqrt{ax} \times \left(1 - \frac{x}{2a} - \frac{x^2}{2.4a^2} - \frac{1.3x^3}{2.4.6a^3} - \frac{1.3.5x^4}{2.4.6.8a^4} - \&c\right);$$

and then the fluent of every term being taken, it gives

$$x\sqrt{ax} \times \left(\frac{2}{3} - \frac{1.x}{5a} - \frac{1.x^2}{4.7a^2} - \frac{1.3x^3}{4.6.9a^3} - \frac{1.3.5x^4}{4.6.8.11a^4} - \&c\right);$$

for the general expression of the semisegment ADE.

And when the point D arrives at the extremity of the diameter, then the space becomes a semicircle, and $x = a$; and then the series above becomes barely

$$a^2\left(\frac{2}{3} - \frac{1}{5} - \frac{1}{4.7} - \frac{1.3}{4.6.9} - \frac{1.3.5.}{4.6.8.11} - \&c\right)$$

for the area of the semicircle whose diameter is a.

 EXAM. 3.

Exam. 3. To find the area of any parabola, whose equation is $a^m z^n = y^{m+n}$.

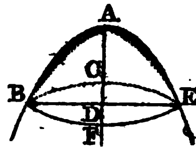
Exam. 4. To find the area of an ellipse.

Exam. 5. To find the area of an hyperbola.

Exam. 6. To find the area between the curve and asymptote of an hyperbola.

Exam. 7. To find the like area in any other hyperbola whose general equation is $x^m y^n = a^{m+n}$.

TO FIND THE SURFACES OF SOLIDS.



99. In the solid formed by the rotation of any curve about its axis, the surface may be considered as generated by the circumference of an expanding circle, moving perpendicularly along the axis, but the expanding circumference moving along the arc or curve of the solid. Therefore, as the fluxion of any generated quantity, is produced by drawing the generating quantity into the fluxion of the line or direction in which it moves, the fluxion of the surface will be found by drawing the circumference of the generating circle into the fluxion of the curve. That is, the fluxion of the surface, BAE, is equal to $\dot{\text{AE}}$ drawn into the circumference BCEF, whose radius is the ordinate DE.

100. But, if c be $= 3{\cdot}1416$, the circumference of a circle whose diameter is 1, $x =$ AD the absciss, $y =$ DE the ordinate, and $z =$ AE the curve; then $2y =$ the diameter BE, and $2cy =$ the circumference BCEF; also, $\dot{\text{AE}} = \dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$: therefore $2cy\dot{z}$ or $2cy\sqrt{\dot{x}^2 + \dot{y}^2}$ is the fluxion of the surface. And consequently if, from the given equation of the curve, the value of $\dot{x}$ or $\dot{y}$ be found, and substituted in this expression $2cy\sqrt{\dot{x}^2 + \dot{y}^2}$, the fluent of the expression being then taken, will be the surface of the solid required.

EXAMPLES.

Exam. 1. To find the surface of a sphere, or of any segment.

In

In this case, AE is a circular arc, whose equation is $y^2 = ax - x^2$, or $y = \sqrt{ax - x^2}$.

The fluxion of this gives $\dot{y} = \frac{a-2x}{2\sqrt{ax-x^2}}\dot{x} = \frac{a-2x}{2y}\dot{x}$;

hence $\dot{y}^2 = \frac{a^2 - 4ax + 4x^2}{4y^2}\dot{x}^2 = \frac{a^2 - 4y^2}{4y^2}\dot{x}^2$; consequently

$\dot{x}^2 + \dot{y}^2 = \frac{a^2\dot{x}^2}{4y^2}$, and $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2} = \frac{a\dot{x}}{2y}$.

This value of $\dot{z}$, the fluxion of a circular arc, may be found more easily thus: In the fig. to art. 95, the two triangles EDC, Eae are equiangular, being each of them equiangular to the triangle ETC: conseq. ED : EC :: Ea : Ee, that is, - - -

$y : \frac{1}{2}a :: \dot{x} : \dot{z} = \frac{a\dot{x}}{2y}$, the same as before.

The value of $\dot{z}$ being found, by substitution is obtained $2cy\dot{z} = ac\dot{x}$ for the fluxion of the spherical surface, generated by the circular arc in revolving about the diameter AD. And the fluent of this gives acx for the said surface of the spherical segment BAE.

But ac is equal to the whole circumference of the generating circle; and therefore it follows, that the surface of any spherical segment, is equal to the same circumference of the generating circle, drawn into x or AD, the height of the segment.

Also when x or AD becomes equal to the whole diameter a, the expression acx becomes aca or ca^2, or 4 times the area of the generating circle, for the surface of the whole sphere.

And these agree with the rules before found in Mensuration of Solids.

EXAM. 2. To find the surface of a spheroid.

EXAM. 3. To find the surface of a paraboloid.

EXAM. 4. To find the surface of an hyperboloid.

TO FIND THE CONTENTS OF SOLIDS.

101. ANY solid which is formed by the revolution of a curve about its axis (see last fig.), may also be conceived to be generated by the motion of the plane of an expanding circle, moving perpendicularly along the axis. And therefore

fore the area of that circle being drawn into the fluxion of the axis, will produce the fluxion of the solid. That is, $\dot{\text{AD}} \times$ area of the circle BCF, whose radius is DE, or diameter BE, is the fluxion of the solid, by art. 9.

102. Hence, if AD $= x$, DE $= y$, $c = 3{\cdot}1416$; because cy^2 is equal to the area of the circle BCF: therefore $cy^2\dot{x}$ is the fluxion of the solid. Consequently if, from the given equation of the curve, the value of either y^2 or x be found, and that value substituted for it in the expression $cy^2\dot{x}$, the fluent of the resulting quantity, being taken, will be the solidity of the figure proposed.

EXAMPLES.

EXAM. 1. To find the solidity of a sphere, or any segment.

The equation to the generating circle being $y^2 = ax - x^2$, where a denotes the diameter, by substitution, the general fluxion of the solid $cy^2\dot{x}$, becomes $cax\dot{x} - cx^2\dot{x}$, the fluent of which gives $\frac{1}{2}cax^2 - \frac{1}{3}cx^3$, or $\frac{1}{6}cx^2\ (3a-2x)$, for the solid content of the spherical segment BAE, whose height AD is x.

When the segment becomes equal to the whole sphere, then $x = a$, and the above expression for the solidity, becomes $\frac{1}{6}ca^3$ for the solid content of the whole sphere.

And these deductions agree with the rules before given and demonstrated in the Mensuration of Solids.

EXAM. 2. To find the solidity of a spheroid.

EXAM. 3. To find the solidity of a paraboloid.

EXAM. 4. To find the solidity of an hyperboloid.

TO FIND LOGARITHMS.

103. It has been proved, art 23, that the fluxion of the hyperbolic logarithm of a quantity, is equal to the fluxion of the quantity divided by the same quantity. Therefore, when any quantity is proposed, to find its logarithm; take the fluxion of that quantity, and divide it by the same quantity; then take the fluent of the quotient, either in a series or otherwise, and it will be the logarithm sought: when corrected as usual, if need be; that is, the hyperbolic logarithm.

104. But, for any other logarithm, multiply the hyperbolic logarithm, above found, by the modulus of the system, for the logarithm sought.

Note.

Note. The modulus of the hyperbolic logarithms, is 1; and the modulus of the common logarithms, is ·43429448190 &c; and, in general, the modulus of any system, is equal to the logarithm of 10 in that system divided by the number 2·3025850929940 &c, which is the hyp. log. of 10. Also, the hyp. log. of any number, is in proportion to the com. log. of the same number, as unity or 1 is to ·43429&c, or as the number 2·302585&c, is to 1; and therefore, if the common log. of any number be multiplied by 2·302585&c, it will give the hyp. log. of the same number; or if the hyp. log. be divided by 2·302585&c, or multiplied by ·43429&c, it will give the common logarithm.

EXAM. 1. To find the log. of $\frac{a+x}{a}$.

Denoting any proposed number z, whose logarithm is required to be found, by the compound expression - - - - $\frac{a+x}{a}$, the fluxion of the number $\dot{z}$, is $\frac{\dot{x}}{a}$, and the fluxion of the log. $\frac{\dot{z}}{z} = \frac{\dot{x}}{a+x} = \frac{\dot{x}}{a} - \frac{x\dot{x}}{a^2} + \frac{x^2\dot{x}}{a^3} - \frac{x^3\dot{x}}{a^4} +$ &c.

Then the fluent of these terms give the logarithm of z or logarithm of $\frac{a+x}{a} = \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4}$ &c.

Writing $-x$ for x, gives log. $\frac{a-x}{a} = -\frac{x}{a} - \frac{x^2}{2a^2} - \frac{x^3}{3a^3} - \frac{x^4}{4a^4}$ &c.

Div. these numb. and subtr. their logs. gives $\Big\}$ log. $\frac{a+x}{a-x} = \frac{2x}{a} + \frac{2x^3}{3a^3} + \frac{2x^5}{5a^5}$ &c.

Also, because $\frac{a}{a \pm x} = 1 \div \frac{a \pm x}{a}$, or log. $\frac{a}{a \pm x} = 0 - \log. \frac{a \pm x}{a}$;

therefore log. of $\frac{a}{a+x}$ is $-\frac{x}{a} + \frac{x^2}{2a^2} - \frac{x^3}{3a^3} + \frac{x^4}{4a^4}$ -&c,

and the log. of $\frac{a}{a-x}$ is $+\frac{x}{a} + \frac{x^2}{2a^2} + \frac{x^3}{3a^3} + \frac{x^4}{4a^4}$ &c,

the prod. gives log. $\frac{a^2}{a^2 - x^2} = \frac{x^2}{a^2} + \frac{x^4}{2a^4} + \frac{x^6}{3a^6} +$ &c.

Now, for an example in numbers, suppose it were required to compute the common logarithm of the number 2. This will be best done by the series,

$$\text{log. of } \frac{a+x}{a-x} = 2m \times \left(\frac{x}{a} + \frac{x^3}{3a^3} + \frac{x^5}{5a^5} + \frac{x^7}{7a^7} \text{ &c.}\right.$$

Making

Making $\frac{a+x}{a-x}=2$, gives $a=3x$; conseq. $\frac{x}{a}=\frac{1}{3}$, and $\frac{x^2}{a^2}=\frac{1}{9}$, which is the constant factor for every succeeding term; also, $2m=2\times\cdot434294482190=\cdot868588964$; therefore the calculation will be conveniently made, by first dividing this number by 3, then the quotients successively by 9, and lastly these quotients in order by the respective numbers 1, 3, 5, 7, 9, &c, and after that, adding all the terms together, as follows:

3)	·868588964			
9)	289529654	1)	·289529654	(·289529654
9)	32169962	3)	32169962	(10723321
9)	3574440	5)	3574440	(714888
9)	397160	7)	397160	(56737
9)	44129	9)	44129	(4903
9)	4903	11)	4903	(446
9)	545	13)	545	(42
9)	61	15)	61	(4

Sum of the terms gives log. 2 = ·301029995

EXAM. 2. To find the log. of $\frac{a+x}{b}$.

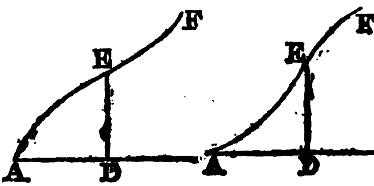
EXAM. 3. To find the log. of $a-x$.

EXAM. 4. To find the log. of 3.

EXAM. 5. To find the log. of 5.

EXAM. 6. To find the log. of 11.

TO FIND THE POINTS OF INFLEXION, OR OF CONTRARY FLEXURE IN CURVES.



106. THE Point of Inflexion in a curve, is that point of it which separates the concave from the convex part, lying between the two; or where the curve changes from concave to convex, or from convex to concave, on the same side of the curve. Such as the point E in the annexed figures, where the former of the two is concave towards

towards the axis AD, from A to E, but convex from E to F; and on the contrary, the latter figure is convex from A to E, and concave from E to F.

106. From the nature of curvature, as has been remarked before at art. 28, it is evident, that when a curve is concave towards an axis, then the fluxion of the ordinate decreases, or is in a decreasing ratio, with regard to the fluxion of the absciss; but, on the contrary, that it increases, or is in an increasing ratio to the fluxion of the absciss, when the curve is convex towards the axis; and consequently those two fluxions are in a constant ratio at the point of inflexion, where the curve is neither convex nor concave; that is, $\dot{x}$ is to $\dot{y}$ in a constant ratio, or $\frac{\dot{y}}{\dot{x}}$ or $\frac{\dot{x}}{\dot{y}}$ is a constant quantity. But constant quantities have no fluxion, or their fluxion is equal to nothing; so that in this case, the fluxion of $\frac{\dot{y}}{\dot{x}}$ or of $\frac{\dot{x}}{\dot{y}}$ is equal to nothing. And hence we have this general rule:

107. Put the given equation of the curve into fluxions; from which find either $\frac{\dot{y}}{\dot{x}}$ or $\frac{\dot{x}}{\dot{y}}$. Then take the fluxion of this ratio, or fraction, and put it equal to 0 or nothing; and from this last equation find also the value of the same $\frac{\dot{x}}{\dot{y}}$ or $\frac{\dot{y}}{\dot{x}}$. Then put this latter value equal to the former, which will form an equation; from which, and the first given equation of the curve, x and y will be determined, being the absciss and ordinate answering to the point of inflexion in the curve, as required.

EXAMPLES.

EXAM. 1. To find the point of inflexion in the curve whose equation is $ax^2 = a^2y + x^2y$.

This equation in fluxions is $2ax\dot{x} = a^2\dot{y} + 2xy\dot{x} + x^2\dot{y}$, which gives $\frac{\dot{x}}{\dot{y}} = \frac{a^2 + x^2}{2ax - 2xy}$. Then the fluxion of this quantity made $= 0$, gives $2x\dot{x}(ax - xy) = (a^2 + x^2) \times (a\dot{x} - \dot{x}y - x\dot{y})$; and this again gives $\frac{\dot{x}}{\dot{y}} = \frac{a^2 + x^2}{a^2 - x^2} \times \frac{x}{a - y}$.

Lastly, this value of $\frac{\dot{x}}{\dot{y}}$ being put equal the former, gives

$\frac{a^2 + x^2}{a^2 - x^2}$

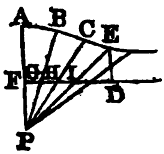
$\frac{a^2+x^2}{a^2-x^2} \cdot \frac{x}{a-y} = \frac{a^2+x^2}{2x} \cdot \frac{1}{a-y}$; and hence $2x^2 = a^2 - x^2$, or $3x^2 = a^2$, and $x = a\sqrt{\frac{1}{3}}$, the absciss.

Hence also, from the original equation, - - - - - - -

$y = \frac{ax^2}{a^2+x^2} = \frac{\frac{1}{3}a^3}{\frac{4}{3}a^2} = \frac{1}{4}a$, the ordinate of the point of inflexion sought.

Exam. 2. To find the point of inflexion in a curve defined by the equation $ay = a\sqrt{ax^2 + xx}$.

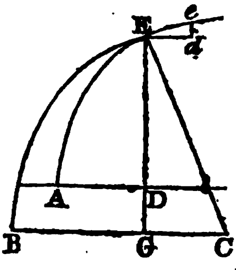
Exam. 3. To find the point of inflexion in a curve defined by the equation $ay^2 = a^2x + x^3$.



Exam. 4. To find the point of inflexion in the Conchoid of Nicomedes, which is generated or constructed in this manner: From a fixed point P, which is called the pole of the conchoid, draw any number of right lines PA, PB, PC, PE, &c, cutting the given line FD in the points F, G, H, I, &c: then make the distances FA, GB, HC, IE, &c, equal to each other, and equal to a given line; then the curve line ABCE &c, will be the conchoid; a curve so called by its inventor Nicomedes.

TO FIND THE RADIUS OF CURVATURE OF CURVES.

108. The Curvature of a Circle is constant, or the same in every point of it, and its radius is the radius of curvature. But the case is different in other curves, every one of which has its curvature continually varying, either increasing or decreasing, and every point having a degree of curvature peculiar to itself; and the radius of a circle which has the same curvature with the curve at any given point, is the radius of curvature at that point; which radius it is the business of this chapter to find.



109. Let AEe be any curve, concave towards its axis AD; draw an ordinate DE to the point E, where the curvature is to be found; and suppose EC perpendicular to the curve, and equal to the radius of curvature sought, or equal to the radius of a circle having the same curvature there, and with that radius describe the said equally-

curved

curved circle BEe; lastly, draw Ed parallel to AD, and de parallel and indefinitely near to DE: thereby making Ed the fluxion or increment of the absciss AD, also de the fluxion of the ordinate DE, and Ee that of the curve AE. Then put $x =$ AD, $y =$ DE, $z =$ AE, and $r =$ CE the radius of curvature; then is E$d = \dot{x}$, $de = \dot{y}$, and E$e = \dot{z}$.

Now, by sim. triangles, the three lines Ed, de, Ee,
or $\dot{x}$, $\dot{y}$, $\dot{z}$,
are respectively as the three - - - GE, GC, CE;
therefore - - - - - - - - - - - GC . $\dot{x}$ = GE . $\dot{y}$;
and the flux of this eq. is GC . $\ddot{x}$ + G$\dot{\text{C}}$. $\dot{x}$ = GE . $\ddot{y}$ + G$\dot{\text{E}}$. $\dot{y}$;
or, because G$\dot{\text{C}}$ = −B$\dot{\text{G}}$, it is GC . $\ddot{x}$ − B$\dot{\text{G}}$. $\dot{x}$ = GE . $\ddot{y}$ + G$\dot{\text{E}}$. $\dot{y}$.

But since the two curves AE and BE have the same curvature at the point E, their abscisses and ordinates have the same fluxions at that point, that is, Ed, or $\dot{x}$ is the fluxion both of AD and BG, and de or $\dot{y}$ is the fluxion both of DE and GE. In the equation above therefore substitute $\dot{x}$ for $\dot{\text{B}}$G, and $\dot{y}$ for G$\dot{\text{E}}$, and it becomes

$$\text{GC}\ddot{x} - \dot{x}\dot{x} = \text{GE}\ddot{y} + \dot{y}\dot{y},$$
$$\text{or GC}\ddot{x} - \text{GE}\ddot{y} = \dot{x}^2 + \dot{y}^2 = \dot{z}^2.$$

Now multiply the three terms of this equation respectively, by these three quantities, $\frac{\dot{y}}{\text{GC}}$, $\frac{\dot{x}}{\text{GE}}$, $\frac{\dot{z}}{\text{CE}}$, which are all equal, and it becomes - - - $\dot{y}\ddot{x} - \dot{x}\ddot{y} = \frac{\dot{z}^3}{\text{CE}}$, or $\frac{\dot{z}^3}{r}$;

and hence is found $r = \frac{\dot{z}^3}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$, for the general value of the radius of curvature, for all curves whatever, in terms of the fluxions of the absciss and ordinate.

110. Further, as in any case either x or y may be supposed to flow equably, that is, either $\dot{x}$ or $\dot{y}$ constant quantities, or $\ddot{x}$ or $\ddot{y}$ equal to nothing, it follows that, by this supposition, either of the terms in the denominator, of the value of r, may be made to vanish. Thus, when $\dot{x}$ is supposed constant, $\ddot{x}$ being then = 0, the value of r is barely - - - - - $\frac{\dot{z}^3}{-\dot{x}\ddot{y}}$; or r is $= \frac{\dot{z}^3}{\dot{y}\ddot{x}}$ when $\dot{y}$ is constant.

EXAMPLES.

EXAM. 1. To find the radius of curvature to any point of

of a parabola, whose equation is $ax = y^2$, its vertex being A, and axis AD.

Now, the equation to the curve being $ax = y^2$, the fluxion of it is $a\dot{x} = 2y\dot{y}$; and the fluxion of this again is $a\ddot{x} = 2\dot{y}^2$, supposing $\dot{y}$ constant; hence then r or

$$\frac{\dot{z}^3}{\dot{y}\ddot{x}} \text{ or } \frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{\dot{y}\ddot{x}} \text{ is } = \frac{(a^2+4y^2)^{\frac{3}{2}}}{2a^2} \text{ or } \frac{(a+4x)^{\frac{3}{2}}}{2\sqrt{a}},$$

for the general value of the radius of curvature at any point E, the ordinate to which cuts off the absciss AD $= x$.

Hence, when the absciss x is nothing, the last expression becomes barely $\frac{1}{2}a = r$, for the radius of curvature at the vertex of the parabola; that is, the diameter of the circle of curvature at the vertex of a parabola, is equal to a, the parameter of the axis.

EXAM. 2. To find the radius of curvature of an ellipse, whose equation is $a^2y^2 = c^2 \,.\, \overline{ax - x^2}$.

$$\text{Ans. } r = \frac{(a^2c^2 + 4(a^2 - c^2) \times (ax - x^2))^{\frac{3}{2}}}{2a^4c}.$$

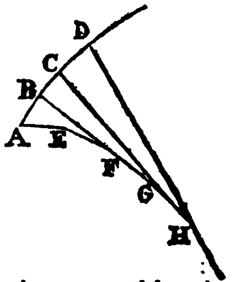
EXAM. 3. To find the radius of curvature of an hyperbola, whose equation is $a^2y^2 = c^2 \,.\, \overline{ax + x^2}$.

EXAM. 4. To find the radius of curvature of the cycloid.
Ans. $r = 2\sqrt{aa - ax}$, where x is the absciss, and a the diameter of the generating circle.

OF INVOLUTE AND EVOLUTE CURVES.

111. AN Evolute is any curve supposed to be evolved or opened, which having a thread wrapped close about it, fastened at one end, and beginning to evolve or unwind the thread from the other end, keeping always tight stretched the part which is evolved or wound off: then this end of the thread will describe another curve, called the Involute. Or, the same involute is described in the contrary way, by wrapping the thread about the curve of the evolute, keeping it at the same time always stretched.

112. Thus

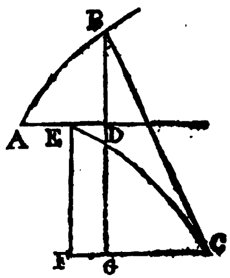


112. Thus, if EFGH be any curve, and AE be either a part of the curve, or a right line: then if a thread be fixed to the curve at H, and be wound or plied close to the curve, &c, from H to A, keeping the thread always stretched tight; the other end of the thread will describe a certain curve ABCD, called an Involute; the first curve EFGH being its evolute. Or, if the thread, fixed at H, be unwound from the curve, beginning at A, and keeping it always tight, it will describe the same involute ABCD.

113. If AE, BF, CG, DH, &c, be any positions of the thread, in evolving or unwinding; it follows, that these parts of the thread are always the radii of curvature, at the corresponding points, A, B, C, D; and also equal to the corresponding lengths AE, AEF, AEFG, AEFGH, of the evolute; that is,

AE = AE is the radius of curvature to the point A,
BF = AF is the radius of curvature to the point B,
CG = AG is the radius of curvature to the point C,
DH = AH is the radius of curvature to the point D.

114. It also follows, from the premises, that any radius of curvature, BF, is perpendicular to the involute at the point B, and is a tangent to the evolute curve at the point F. Also, that the evolute is the locus of the centre of curvature of the involute curve.



115. Hence, and from art. 109, it will be easy to find one of these curves, when the other is given. To this purpose, put

x = AD, the absciss of the involute,
y = DB, an ordinate to the same,
z = AB, the involute curve,
r = BC, the radius of curvature,
v = EF, the absciss of the evolute, EC,
u = FC, the ordinate of the same, and
a = AE, a certain given line.

Then

Then, by the nature of the radius of curvature, it is

$r = \frac{\dot{z}^3}{\dot{y}\ddot{x} - \dot{x}\ddot{y}} = \text{BC} = \text{AE} + \text{EC}$; also, by sim. triangles.

$\dot{z} : \dot{x} :: r : \text{GB} = \frac{r\dot{x}}{\dot{z}} = \frac{\dot{x}\dot{z}^2}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$;

$\dot{z} : \dot{y} :: r : \text{GC} = \frac{r\dot{y}}{\dot{z}} = \frac{\dot{y}\dot{z}}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$.

Hence $\text{EF} = \text{GB} - \text{DB} = \frac{\dot{x}\dot{z}^2}{\dot{y}\ddot{x} - \dot{x}\ddot{y}} - y = v$;

and $\text{FC} = \text{AD} - \text{AE} + \text{GC} = x - a + \frac{\dot{y}\dot{z}^2}{\dot{y}\ddot{x} - \dot{x}\ddot{y}} = u$;
which are the values of the absciss and ordinate of the evolute curve EC; from which therefore these may be found, when the involute is given.

On the contrary, if v and u, or the evolute, be given: then, putting the given curve $\text{EC} = s$, since $\text{CB} = \text{AE} + \text{EC}$, or $r = a + s$, this gives r the radius of curvature. Also, by similar triangles, there arise these proportions, viz.

$$\dot{s} : \dot{v} :: r : \frac{r\dot{v}}{\dot{s}} = \frac{a+s}{\dot{s}}\dot{v} = \text{GB},$$

$$\text{and } \dot{s} : \dot{u} :: r : \frac{r\dot{u}}{\dot{s}} = \frac{a+s}{\dot{s}}\dot{u} = \text{GC};$$

$$\text{theref. AD} = \text{AE} + \text{FC} - \text{GC} = a + u - \frac{a+s}{\dot{s}}\dot{u} = x,$$

$$\text{and DB} = \text{GB} - \text{GD} = \text{GB} - \text{EF} = \frac{a+s}{\dot{s}}\dot{v} - v = y;$$

which are the absciss and ordinate of the involute curve, and which may therefore be found, when the evolute is given. Where it may be noted, that $\dot{s}^2 = \dot{v}^2 + \dot{u}^2$, and $\dot{z}^2 = \dot{x}^2 + \dot{y}^2$. Also, either of the quantities x, y, may be supposed to flow equably, in which case the respective second fluxion, $\ddot{x}$ or $\ddot{y}$, will be nothing, and the corresponding term in the denominator $\dot{y}\ddot{x} - \dot{x}\ddot{y}$ will vanish, leaving only the other term in it; which will have the effect of rendering the whole operation simpler.

116. EXAMPLES.

EXAM. 1. To determine the nature of the curve by whose evolution the common parabola AB is described.

Here

Here the equation of the given involute AB, is $cx = y^2$ where c is the parameter of the axis AD. Hence then $y = \sqrt{cx}$, and $\dot{y} = \frac{1}{2}\dot{x}\sqrt{\frac{c}{x}}$, also $\ddot{y} = \frac{-\dot{x}^2}{4x}\sqrt{\frac{c}{x}}$ by making $\dot{x}$ constant. Consequently the general values of v and u, or of the absciss and ordinate, EF and FC, above given, become, in that case,

$$\text{EF} = v = \frac{\dot{z}^2}{-\ddot{y}} - y = \frac{\dot{x}^2 + \dot{y}^2}{-\ddot{y}} - y = 4x\sqrt{\frac{x}{c}}\text{; and}$$

$$\text{FC} = u = x - a + \frac{\dot{y}\dot{z}^2}{-\dot{x}\ddot{y}} = 3x + \tfrac{1}{2}c - a.$$

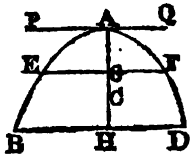
But the value of the quantity a or AE, by exam. 1 to art. 75, was found to be $\frac{1}{2}c$; consequently the last quantity, FC or u, is barely $= 3x$.

Hence then, comparing the values of v and u, there is found $3v\sqrt{c} = 4u\sqrt{x}$, or $27cv^2 = 16u^3$; which is the equation between the absciss and ordinate of the evolute curve EC, showing it to be the semicubical parabola.

Exam. 2. To determine the evolute of the common cycloid. Ans. another cycloid, equal to the former.

TO FIND THE CENTRE OF GRAVITY.

117. By referring to prop. 42, &c, in Mechanics, it is seen what are the principles and nature of the Centre of Gravity in any figure, and how it is generally expressed. It there appears, that if PAQ be a line, or plane, drawn through any point, as suppose the vertex of any body, or figure, ABD, and if - - - s denote any section EF of the figure, d = AG, its distance below PQ, and b = the whole body or figure ABD; then the distance AC, of the centre of gravity below PQ, is universally denoted by $\frac{\text{sum of all the } ds}{b}$; whether ABD be a line, or a plane surface, or a curve superficies, or a solid.



But

But the sum of all the $d s$, is the same as the fluent of $d\dot{b}$, and b is the same as the fluent of $\dot{b}$; therefore the general expression for the distance of the centre of gravity, is AC $= \frac{\text{fluent of } x\dot{b}}{\text{fluent of } \dot{b}} = \frac{\text{fluent } x\dot{b}}{b}$; putting $x = d$ the variable distance AG. Which will divide into the following four cases.

118. CASE 1. When AE is some line, as a curve suppose. In this case $\dot{b}$ is $= \dot{z}$ or $\sqrt{\dot{x}^2 + \dot{y}^2}$, the fluxion of the curve; and $b = z$: theref. AC $= \frac{\text{fluent of } x\dot{z}}{z} = \frac{\text{fluent of } x\sqrt{\dot{x}^2 + \dot{y}^2}}{z}$ is the distance of the centre of gravity in a curve.

119. CASE. 2. When the figure ABD is a plane; then $\dot{b} = y\dot{x}$; therefore the general expression becomes AC $= \frac{\text{fluent of } yx\dot{x}}{\text{fluent of } y\dot{x}}$ for the distance of the centre of gravity in a plane.

120. CASE 3. When the figure is the superficies of a body generated by the rotation of a line AEB, about the axis AH. Then, putting $c = 3{\cdot}14159$ &c, $2cy$ will denote the circumference of the generating circle, and $2cy\dot{z}$ the fluxion of the surface; therefore AC $= \frac{\text{fluent of } 2cyx\dot{z}}{\text{fluent of } 2cy\dot{z}} = \frac{\text{fluent of } yx\dot{z}}{\text{fluent of } y\dot{z}}$ will be the distance of the centre of gravity for a surface generated by the rotation of a curve line z.

121. CASE. 4. When the figure is a solid generated by the rotation of a plane ABH, about the axis AH.

Then, putting $c = 3{\cdot}14159$ &c, it is cy^2 = the area of the circle whose radius is y, and $cy^2\dot{x} = \dot{b}$, the fluxion of the solid; therefore - - - - - - - - - - - - - - - -

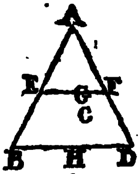
AC $= \frac{\text{fluent of } x\dot{b}}{\text{fluent of } \dot{b}} = \frac{\text{fluent of } cy^2x\dot{x}}{\text{fluent of } cy^2\dot{x}} = \frac{\text{fluent of } y^2x\dot{x}}{\text{fluent of } y^2\dot{x}}$ is the distance of the centre of gravity below the vertex in a solid.

122. EXAMPLES.

EXAM. 1. Let the figure proposed be the isosceles triangle ABD.

It is evident that the centre of gravity C, will be somewhere

where in the perpendicular AH. Now, if a denote AH, $c =$ BD, $x =$ AG, and $y =$ EF any line parallel to the base BD: then as $a : c :: x : y = \frac{cx}{a}$; therefore, by the 2d Case, AC $= \frac{\text{fluent } yx\dot{x}}{\text{fluent } y\dot{x}} = \frac{\text{fluent } x^2\dot{x}}{\text{fluent } x\dot{x}} = \frac{\frac{1}{3}x^3}{\frac{1}{2}x^2}$ $= \frac{2}{3}x = \frac{2}{3}$ AH, when x becomes $=$ AH: consequently GH $=$ $\frac{1}{3}$AH.



In like manner, the centre of gravity of any other plane triangle, will be found to be at $\frac{1}{3}$ of the altitude of the triangle; the same as it was found in prop. 43, Mechanics.

EXAM. 2. In a parabola; the distance from the vertex is $\frac{3}{5}x$, or $\frac{3}{5}$ of the axis.

EXAM. 3. In a circular arc; the distance from the centre of the circle, is $\frac{cr}{a}$; where a denotes the arc, c its chord, and r the radius.

EXAM. 4. In a circular sector; the distance from the centre of the circle, is $\frac{2cr}{3a}$: where a, c, r, are the same as in exam. 3.

EXAM. 5. In a circular segment; the distance from the centre of the circle is $\frac{c^3}{12a}$; where c is the chord, and a the area, of the segment.

EXAM. 6. In a cone, or any other pyramid; the distance from the vertex is $\frac{3}{4}x$, or $\frac{3}{4}$ of the altitude.

EXAM. 7. In the semisphere, or semispheroid; the distance from the centre is $\frac{3}{8}r$, or $\frac{3}{8}$ of the radius: and the distance from the vertex $\frac{5}{8}$ of the radius.

EXAM. 8. In the parabolic conoid; the distance from the base is $\frac{1}{3}x$, or $\frac{1}{3}$ of the axis. And the distance from the vertex $\frac{2}{3}$ of the axis.

EXAM. 9. In the segment of a sphere, or of a spheroid; the distance from the base is $\frac{2a - x}{6a - 4x}x$; where x is the height of the segment, and a the whole axis, or diameter of the sphere.

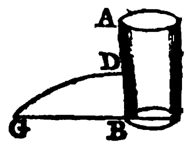
EXAM. 10. In the hyperbolic conoid; the distance from the base is $\frac{2a + x}{6a + 4x}x$; where x is the height of the conoid, and a the whole axis or diameter.

PRACTICAL

123. PRACTICAL QUESTIONS.

QUESTION I.

A LARGE vessel, of 10 feet, or any other given depth, and of any shape, being kept constantly full of water, by means of a supplying cock, at the top; it is proposed to assign the place where a small hole must be made in the side of it, so that the water may spout through it to the greatest distance on the plane of the base.

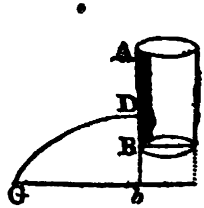


Let AB denote the height or side of the vessel; D the required hole in the side, from which the water spouts, in the parabolic curve DG, to the greatest distance BG, on the horizontal plane.

By the scholium to prop. 68, Hydraulics, the distance BG is always equal to $2\sqrt{\text{AD}\,.\,\text{DB}}$, which is equal to $2\sqrt{(x-a)}$ or $2\sqrt{ax-x^2}$, if a be put to denote the whole height AB of the vessel, and $x = \text{AD}$, the depth of the hole. Hence $2\sqrt{ax - x^2}$, or $ax - x^2$, must be a maximum. In fluxions, $a\dot{x} - 2x\dot{x} = 0$, or $a - 2x = 0$, and $2x = a$, or $x = \frac{1}{2}a$. So that the hole D must be in the middle between the top and bottom; the same as before found at the end of the scholium above quoted.

124. QUESTION II.

If the same vessel, as in Quest. 1, stand on high, with its bottom a given height above a horizontal plane below; it is proposed to determine where the small hole must be made, so as to spout farthest on the said plane.



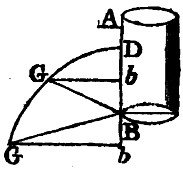
Let the annexed figure represent the vessel as before, and bG the greatest distance spouted by the fluid, DG, on the plane bG.

Here, as before, $b\text{G} = 2\sqrt{\text{AD}\,.\,\text{D}b} = 2\sqrt{x(c-x)} = 2\sqrt{cx - x^2}$, by putting $\text{A}b = c$, and $\text{AD} = x$. So that $2\sqrt{cx - x^2}$ or $cx - x_2$ must be a maximum. And hence, like as in the former question, - - - $x = \frac{1}{2}c = \frac{1}{2}\text{A}b$. So that the hole D must be made in the middle

middle between the top of the vessel, and the given plane, that the water may spout farthest.

125. QUESTION III.

But if the same vessel, as before, stand on the top of an inclined plane, making a given angle, as suppose of 30 degrees, with the horizon; it is proposed to determine the place of the small hole, so as the water may spout the farthest on the said inclined plane.



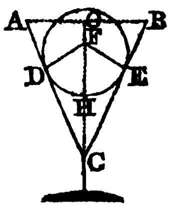
Here again (D being the place of the hole, and BG the given inclined plane), $b\text{G} = 2\sqrt{\text{AD} \cdot \text{D}b} = 2\sqrt{x(a - x \pm z)}$, putting $z = \text{B}b$, and, as before, $a = \text{AB}$; and $x = \text{AD}$. Then $b\text{G}$ must still be a maximum, as also $\text{B}b$, being in a given ratio to the maximum BG, on account of the given angle B. Therefore $ax - x^2 \pm xz$, as well as z, is a maximum. Hence, by art. 54 of the Fluxions, $a\dot{x} - 2x\dot{x} \pm z\dot{x} = 0$, or $a - 2x \pm z = 0$; conseq. $\pm z = 2x - a$; and hence $b\text{G} = 2\sqrt{x(a - x \pm z)}$ becomes barely $2x$. But as the given angle $\text{GB}b$ is $= 30°$, the sine of which is $\frac{1}{2}$; therefore $\text{BG} = 2\text{B}b$ or $2z$, and $b\text{G}^2 = \text{BG}^2 - \text{B}b^2 = 3z^2 = 3\,(2x - a)^2$, or $b\text{G} = \pm\,(2x - a)\sqrt{3}$.

Putting now these two values of $b\text{G}$ equal to each other, gives the equation $2x = \pm\,(2x - a)\sqrt{3}$, from which is found $x = \frac{\frac{1}{2}a\sqrt{3}}{\sqrt{3} \pm 1} = \frac{3 \pm \sqrt{3}}{4}\,a$, the value of AD required.

Note. In the Select Exercises, page 252, this answer is brought out $\frac{6 + \sqrt{6}}{10}\,a$, by taking the velocity proportional to the root of *half* the altitude only.

126. QUESTION IV.

It is required to determine the size of a ball, which, being let fall into a conical glass full of water, shall expel the most water possible from the glass; its depth being 6, and diameter 5 inches.



Let ABC represent the cone of the glass, and DHE the ball, touching the sides in the points D and E, the centre of the ball being at some points F in the axis GC of the cone.

Put $AG = GB = 2\frac{1}{2} = a$,
$CG = 6 = b$,
$AC = \sqrt{AG^2 + GC^2} = 6\frac{1}{2} = c$,
$AD = FE = FH = x$ the radius of the ball.

The two triangles ACG and DCF are equiangular; theref. $AG : AC :: DF : FC$, that is, $a : c :: x : \frac{cx}{a} = FC$; hence $GF = GC - FC = b - \frac{cx}{a}$, and $GH = GF + FH = b + x - \frac{cx}{a}$, the height of the segment immersed in the water. Then (by rule 1 for the spherical segment, p. 427 vol. 1.), the content of the said immersed segment will be $(6DF - 2GH) \times GH^2 \times \cdot 5236 = (2x - b + \frac{cx}{a}) \times (x + b - \frac{cx}{a})^2 \times 1\cdot 0472$, which must be a maximum by the question; the fluxion of this made $= 0$, and divided by $2\dot{x}$ and the common factors, gives $\frac{2a+c}{a} \times (b - \frac{c-a}{a}x) - (\frac{2a+c}{a}x - b) \times \frac{c-a}{a} \times 2 = 0$; this reduced gives $x = \frac{abc}{(c-a) \times (c+2a)} = 2\frac{14}{92}$, the radius of the ball. Consequently its diameter is $4\frac{11}{46}$ inches, as required.

PRACTICAL EXERCISES CONCERNING FORCES; WITH THE RELATION BETWEEN THEM AND THE TIME, VELOCITY, AND SPACE DESCRIBED.

Before entering on the following problems, it will be convenient here, to lay down a synopsis of the theorems which express the several relations between any forces, and their corresponding times, velocities, and spaces, described; which are all comprehended in the following 12 theorems, as collected from the principles in the foregoing parts of this work.

Let f, F, be any two constant accelerative forces, acting on any body, during the respective times t, T, at the end of which are generated the velocities v, V, and described the spaces s, S. Then, because the spaces are as the times and velocities conjointly, and the velocities as the forces and times; we shall have,

I. In

1. *In Constant Forces.*

$$1.\quad \frac{s}{S} = \frac{tv}{TV} = \frac{t^2f}{T^2F} = \frac{v^2F}{V^2f}.$$

$$2.\quad \frac{v}{V} = \frac{ft}{FT} = \frac{sT}{St} = \sqrt{\frac{fs}{FS}}.$$

$$3.\quad \frac{t}{T} = \frac{Fv}{fV} = \frac{sV}{Sv} = \sqrt{\frac{Fs}{fS}}.$$

$$4.\quad \frac{f}{F} = \frac{Tv}{tV} = \frac{T^2s}{t^2S} = \frac{v^2S}{V^2s}.$$

And if one of the forces, as F, be the force of gravity at the surface of the earth, and be called 1, and its time T be $= 1''$; then it is known by experiment that the corresponding space S is $= 16\frac{1}{12}$ feet, and consequently its velocity V $= 2$S $= 32\frac{1}{6}$, which call $2g$, namely, $g = 16\frac{1}{12}$ feet, or 193 inches. Then the above four theorems, in this case, become as here below:

$$5.\quad s = \tfrac{1}{2}tv = gft^2 = \frac{v^2}{4gf}.$$

$$6.\quad v = \frac{2s}{t} = 2gft = \sqrt{4gfs}.$$

$$7.\quad t = \frac{2s}{v} = \frac{v}{2gf} = \sqrt{\frac{s}{gf}}.$$

$$8.\quad f = \frac{v}{2gt} = \frac{s}{gt^2} = \frac{v^2}{4gs}.$$

And from these are deduced the following four theorems, for variable forces, viz.

II. *In Variable Forces.*

$$9.\quad \dot{s} = v\dot{t} = \frac{v\dot{v}}{2gf}.$$

$$10.\quad \dot{v} = 2gf\dot{t} = \frac{2gf\dot{s}}{v}.$$

$$11.\quad \dot{t} = \frac{\dot{s}}{v} = \frac{\dot{v}}{2gf}.$$

$$12.\quad f = \frac{v\dot{v}}{2g\dot{s}} = \frac{\dot{v}}{2g\dot{t}}.$$

In these last four theorems, the force f, though variable, is supposed to be constant for the indefinitely small time t, and they are to be used in all cases of variable forces, as the former ones in constant forces; namely from the circumstances of the problem under consideration, an expression is deduced for the value of the force f, which being substituted in one of these theorems, that may be proper to the case in hand; the equation thence resulting will determine the corresponding values of the other quantities, required in the problem.

When a motive force happens to be concerned in the question, it may be proper to observe, that the motive force m, of a body, is equal to fq, the product of the accelerative force, and the quantity of matter in it q; and the relation between these three quantities being universally expressed by this equation $m = qf$, it follows that, by means of it, any one of the three may be expelled out of the calculation, or else brought into it.

Also, the momentum, or quantity of motion in a moving body, is qv, the product of the velocity and matter.

It is also to be observed, that the theorems equally hold good for the destruction of motion and velocity, by means of retarding forces, as for the generation of the same, by means of accelerating forces.

And to the following problems, which are all resolved by the application of these theorems, it has been thought proper to subjoin their solutions, for the better information and convenience of the student.

PROBLEM I.

To determine the time and velocity of a body descending, by the force of gravity, down an inclined plane; the length of the plane being 20 feet, and its height 1 foot.

Here, by Mechanics, the force of gravity being to the force down the plane, as the length of the plane is to its height, therefore as $20 : 1 :: 1$ (the force of gravity) : $\frac{1}{20} = f$, the force on the plane.

Therefore, by theor. 6, v or $\sqrt{4gfs}$ is $\sqrt{4 \times 16\frac{1}{12} \times \frac{1}{20} \times 20} = \sqrt{4 \times 16\frac{1}{12}} = 2 \times 4\frac{1}{96}$ or $8\frac{1}{48}$ feet nearly, the last velocity per second. And,

By theor. 7, t or $\sqrt{\frac{s}{gf}}$ is $\sqrt{\frac{20}{16\frac{1}{12} \times \frac{1}{20}}} = \sqrt{\frac{400}{16\frac{1}{12}}} = \frac{20}{4\frac{1}{96}} = 4\frac{76}{77}$ seconds, the time of descending.

PROBLEM

PROBLEM II.

If a canon ball be fired with a velocity of 1000 *feet per second, up a smooth inclined plane, which rises* 1 *foot in* 20 *: it is proposed to assign the length which it will ascend up the plane, before it stops and begins to return down again, and the time of its ascent.*

Here $f = \frac{1}{20}$ as before.

Then, by theor. 5, $s = \frac{v^2}{4gf} = \frac{1000^2}{4 \times 16\frac{1}{12} \times \frac{1}{20}} = \frac{60000000}{193}$ $= 310880\frac{160}{193}$ feet, or nearly 59 miles, the distance moved.

And, by theor. 7, $t = \frac{v}{2gf} = \frac{1000}{2 \times 16\frac{1}{12} \times \frac{1}{20}} = \frac{120000}{193} =$ $621'' \frac{147}{193} = 10' \, 21'' \frac{147}{193}$, the time of ascent.

PROBLEM III.

If a ball be projected up a smooth inclined plane, which rises 1 *foot in* 10, *and ascend* 100 *feet before it stop : required the time of ascent, and the velocity of projection.*

First, by theor. 6, $v = \sqrt{4gfs} = \sqrt{4 \times 16\frac{1}{12} \times \frac{1}{10} \times 100} = 8\frac{1}{48}\sqrt{10} = 25\cdot36408$ feet per second, the velocity.

And, by theor. 7, $t = \sqrt{\frac{s}{gf}} = \sqrt{\frac{100}{16\frac{1}{12} \times \frac{1}{10}}} = \frac{10}{4\frac{1}{96}}\sqrt{10}$ $= \frac{192}{77}\sqrt{10} = 7\cdot88516$ seconds, the time in motion.

PROBLEM IV.

If a ball be observed to ascend up a smooth inclined plane, 100 *feet in* 10 *seconds, before it stop, to return back again : required the velocity of projection, and the angle of the plane's inclination.*

First, by theor. 6, $v = \frac{2s}{t} = \frac{200}{10} = 20$ feet per second, the velocity.

And, by theor. 8, $f = \frac{s}{gt^2} = \frac{100}{16\frac{1}{12} \times 100} = \frac{12}{193}$. That is, the length of the plane is to its height, as 193 to 12.

Therefore 193 : 12 : : 100 : 6·2176 the height of the plane, or the sine of elevation to radius 100, which answers to 3° 34′, the angle of elevation of the plane.

PROBLEM

PROBLEM V.

By a mean of several experiments, I have found, that a cast iron ball, of 2 inches diameter, fired perpendicularly into the face or end of a block of elm wood, or in the direction of the fibres, with a velocity of 1500 feet per second, penetrated 13 inches deep into its substance. It is proposed then to determine the time of the penetration, and the resisting force of the wood, as compared to the force of gravity, supposing that force to be a constant quantity.

First, by theor. 7, $t = \frac{2s}{v} = \frac{2 \times 13}{1500 \times 12} = \frac{1}{692}$ part of a second, the time in penetrating.

And, by theor. 8, $f = \frac{v^2}{4gs} = \frac{1500^2}{4 \times 16\frac{1}{12} \times \frac{13}{12}} = \frac{81000000}{13 \times 193}$ $= 32284$. That is, the resisting force of the wood, is to the force of gravity, as 32284 to 1.

But this number will be different, according to the diameter of the ball, and its density or specific gravity. For, since f is as $\frac{v^2}{s}$ by theor. 4, the density and size of the ball remaining the same; if the density, or specific gravity, n, vary, and all the rest be constant, it is evident that f will be as n; and therefore f as $\frac{nv^2}{s}$ when the size of the ball only is constant. But when only the diameter d varies, all the rest being constant, the force of the blow will vary as d^3, or as the magnitude of the ball; and the resisting surface, or force of resistance, varies as d^2; therefore f is as $\frac{d^3}{d^2}$, or as d only when all the rest are constant. Consequently f is as $\frac{dnv^2}{s}$ when they are all variable.

And so $\frac{f}{F} = \frac{dnv^2S}{DNV^2s}$ and $\frac{s}{S} = \frac{dnv^2F}{DNV^2f}$; where f denotes the strength or firmness of the subtance penetrated, and is here supposed to be the same, for all balls and velocities, in the same substance, which is either accurately or nearly so. See page 581, &c, vol. 1, of my Tracts.

Hence, taking the numbers in the problem, it is - - -

$f = \frac{dnv^2}{s} = \frac{\frac{6}{12} \times 7\frac{1}{3} \times 1500^2}{\frac{13}{12}} = \frac{44 \times 1500^2}{39} = 2538462$ the value of f for elm wood. Where the specific gravity of the

the ball is taken $7\frac{1}{3}$, which is a little less than that of solid cast iron, as it ought, on account of the air bubble which is found in all cast balls.

PROBLEM VI.

To find how far a 24lb ball of cast iron will penetrate into a block of sound elm, when fired with a velocity of 1600 feet per second.

Here, because the substance is the same as in the last problem, both of the balls and wood, $N = n$, and $F = f$; therefore $\frac{S}{s} = \frac{DV^2}{dv^2}$, or $S = \frac{DV^2 s}{dv^2} = \frac{5{\cdot}55 \times 1600^2 \times 13}{2 \times 1500^2} = 41\frac{2}{45}$ inches nearly, the penetration required.

PROBLEM VII.

It was found by Mr. Robins, (vol. i. p. 273, *of his works*), *that an 18-pounder ball, fired with a velocity of 1200 feet per second, penetrated 34 inches into sound dry oak. It is required then to ascertain the comparative strength or firmness of oak and elm.*

The diameter of an 18lb ball is 5·04 inches = D. Then, by the numbers given in this problem for oak, and in prob. 5, for elm, we have - - - - - - - - - - - - - -

$$\frac{f}{F} = \frac{dv^2 s}{DV^2 s} = \frac{2 \times 1500^2 \times 34}{5{\cdot}04 \times 1200^2 \times 13} = \frac{100 \times 17}{5{\cdot}04 \times 16 \times 13} = \frac{1700}{1048} \text{ or } = \frac{8}{5}$$

nearly.

From which it would seem, that elm timber resists more than oak, in the ratio of about 8 to 5; which is not probable, as oak is a much firmer and harder wood. But it is to be suspected that the great penetration in Mr. R's experiment was owing to the splitting of his timber in some degree.

PROBLEM VIII.

A 24-pounder ball being fired into a bank of firm earth, with a velocity of 1300 feet per second, penetrated 15 feet. It is required then to ascertain the comparative resistance of elm and earth.

Comparing the numbers here with those in prob. 5, it is

is $\frac{f}{F} = \frac{dv^2s}{DV^2s} = \frac{2 \times 1500^2 \times 15 \times 12}{5{\cdot}55 \times 1300^2 \times 13} = \frac{15^2 \times 24}{13^3 \times 0{\cdot}37} = \frac{1800}{271} = \frac{20}{3}$ nearly $= 6\frac{2}{3}$ nearly. That is, elm timber resists about $6\frac{2}{3}$ times more than earth.

PROBLEM IX.

To determine how far a leaden bullet, of $\frac{3}{4}$ of an inch diameter, will penetrate dry elm; supposing it fired with a velocity of 1700 *feet per second, and that the lead does not charge its figure by the stroke against the wood.*

HERE $D = \frac{3}{4}$, $N = 11\frac{1}{3}$, $n = 7\frac{1}{3}$. Then, by the numbers and theorem in prob. 5, it is $S =$ - - - - - - - - - - -

$$\frac{DNV^2s}{dnv^2} = \frac{\frac{3}{4} \times 11\frac{1}{3} \times 1700^2 \times 13}{2 \times 7\frac{1}{3} \times 1500^2} = \frac{17^3 \times 13}{200 \times 33} = \frac{63869}{6600} =$$

$9\frac{2}{3}$ inches nearly, the depth of penetration.

But as Mr. Robins found this penetration, by experiment, to be only 5 inches; it follows, either that his timber must have resisted about twice as much; or else, which is much more probable, that the defect in his penetration arose from the change of figure in the leaden ball he used, from the blow against the wood.

PROBLEM X.

A one pound ball, projected with a velocity of 1500 *feet per second, having been found to penetrate* 13 *inches deep into dry elm: It is required to ascertain the time of passing through every single inch of the* 13, *and the velocity lost at each of them; supposing the resistance of the wood constant or uniform.*

THE velocity v being 1500 feet, or $1500 \times 12 = 18000$ inches, and velocities and times being as the roots of the spaces, in constant retarding forces, as well as in accelerating ones, and t being $= \frac{2s}{v} = \frac{26}{12 \times 1500} = \frac{13}{9000} = \frac{1}{692}$ part of a second, the whole time of passing through the 13 inches; therefore as

$\sqrt{13}$

$$\sqrt{13} : \sqrt{13} - \sqrt{12} :: v :$$

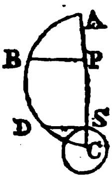
veloc. lost	Time in the
$\frac{\sqrt{13}-\sqrt{12}}{\sqrt{13}} v = 58{\cdot}9 :: t :$	$\frac{\sqrt{13}-\sqrt{12}}{\sqrt{13}} t = {\cdot}00005$ 1st inch.
$\frac{\sqrt{12}-\sqrt{11}}{\sqrt{13}} v = 61{\cdot}4 :: t :$	$\frac{\sqrt{12}-\sqrt{11}}{\sqrt{13}} t = {\cdot}00006$ 2d
$\frac{\sqrt{11}-\sqrt{10}}{\sqrt{13}} v = 64{\cdot}2$ &c	$\frac{\sqrt{11}-\sqrt{10}}{\sqrt{13}} t = {\cdot}00006$ 3d
$\frac{\sqrt{10}-\sqrt{9}}{\sqrt{18}} v = 67{\cdot}5$	$\frac{\sqrt{10}-\sqrt{9}}{\sqrt{13}} t = {\cdot}00007$ 4th
$\frac{\sqrt{9}-\sqrt{8}}{\sqrt{13}} v = 71{\cdot}4$	$\frac{\sqrt{9}-\sqrt{8}}{\sqrt{13}} t = {\cdot}00007$ 5th
$\frac{\sqrt{8}-\sqrt{7}}{\sqrt{13}} v = 76{\cdot}0$	$\frac{\sqrt{8}-\sqrt{7}}{\sqrt{13}} t = {\cdot}00007$ 6th
$\frac{\sqrt{7}-\sqrt{6}}{\sqrt{13}} v = 81{\cdot}7$	$\frac{\sqrt{7}-\sqrt{6}}{\sqrt{13}} t = {\cdot}00008$ 7th
$\frac{\sqrt{6}-\sqrt{5}}{\sqrt{13}} v = 88{\cdot}8$	$\frac{\sqrt{6}-\sqrt{5}}{\sqrt{13}} t = {\cdot}00008$ 8th
$\frac{\sqrt{5}-\sqrt{4}}{\sqrt{13}} v = 98{\cdot}2$	$\frac{\sqrt{5}-\sqrt{4}}{\sqrt{13}} t = {\cdot}00009$ 9th
$\frac{\sqrt{4}-\sqrt{3}}{\sqrt{13}} v = 111{\cdot}4$	$\frac{\sqrt{4}-\sqrt{3}}{\sqrt{13}} t = {\cdot}00011$ 10th
$\frac{\sqrt{3}-\sqrt{2}}{\sqrt{13}} v = 132{\cdot}2$	$\frac{\sqrt{3}-\sqrt{2}}{\sqrt{13}} t = {\cdot}00013$ 11th
$\frac{\sqrt{2}-\sqrt{1}}{\sqrt{13}} v = 172{\cdot}3$	$\frac{\sqrt{2}-\sqrt{1}}{\sqrt{13}} t = {\cdot}00017$ 12th
$\frac{\sqrt{1}-\sqrt{0}}{\sqrt{13}} v = 416{\cdot}0$	$\frac{\sqrt{1}-\sqrt{0}}{\sqrt{13}} t = {\cdot}00040$ 13th
Sum $1500{\cdot}0$	Sum $\frac{1}{692}$ or ${\cdot}00144$ sec.

Hence, as the motion lost at the beginning is very small; and consequently the motion communicated to any body, as an inch plank, in passing through it, is very small also; we can conceive how such a plank may be shot through, when standing upright, without oversetting it.

PROBLEM XI.

The force of attraction, above the earth, being inversely as the square of the distance from the centre; it is proposed to determine the time, velocity, and other circumstances, attending a heavy body falling from any given height; the descent at the earth's surface being $16\frac{1}{12}$ feet, or 193 inches, in the first second of time.

Put



$r =$ CS the radius of the earth,
$a =$ CA the dist. fallen from,
$x =$ CP any variable distance,
$v =$ the velocity at P,
$t =$ time of falling there, and
$g = 16\frac{1}{12}$, half the veloc. or force at S,
$f =$ the force at the point P.

Then we have the three following equations, viz.

$x^2 : r^2 :: 1 : f = \frac{r^2}{x^2}$ the force at P, when the force of gravity is considered as 1;

$\dot{t}v = -\dot{x}$, because x decreases; and

$v\dot{v} = -2gf\dot{x} = -\frac{2gr^2\dot{x}}{x^2}$.

The fluents of the last equation give $v^2 = \frac{4gr^2}{x}$. But when $x = a$, the velocity $v = 0$; therefore, by correction, $v^2 = \frac{4gr^2}{x} - \frac{4gr^2}{a} = 4gr^2 \times \frac{a-x}{ax}$; or $v = \sqrt{(\frac{4gr^2}{a} \times \frac{a-x}{x})}$, a general expression for the velocity at any point P.

When $x = r$, this gives $v = \sqrt{(4gr \times \frac{a-r}{a})}$ for the greatest velocity, or the velocity when the body strikes the earth.

When a is very great in respect of r, the last velocity becomes $(1 - \frac{r}{2a}) \times \sqrt{4gr}$ very nearly, or nearly $\sqrt{4gr}$ only, which is accurately the greatest velocity by falling from an infinite height. And this, when $r = 3965$ miles, is 6·9506 miles per second. Also, the velocity acquired in falling from the

the distance of the sun, or 12000 diameters of the earth, is 6·9505 miles per second. And the velocity acquired in falling from the distance of the moon, or 30 diameters, is 6·8972 miles per second.

Again, to find the time; since $\dot{t}v = -\dot{x}$, therefore $\dot{t} = \frac{-\dot{x}}{v} = \sqrt{\frac{a}{4gr^2}} \times \frac{-x\dot{x}}{\sqrt{ax-xx}}$; the correct fluent of which gives $t = \sqrt{\frac{a}{4gr^2}} \times (\sqrt{ax-xx} +$ arc to diameter a and vers. $a - x)$; or the time of falling to any point P $= \frac{1}{2r}\sqrt{\frac{a}{g}} \times (\text{AB} + \text{BP})$. And when $x = r$, this becomes $t = \frac{1}{2}\sqrt{\frac{a}{g}} \times \frac{\text{AD} + \text{DS}}{\text{SC}}$ for the whole time of falling to the surface at S; which is evidently infinite when a or AC is infinite, though the velocity is then only the finite quantity $\sqrt{4gr}$.

When the height above the earth's surface is given $= g$; because r is then nearly $= a$, and AD nearly $=$ DS, the time t for the distance g will be nearly - - - - - - - - - -
$\sqrt{\frac{1}{4gr^2}} \times 2\text{DS} = \sqrt{\frac{1}{4gr}} \times \sqrt{4gr} = 1''$, as it ought to be.

If a body, at the distance of the moon at A, fall to the earth's surface at S. Then $r = 3965$ miles, $a = 60r$, and $t = 416806'' = 4$ da. 19 h. 46′ 46″, which is the time of falling from the moon to the earth.

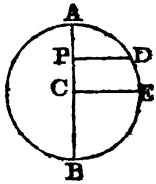
When the attracting body is considered as a point C; the whole time of descending to C will be - - - - - - - - - -
$\frac{1}{2r}\sqrt{\frac{a}{g}} \times \text{ABDC} = \frac{\cdot7854a}{r}\sqrt{\frac{a}{g}} = \frac{10a}{51r}\sqrt{a} = \frac{\cdot7854}{r}\sqrt{\frac{a^3}{g}}$.

Hence, the times employed by bodies, in falling from quiescence to the centre of attraction, are as the square roots of the cubes of the heights from which they respectively fall.

PROBLEM XII.

The force of attraction below the earth's surface being directly as the distance from the centre; it is proposed to determine the circumstances of velocity, time, and space fallen by a heavy body from the surface, through a perforation made straight to the centre of the earth: abstracting from the effect of the earth's rotation, and supposing it to be a homogeneous sphere of 3965 miles radius.

Put



Put $r =$ AC the radius of the earth,
$x =$ CP the dist. from the centre,
$v =$ the velocity at P,
$t =$ the time there,
$g = 16\frac{1}{12}$, half the force at A,
$f =$ the force at P.
Then CA : CP :: 1 : f; and the three equations are $rf = x$, and $v\dot{v} = -2gf\dot{x}$, and $\dot{t}v = -\dot{x}$.

Hence $f = \frac{x}{r}$, and $v\dot{v} = \frac{-2gx\dot{x}}{r}$; the correct fluent of which gives $v = \sqrt{(2g \times \frac{r^2 - x^2}{r})} = \text{PD}\sqrt{\frac{2g}{r}} = \text{PD}\sqrt{\frac{2g}{\text{CE}}}$, the velocity at the point P; where PD and CE are perpendicular to CA. So that the velocity at any point P, is as the perpendicular or sine PD at that point.

When the body arrives at C, then $v = \sqrt{2gr} = \sqrt{2g \cdot \text{AC}}$ = 25950 feet or 4·9148 miles per second, which is the greatest velocity, or that at the centre C.

Again, for the time; $\dot{t} = \frac{-\dot{x}}{v} = \sqrt{\frac{r}{2g}} \times \frac{-\dot{x}}{\sqrt{r^2 - x^2}}$; and the fluents give $t = \sqrt{\frac{r}{2g}} \times$ arc to cosine $\frac{x}{r} = \sqrt{\frac{1}{2gr}} \times$ arc AD. So that the time of descent to any point P, is as the corresponding arc AD.

When P arrives at C, the above becomes $t =$ - - - - $\sqrt{\frac{1}{2gr}} \times$ quadrant AE $= \frac{\text{AE}}{\text{AC}}\sqrt{\frac{r}{2g}} = 1·5708\sqrt{\frac{r}{2g}} = 1267\frac{1}{4}$ seconds $= 21' \, 7''\frac{1}{4}$, for the time of falling to the centre C.

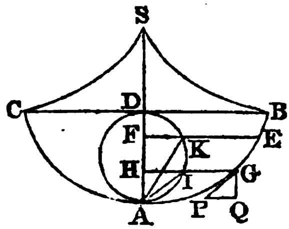
The time of falling to the centre is the same quantity $1·5708\sqrt{\frac{r}{2g}}$, from whatever point in the radius AC the body begins to move. For, let n be any given distance from C at which the motion commences: then by correction, $v = \sqrt{(\frac{2g}{r} \cdot \overline{n^2 - x^2})}$, and hence $\dot{t} = \sqrt{\frac{r}{2g}} \times \frac{-\dot{x}}{\sqrt{n^2 - x^2}}$, the fluents of which give $t = \sqrt{\frac{r}{2g}} \times$ arc to cosine $\frac{x}{n}$; which, when $x = 0$, gives $t = \sqrt{\frac{r}{2g}} \times$ quadrant $= 1·5708\sqrt{\frac{r}{2g}}$, for the time of descent to the centre C, the same as before.

As

As an equal force, acting in contrary directions, generates or destroys an equal quantity of motion, in the same time; it follows that, after passing the centre, the body will just ascend to the opposite surface at B, in the same time in which it fell to the centre from A. Then from B it will return again in the same manner, through C to A; and so oscillate continually between A and B, the velocity being always equal at equal distances from C on both sides; and the whole time of a double oscillation, or of passing from A and arriving at A again, will be quadruple the time of passing over the radius AC, or $= 2 \times 3\cdot1416\sqrt{\frac{r}{2g}} = 1\text{h}.\ 24'\ 29''$.

PROBLEM XIII.

To find the Time of a Pendulum vibrating in the Arc of a Cycloid.



Let
S be the point of suspension;
SA, the length of pendulum;
CAB, the whole cycloidal arc;
AIKD, the generating circle, to which FKE, HIG are perpendiculars.
SC, SB two other equal semicloids, on which the thread wrapping, the end A is made to describe the cycloid BAC.

Now, by the nature of the cycloid, AD = DS; and SA = 2AD = SC = SB = SA = AB. Also, if at any point G be drawn the tangent GP; also GQ parallel and PQ perpendicular to AD. Then PG is parallel to the chord AI by the nature of the curve. And, by the nature of forces, the force of gravity : force in direction GP : : GP : GQ : : AI : AH : : AD : AI; in like manner, the force of gravity: force in the curve at E : : AD : AK; that is, the accelerative force in the curve, is every where as the corresponding chord AI or AK of the circle, or as the arc AG or AE of the cycloid, since AG is always = 2AI, by the nature of the curve. So that the process and conclusions, for the velocity and time of describing any arc in this case, will be the very same as in the last problem, the nature of the forces being the same, viz. as the distance to be passed over to the lowest point A.

From

From which it follows, that the time of a semi-vibration, in all arcs, AG, AE, &c, is the same constant quantity $1{\cdot}5708\sqrt{\frac{r}{2g}} = 1{\cdot}5708\sqrt{\frac{AS}{2g}} = 1{\cdot}5708\sqrt{\frac{l}{2g}}$; and the time of a whole vibration from B to C, or from C to B, is $3{\cdot}1416\frac{l}{2g}$; where $l = AS = AB$ is the length of the pendulum, $g = 16\frac{1}{12}$ feet or 193 inches, and 3·1416 the circumference of a circle whose diameter is 1.

Since the time of a body's falling by gravity through $\frac{1}{2}l$, or half the length of the pendulum, by the nature of descents, is $\sqrt{\frac{l}{2g}}$, which being in proportion to $3{\cdot}1416\sqrt{\frac{l}{2g}}$, as 1 is to 3·1416; therefore the diameter of a circle is to its circumference, as the time of falling through half the length of a pendulum, is to the time of one vibration.

If the time of the whole vibration be 1 second, this equation arises, viz. $1'' = 3{\cdot}1416\sqrt{\frac{l}{2g}}$; hence $l = \frac{2g}{3{\cdot}1416^2} = \frac{g}{4{\cdot}9348}$, and $g = 3{\cdot}1416^2 \times \frac{1}{2}l = 4{\cdot}9348l$. So that if one of these, g or l, be given by experiment, these equations will give the other. When g, for instance, is supposed to be given $= 16\frac{1}{12}$ feet, or 193 inches; then is $l = \frac{g}{4{\cdot}9348} = 39{\cdot}11$, the length of a pendulum to vibrate seconds. Or if $l = 39\frac{1}{8}$, the length of the seconds pendulum for the latitude of London, by experiment; then is $g = 4{\cdot}9348l = 193{\cdot}07$ inches $= 16\frac{1}{12}\frac{07}{00}$ feet, or nearly $16\frac{1}{12}$ feet, for the space descended by gravity in the first second of time in the latitude of London; also agreeing with experiment.

Hence the times of vibration of pendulums, are as the square roots of their lengths; and the number of vibrations made in a given time, is reciprocally as the square roots of the lengths. And hence also, the length of a pendulum vibrating n times in a minute, or $60''$, is $l = 39\frac{1}{8} \times \frac{60^2}{n^2} = \frac{140850}{nn}$.

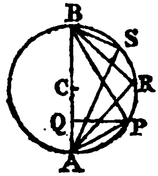
When a pendulum vibrates in a circular arc; as the length of the string is constantly the same, the time of vibration will be longer than in a cycloid; but the two times will approach nearer together as the circular arc is smaller; so that

when

when it is very small, the times of vibration will be nearly equal. And hence it happens that $39\frac{1}{8}$ inches is the length of a pendulum vibrating seconds, in the very small arc of a circle.

PROBLEM XIV.

To determine the Time of a Body descending down the Chord of a Circle.



Let C be the centre; AB the vertical diameter; AP any chord, down which a body is to descend from P to A; and PQ perpendicular to AB.

Now, as the natural force of gravity in the vertical direction BA, is to the force urging the body down the plane PA, as the length of the plane AP, is to its height AQ; therefore the velocity in PA and QA, will be equal at all equal perpendicular distances below PQ; and consequently the time in PA : time in QA :: PA : QA :: BA : PA; but time in BA : time in QA :: $\sqrt{}$ BA : $\sqrt{}$ QA :: BA : PA; hence, as three of the terms in each proportion are the same, the fourth terms must be equal, namely the time in BA = the time PA.

And, in like manner, the time in BP = the time in BA. So that, in general, the times of descending down all the chords BA, BP, BR, BS, &c, or PA, RA, SA, &c, are all equal, and each equal to the time of falling freely through the diameter; as before found at art. 131, Mechanics. Which time is $\sqrt{\frac{2r}{g}}$, where $g = 16\frac{1}{12}$ feet, and r = the radius AC; for $\sqrt{g} : \sqrt{2r} :: 1'' : \sqrt{\frac{2r}{g}}$.

PROBLEM XV.

To determine the Time of filling the Ditches of a Work with Water, at the Top, by a Sluice of 2 Feet square; the Head of Water above the Sluice being 10 Feet, and the Dimensions of the Ditch being 20 Feet wide at Bottom, 22 at Top, 9 deep, and 1000 Feet long.

The capacity of the ditch is 189000 cubic feet.

But $\sqrt{g} : \sqrt{10} :: 2g : 2\sqrt{10g}$ the velocity of the water through the sluice, the area of which is 4 square feet; therefore

therefore $8\sqrt{10g}$ is the quantity per second running through it; and consequently $8\sqrt{10g} : 189000 :: 1'' : \frac{23625}{\sqrt{10g}} = 1863''$ or $31'\ 3''$ nearly, which is the time of filling the ditch.

PROBLEM XVI.

To determine the Time of emptying a Vessel of Water by a Sluice in the Bottom of it, or in the Side near the Bottom: the Height of the Aperture being very small in respect of the Altitude of the Fluid.

PUT a = the area of the aperture or sluice;
$2g = 32\frac{1}{6}$ feet, the force of gravity;
d = the whole depth of water;
x = the variable altitude of the surface above the aperture;
A = the area of the surface of the water.

Then $\sqrt{g} : \sqrt{x} :: 2g : 2\sqrt{gx}$ the velocity with which the fluid will issue at the sluice; and hence $A : a :: 2\sqrt{gx} : \frac{2a\sqrt{gx}}{A}$, the velocity with which the surface of the water will descend at the altitude x, or the space it would descend in 1 second with the velocity there. Now, in descending the space $\dot{x}$, the velocity may be considered as uniform; and uniform descents are as their times; therefore $\frac{2a\sqrt{gx}}{A} : \dot{x} :: 1'' : \frac{A\dot{x}}{2a\sqrt{gx}}$ the time of descending $\dot{x}$ space, or the fluxion of the time of exhausting. That is, $\dot{t} = \frac{-A\dot{x}}{2a\sqrt{gx}}$; which is made negative, because x is a decreasing quantity, or its fluxion negative.

Now, when the nature or figure of the vessel is given, the area A will be given in terms of x; which value of A being substituted into this fluxion of the time, the fluent of the result will be the time of exhausting sought.

So if, for example, the vessel be any prism, or everywhere of the same breadth; then A is a constant quantity, and therefore the fluent is $-\frac{A}{a}\sqrt{\frac{x}{g}}$. But when $x = d$, this becomes $-\frac{A}{a}\sqrt{\frac{d}{g}}$, and should be 0; therefore the correct fluent is $t = \frac{A}{a} \times \frac{\sqrt{d} - \sqrt{x}}{\sqrt{g}}$ for the time of the surface descending

scending till the depth of the water be x. And when $x = 0$, the whole time of exhausting is barely $\frac{A}{a}\sqrt{\frac{d}{g}}$.

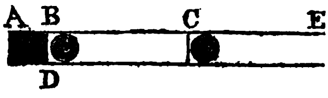
Hence, if A be $=$ 10000 square feet, $a = 1$ square foot, and $d = 10$ feet; the time is $7885\frac{1}{3}$ seconds, or 2h. 11′ 25″$\frac{1}{3}$.

Again, if the vessel be a ditch, or canal, of 20 feet broad at the bottom, 22 at the top, 9 deep, and 1000 feet long; then is $90 : 90 + x :: 20 : \frac{90+x}{9} \times 2$ the breadth of the surface of the water when its depth in the canal is x; and therefore $A = \frac{90+x}{9} \times 2000$ is the surface at that time. Consequently $\dot{t}$ or $\frac{-A\dot{x}}{2a\sqrt{gx}} = 1100 \times \frac{90+x}{9} \times \frac{-\dot{x}}{a\sqrt{gx}}$ is the fluxion of the time; the correct fluent of which, when $x = 0$, is $1000 \times \frac{180 + \frac{2}{3}d}{9a} \times \sqrt{\frac{d}{g}} = \frac{1000 \times 186 \times 3}{9 \times 4\frac{1}{12}} =$ 15459″$\frac{2}{3}$ nearly, or 4h. 17′ 39″$\frac{2}{3}$, being the whole time of exhausting by a sluice of 1 foot square.

PROBLEM XVII.

To determine the Velocity with which a Ball is discharged from a Given Piece of Ordnance, with a Given Charge of Gunpowder.

Let the annexed figure represent the bore of the gun; AD being the part filled with gunpowder. And put



$a =$ AB, the part at first filled with powder and the bag;
$b =$ AE, the whole length of the gunbore;
$c = \cdot 7854$, the area of a circle whose diameter is 1;
$d =$ BD, the diameter of the ball;
$e =$ the specific gravity of the ball, or weight of 1 cubic foot;
$g = 16\frac{1}{12}$ feet, descended by a body in 1 second;
$m = 230$ ounces, the pressure of the atmosphere on a sq. inch;
n to 1 the ratio of the first force of the fired powder, to the pressure of the atmosphere;
$w =$ the weight of the ball. Also, let
$x =$ AC, be any variable distance of the ball from A, in moving along the gunbarrel.

 First,

First, cd^2 is = the area of the circle BD of the ball; there mcd^2 is the pressure of the atmosphere on BD; conseq. $mncd^2$ is the first force of the powder on BD.

But the force of the inflamed powder is proportional to its density, and the density is inversely as the space it fills; therefore the force of the powder on the ball at B, is to the force on the same at C, as AC is to AB; that is, - - - -

$x : a :: mncd^2 : \frac{mnacd^2}{x} = F$, the motive force at C:

conseq. $\frac{F}{w} = \frac{mnacd^2}{wx} = f$, the accelerating force there.

Hence, theor. 10 of forces gives $v\dot{v} = 2gf\dot{x} = \frac{2gmnacd^2}{w} \times \frac{\dot{x}}{x}$;

the fluent of which is $v^2 = \frac{4gmnacd^2}{w} \times$ hyp. log. of x.

But when $v = 0$, then $x = a$; theref. by correction,

$v^2 = \frac{4gmnacd^2}{w} \times$ hyp. log. $\frac{x}{a}$ is the correct fluent; conseq.

$v = \sqrt{(\frac{4gmnacd^2}{w} \times \text{hyp. log. } \frac{x}{a})}$ is the vel. of the ball at C.

and $v = \sqrt{(\frac{4gmnhcd^2}{w} \times \text{hyp. log. } \frac{b}{a})}$ the velocity with which the ball issues from the muzzle at E; where h denotes the length of the cylinder filled with powder; and a the length to the hinder part of the ball, which will be more than h when the powder does not touch the ball.

Or, by substituting the numbers for g and m, and changing the hyperbolic logarithms for the common ones, then $v = \sqrt{(\frac{2230nhd^2}{w} \times \text{com. log.} \frac{b}{a})}$, the velocity at E, in feet.

But, the content of the ball being $\frac{2}{3}cd^3$, its weight is - -

$w = \frac{\frac{2}{3}cd^3e}{12^3} = \frac{ced^3}{2592} = \frac{ed^3}{3300}$; which being substituted for w, in the value of v, it becomes

$v = 2713 \sqrt{(\frac{nh}{de} \times \text{com. log.} \frac{b}{a})}$, the velocity at E.

When the ball is of cast iron; taking $e=7368$, the rule becomes

$v = 100 \sqrt{(\frac{nh}{10d} \times \log. \frac{b}{a})}$ for the veloc. of the cast-iron ball.

Or, when the ball is of lead; then - - - - - - - - -

$v = 80\frac{3}{4} \sqrt{(\frac{nh}{10d} \times \log. \frac{b}{a})}$ for the veloc. of the leaden ball.

Corol.

Corol. From the general expression for the velocity v, above given, may be derived what must be the length of the charge of powder a, in the gun-barrel, so as to produce the greatest possible velocity in the ball; namely, by making the value of v a maximum, or, by squaring and omitting the constant quantities, the expression $a \times$ hyp. log. of $\frac{b}{a}$ a maximum, or its fluxion equal to nothing; that is $\dot{a} \times$ hyp. log. $\frac{b}{a} - \dot{a} = 0$, or hyp. log. of $\frac{b}{a} = 1$; hence $\frac{b}{a} = 2{\cdot}71828$, the number whose hyp. log. is 1. So that $a : b :: 1 : 2{\cdot}71828$, or as 4 to 11 nearly, or nearer as 7 to 19; that is, the length of the charge, to produce the greatest velocity, is the $\frac{4}{11}$th part of the length of the bore, or nearer $\frac{7}{19}$ of it.

But actual experiment it is found, that the charge for the greatest velocity, is but little less than that which is here computed from theory; as may be seen by turning to page 252 of my volume of Tracts, where the corresponding parts are found to be, for four different lengths of gun, thus, $\frac{3}{10}$, $\frac{3}{12}$, $\frac{3}{16}$, $\frac{3}{26}$; the parts here varying, as the gun is longer, which allows time for the greater quantity of powder to be fired, before the ball is out of the bore.

SCHOLIUM.

In the calculation of the foregoing problem, the value of the constant quantity n remains to be determined. It denotes the first strength or force of the fired gunpowder, just before the ball is moved out of its place. This value is assumed, by Mr. Robins, equal to 1000, that is, 1000 times the pressure of the atmosphere, on any equal spaces.

But the value of the quantity n may be derived much more accurately, from the experiments related in my Tracts, by comparing the velocities there found by experiment, with the rule for the value of v, or the velocity, as above computed by theory, viz. - - - - - - - - - - - - - - - -

$$v = 100\sqrt{\left(\frac{na}{10d} \times \text{log. of} \frac{b}{a}\right)}, \text{ or } = 100\sqrt{\left(\frac{nh}{10d} \times \text{log. of} \frac{b}{a}\right)}.$$

Now, supposing that v is a given quantity, as well as all the other quantities, excepting only the number n, then by reducing this equation, the value of the letter n is found to be as follows, viz. - - - - - - - - - - - - - - - - - -

$$n = \frac{dvv}{1000a} \div \text{com. log. of } \frac{b}{a}, \text{ or } = \frac{dvv}{1000h} \div \text{ log. of } \frac{b}{a},$$

when h is different from a.

Now

Now, to apply this to the experiments. By page 240 of the Tracts, the velocity of the ball, of 1·96 inches diameter, with 4 ounces of powder, in the gun No. 1, was 1100 feet per second; and, by pa 494, vol. 1, the length of the gun, when corrected for the spheroidal hollow in the bottom of the bore, was 28·53; also, by page 228, the length of the charge, when corrected in like manner, was 3·45 inches of powder and bag together, but 2·54 of powder only: so that the values of the quantities in the rule, are thus: $a = 3·45$; $b = 28·53$; $d = 1·96$; $h = 2·54$; and $v = 1100$: then, by substituting these values instead of the letters, in the theorem $n = \frac{dvv}{1000a} \div$ com. log. of $\frac{b}{a}$, it comes out $n = 750$, when h is considered as the same as a. And so on, for the other experiments there treated of.

It is here to be noted however, that there is a circumstance in the experiments delivered in the Tracts, just mentioned, which will alter the value of the letter a in this theorem, which is this, viz. that a denotes the distance of the shot from the bottom of the bore; and the length of the charge of powder alone ought to be the same thing: but, in the experiments, that length included, besides the length of real powder, the substance of the thin flannel-bag in which it was always contained, of which the neck at least extended a considerable length, being the part where the open end was wrapped and tied close round with a thread. This circumstance causes the value of n, as found by the theorem above, to come out less than it ought to be, for it shows the strength of the inflamed powder when just fired, and when the flame fills the whole space a before occupied both by the real powder and the bag, whereas it ought to show the first strength of the flame when it is supposed to be contained in the space only occupied by the powder alone, without the bag. The formula will therefore bring out the value of n too little, in proportion as the real space filled by the powder is less than the space filled both by the powder and its bag. In the same proportion therefore must we increase the formula, that is, in the proportion of h, the length of real powder, to a the length of powder and bag together. When the theorem is so corrected, it becomes $\frac{dvv}{1000h} \div$ com. log. of $\frac{b}{a}$.

Now, by pa. 228 of the Tracts, there are given both the lengths of all the charges, or values of a, including the bag, and also the length of the neck and bottom of the bog, which is 0·91 of an inch, which therefore must be subtracted from all

all the values of a, to give the corresponding values of h. This in the example above reduces 3·45 to 2·54.

Hence, by increasing the above result 750, in proportion of 2·54 to 3·45, it becomes 1018. And so on for the other experiments.

But it will be best to arrange the results in a table, with the several dimensions, when corrected, from which they are computed, as here below.

Table of Velocities of Balls and First Force of Powder, &c.

Gun.		Charge of Powder.			Velocity or value of v.	First force, or value of n
No.	Length, or value of b.	Weight in ounces.	Length or value of a.	of h.		
	inches.	4	3·45	2·54	1100	1018
1	28·53	8	5·99	5·08	1430	1164
		16	11·07	10·16	1430	967
		4	3·45	2·54	1180	1077
2	38·43	8	5·99	5·08	1580	1193
		16	11·07	10·16	1660	984
		4	3·45	2·54	1300	1067
3	57·70	8	5·99	5·08	1790	1256
		16	11·07	10·16	2000	1076
		4	3 45	2·54	1370	1060
4	80·23	8	5·99	5·08	1940	1289
		16	11·07	10·16	2200	1085

Where it may be observed, that the numbers in the column of velocities, 1430 and 2200, are a little increased, as, from a view of the table of experiments, they evidently required to be. Also the value of the letter d is constantly 1·96 inch.

Hence it appears, that the value of the letter n, used in the theorem, though not yet greatly different from the number 1000, assumed by Mr Robins, is rather various, both for the different lengths of the gun, and for the different charges with the same gun.

But

But this diversity in the value of the quantity n, or the first force of the inflamed gunpowder, is probably owing in some measure to the omission of a material datum in the calculation of the problem, namely, the weight of the charge of powder, which has not all been brought into the computation. For it is manifest, that the elastic fluid has not only the ball to move and impel before it, but its own weight of matter also. The computation may therefore be renewed, in the ensuing problem, to take that datum into the account.

PROBLEM XVIII.

To determine the same as in the last Problem; taking both the Weight of Powder and the Ball into the Calculation.

Besides the notation used in the last problem, let $2p$ denote the weight of the powder in the charge, with the flannel bag in which it was inclosed.

Now, because the inflamed powder occupies at all times the part of the gun bore which is behind the ball, its centre of gravity, or the middle part of the same, will move with only half the velocity that the ball moves with; and this will require the same force as half the weight of the powder, &c, moved with the whole velocity of the ball. Therefore, in the conclusion derived in the last problem, we are now, instead of w, to substitute the quantity $p + w$; and when that is done the last velocity will come out, $v = \surd\left(\frac{2230nhd^2}{p+w} \times \text{com. log.}\ \frac{b}{a}\right)$.

And from this equation is found the value of n, which is $n = \frac{p+w}{2230hd^2}v^2 \div \text{log. of}\ \frac{b}{a},\ = \frac{p+w}{8567h}v^2 \div \text{log. of}\ \frac{b}{a}$, by substituting for d its value 1·96, the diameter of the ball.

Now as to the ball, its medium weight was 16 oz. 13 dr. = 16·81 oz. And the weights of the bags containing the several charges of powder, viz. 4 oz, 8 oz, 16 oz, were 8 dr, 12 dr, and 1 oz. 5 dr; then, adding these to the respective contained weights of powder, the sums, 4 5 oz, 8·75 oz, 17·31 oz, are the values of $2p$, or the weights of the powder and bags; the halves of which, or 2·25, and 4·38, and 8·66, are the values of the quantity p for those three charges; and these being added to 16·81, the constant weight of the ball, there are obtained the three values of $p + w$ for the three charges of powder, which values therefore are 19·06 oz, and 21·19 oz, and 25·47 oz. Then, by calculating the values of the first force n, by the last rule above, with these new data, the whole will be found as in the following table..

The

The Gun.		Charge of Powder.			Weight of ball and charge, or values of $p + w$.	Velocity, or the values of v.	First force or the value of n.
No.	Length or value of b.	Weight in ounces.	Length or value of a.	Length or value of h.			
1	inches 28·53	4	3·45	2·54	19·06	1100	1155
		8	5·99	5·08	21·19	1430	1470
		16	11·07	10·16	25·47	1430	1456
2	38·43	4	3·45	2·54	19·06	1180	1167
		8	5·99	5·08	21·19	1580	1506
		16	11·07	10·16	25·47	1660	1492
3	57·70	4	3·45	2·54	19·06	1300	1210
		8	5·99	5·08	21·19	1790	1586
		16	11·07	10·16	25·47	2000	1646
4	80·23	4	3·45	2·54	19·06	1370	1203
		8	5·99	5·08	21·19	1940	1627
		16	11·07	10·16	25·47	2200	1648

And here it appears that the values of n, the first force of the charge, are much more uniform and regular than by the former calculations in the preceding problem, at least in all excepting the smallest charge, 4 oz, in each gun; which it would seem must be owing to some general cause or causes. Nor have we long to search, to find out what those causes may be. For when it is considered that these numbers for the value of n, in the last column of the table, ought to exhibit the first force of the fired powder, when it is supposed to occupy the space only in which the bare powder itself lies; and that whereas it is manifest that the condensed fluid of the charge in these experiments, occupies the whole space between the ball and the bottom of the gun bore, or the whole space taken up by the powder and the bag or cartridge together, which exceeds the former space, or that of the powder alone, at least in the proportion of the circle of the gun bore, to the same as diminished by the thickness of the surrounding flannel of the bag that contained the powder; it is manifest that the force was diminished on that account. Now by gently compressing a number of folds of the flannel together, it has been found that the thickness of the single flannel was equal to the 40th part of an inch; the double of which, $\frac{1}{20}$ or ·05 of an inch, is therefore the quantity

quantity by which the diameter of the circle of the powder within the bag, was less than that of the gun bore. But the diameter of the gun bores was 2·02 inches; therefore, deducting the ·05, the remainder 1·97 is the diameter of the powder cylinder within the bag: and because the areas of circles are to each other as the spaces of their diameters, and the squares of these numbers, 1·97 and 2·02, being to each other as 388 to 408, or as 97 to 102; therefore, on this account alone, the numbers before found, for the value of n, must be increased in the ratio of 97 to 102.

But there is yet another circumstance, which occasions the space at first occupied by the inflamed powder to be larger than that at which it has been taken in the foregoing calculations, and that is the difference between the content of a sphere and cylinder. For the space supposed to be occupied at first by the elastic fluid, was considered as the length of a cylinder measured to the hinder part of the curve surface of the ball, which is manifestly too little by the difference between the content of half the ball and a cylinder of the same length and diameter, that is, by a cylinder whose length is $\frac{1}{3}$ the semidiameter of the ball. Now that diameter was 1·96 inches; the half of which is 0·98, and $\frac{1}{3}$ of this is 0·33 nearly. Hence then it appears that the lengths of the cylinders, at first filled by the dense fluid, viz. 3·45, and 5·99, and 11·07, have been all taken too little by 0·33; and hence it follows that, on this account also, all the numbers before found for the value of the first force n, must be further increased in the ratios of 3·45 and 5·99 and 11·07, to the same numbers increased by 0·33, that is, to the numbers 3·78 and 6·32 and 11·40.

Compounding now these last ratios with the foregoing one, viz. 97 to 102, it produces these three, viz. the ratios of 334 and 581 and 1074, respectively to 385 and 647 and 1163. Therefore increasing the last column of numbers, for the value of n, viz. those of the 4 oz. charge in the ratio of 334 to 385, and those of the 8 oz charge in the ratio of 581 to 647, and those of the 16 oz. charge in the ratio of 1074 to 1163, with every gun, they will be reduced to the numbers in the annexed table; where the numbers are still larger and more regular than before.

Powder.	The Guns.			
	1	2	3	4
oz.				
4	1372	1387	1438	1430
8	1637	1677	1766	1812
16	1577	1616	1782	1784

Thus

Thus then at length it appears that the first force of the inflamed gunpowder, when occupying only the space at first filled with the powder, is about 1800, that is 1800 times the elasticity of the natural air, or pressure of the atmosphere in the charges with 8 oz. and 16 oz. of powder, in the two longer guns; but somewhat less in the two shorter, probably owing to the gradual firing of gunpowder in some degree; and also less in the lowest charge 4 oz, in all the guns, which may probably be owing to the less degree of heat in the small charge. But besides the foregoing circumstances that have been noticed, or used in the calculations, there are yet several others that might and ought to be taken into the account, in order to a strict and perfect solution of the problem; such as, the counter pressure of the atmosphere, and the resistance of the air on the fore part of the ball while moving along the bore of the gun; the loss of the elastic fluid by the vent and windage of the gun; the gradual firing of the powder; the unequal density of the elastic fluid in the different parts of the space it occupies between the ball and the bottom of the bore; the difference between pressure and percussion when the ball is not laid close to the powder; and perhaps some others: on all which accounts it is probable that, instead of 1800, the first force of the elastic fluid is not less than 2000 times the strength of natural air.

Corol. From the theorem last used for the velocity of the ball and elastic fluid, viz. $v = \sqrt{(\frac{2230hd^2}{p+w}n \div \log. \frac{b}{a})} = \sqrt{\frac{8567hn}{p+w} \div \log. \frac{b}{a})}$, we may find the velocity of the elastic fluid alone, viz. by taking w, or the weight of the ball, $= 0$ in the theorem, by which it becomes barely $v = \sqrt{(\frac{8567hn}{p} \div \log. \frac{b}{a})}$, for that velocity. And by computing the several preceding examples by this theorem, supposing the value of n to be 2000, the conclusions come out a little various, being between 4000 and 5000, but most of them nearer to the latter number. So that it may be concluded that the velocity of the flame, or of the fired gun-powder, expands itself at the muzzle of the gun, at the rate of about 5000 feet per second nearly.

ON THE MOTION OF BODIES IN FLUIDS.

PROBLEM XIX.

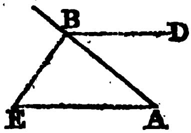
To determine the Force of Fluids in Motion; and the Circumstances attending Bodies Moving in Fluids.

1. It is evident that the resistance to a plane, moving perpendicularly through an infinite fluid, at rest, is equal to the pressure or force of the fluid on the plane at rest, and the fluid moving with the same velocity, and in the contrary direction, to that of the plane in the former case. But the force of the fluid in motion, must be equal to the weight or pressure which generates that motion; and which, it is known, is equal to the weight or pressure of a column of the fluid, whose base is equal to the plane, and its altitude equal to the height through which a body must fall, by the force of gravity, to acquire the velocity of the fluid: and that altitude is, for the sake of brevity, called the altitude due to the velocity. So that, if a denote the area of the plane, v the velocity, and n the specific gravity of the fluid; then, the altitude due to the velocity v being $\frac{v^2}{4g}$, the whole resistance, or motive force m, will be $a \times n \times \frac{v^2}{4g} = \frac{anv^2}{4g}$; g being $16\frac{1}{12}$ feet. And hence, *cæteris paribus*, the resistance is as the square of the velocity.

2. This ratio, of the square of the velocity, may be otherwise derived thus. The force of the fluid in motion, must be as the force of one particle multiplied by the number of them; but the force of a particle is as its velocity; and the number of them striking the plane in a given time, is also as the velocity; therefore the whole force is as $v \times v$ or v^2, that is, as the square of the velocity.

3. If the direction of motion, instead of being perpendicular to the plane, as above supposed, be inclined to it in any angle, the sine of that angle being s, to the radius 1: then the resistance to the plane, or the force of the fluid against

against the plane, in the direction of the motion, as assigned above, will be diminished in the triplicate ratio of radius to the sine of the angle of inclination, or in the ratio of 1 to s^3. For AB being the direction of the plane, and BD that of the motion making the angle ABD, whose sine is s; the number of particles, or quantity of the fluid striking the plane, will be diminished in the ratio of 1 to s, or of radius to the sine of the angle B of inclination; and the force of each particle will also be diminished in the same ratio of 1 to s: so that, on both these accounts, the whole resistance will be diminished in the ratio of 1 to s^2, or in the duplicate ratio of radius to the sine of the said angle. But again, it is to be considered that this whole resistance is exerted in the direction BE perpendicular to the plane; and any force in the direction BE, is to its effect in the direction AE, parallel to BD, as AE to BE, that is as 1 to s. So that finally, on all these accounts, the resistance in the direction of motion, is diminished in the ratio of 1 to s^3, or in the triplicate ratio of radius to the sine of inclination. Hence, comparing this with article 1, the whole resistance, or the motive force on the plane, will be $m = \frac{anv^2s^3}{4g}$.



4. Also, if w denote the weight of the body, whose plane face a is resisted by the absolute force m; then the retarding force f, or $\frac{m}{w}$, will be $\frac{anv^2s^3}{4gw}$.

5. And if the body be a cylinder, whose face or end is a, and diameter d, or radius r, moving in the direction of its axis; because then $s = 1$, and $a = pr^2 = \frac{1}{4}pd^2$, where $p = 3{\cdot}1416$; the resisting force m will be - - - - - $\frac{npd^2v^2}{16g} = \frac{npr^2v^2}{4g}$, and the retarding force $f = \frac{npd^2v^2}{16gw} = \frac{npr^2v^2}{4gw}$.

6. This is the value of the resistance when the end of the cylinder is a plane perpendicular to its axis, or to the direction of motion. But were its face a conical surface, or an elliptic section, or any other figure every where equally inclined to the axis, the sine of inclination being s: then the number of particles of the fluid striking the face being still the same, but the force of each, opposed to the direction

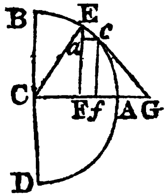
of

of motion, diminished in the duplicate ratio of radius to the sine of inclination, the resisting force m would be $\frac{npd^2v^2s^2}{16g} = \frac{npr^2v^2s^2}{4g}$.

But if the body were terminated by an end or face of any other form, as a spherical one, or such like, where every part of it has a different inclination to the axis; then a further investigation becomes necessary, such as in the following proposition.

PROBLEM XX.

To determine the Resistance of a Fluid to any Body, moving in it, of a Curved End; as a Sphere, or a Cylinder with a Hemispherical End, &c.



1. Let BEAD be a section through the axis CA of the solid, moving in the direction of that axis. To any point of the curve draw the tangent EG, meeting the axis produced in G: also, draw the perpendicular ordinates EF, ef, indefinitely near each other; and draw ae parallel to CG.

Putting CF $= x$, EF $= y$, BE $= z$, $s =$ sine $\angle$ G to radius 1, and p, $= 3\cdot1416$: then $2py$ is the circumference whose radius is EF, or the circumference described by the point E, in revolving about the axis CA; and $2py \times$ Ee or $2py\dot z$ is the fluxion of the surface, or it is the surface described by Ee, in the said revolution about CA, and which is the quantity represented by a in art. 3 of the last problem: hence $\frac{nv^2s^3}{4g} \times 2py\dot z$ or $\frac{pnv^2s^3}{2g} \times y\dot z$ is the resistance on that ring, or the fluxion of the resistance to the body, whatever the figure of it may be. And the fluent of which will be the resistance required.

2. In the case of a spherical form: putting the radius CA or CB $= r$, we have $y = \sqrt{r^2 - x^2}$, $s = \frac{EF}{EG} = \frac{CF}{CE} = \frac{x}{r}$, and $y\dot z$, or EF $\times$ Ec $=$ CE $\times$ ae $= r\dot x$; therefore the general fluxion $\frac{pnv^2}{2g} \times s^3y\dot z$ becomes $\frac{pnv^2}{2g} \times \frac{x^3}{r^3} \times r\dot x = \frac{pnv^2}{2gr^2} \times x^3\dot x$; the

the fluent of which, or $\frac{pnv^2}{8gr^2}x^4$, is the resistance to the spherical surface generated by BE. And when x or CF is $= r$ or CA, it becomes $\frac{pnv^2r^2}{8g}$ for the resistance on the whole hemisphere; which is also equal to $\frac{pnv^2d^2}{32g}$, where $d = 2r$ the diameter.

3. But the perpendicular resistance to the circle of the same diameter d or BD, by art. 5 of the preceding problem, is $\frac{pnv^2d^2}{16g}$; which, being double the former, shows that the resistance to the sphere, is just equal to half the direct resistance to a great circle of it, or to a cylinder of the same diameter.

4. Since $\frac{1}{6}pd^3$ is the magnitude of the globe; if N denote its density or specific gravity, its weight w will be $= \frac{1}{6}pd^3$N, and therefore the retardive force f or $\frac{m}{w} = \frac{pnv^2d^2}{32g} \times \frac{6}{p\text{N}d^3}$ $= \frac{3nv^2}{16g\text{N}d}$; which is also $= \frac{v^2}{4gs}$ by art. 8 of the general theorems in page 380; hence then $\frac{3n}{4\text{N}d} = \frac{1}{s}$, and $s = \frac{\text{N}}{n} \times \frac{4}{3}d$; which is the space that would be described by the globe, while its whole motion is generated or destroyed by a constant force which is equal to the force of resistance, if no other force acted on the globe to continue its motion. And if the density of the fluid were equal to that of the globe, the resisting force is such, as, acting constantly on the globe without any other force, would generate or destroy its motion in describing the space $\frac{4}{3}d$, or $\frac{4}{3}$ of its diameter, by that accelerating or retarding force.

5. Hence the greater velocity that a globe will acquire by descending in a fluid, by means of its relative weight in the fluid, will be found by making the resisting force equal to that weight. For, after the velocity is arrived at such a degree, that the resisting force is equal to the weight that urges it, it will increase no longer, and the globe will afterwards continue to descend with that velocity uniformly. Now, N and n being the separate specific gravities of the globe and fluid, N $- n$ will be the relative gravity of the globe in the fluid, and therefore $w = \frac{1}{6}pd^3(\text{N} - n)$ is the weight

weight by which it is urged; also $m = \text{- - - - - - -}$ $\frac{pnv^2d^2}{32g}$ is the resistance; consequently $\frac{pnv^2d^2}{32g} = \frac{1}{6}pd^3(N-n)$ when the velocity becomes uniform: from which equation is found $v = \sqrt{(4g \cdot \frac{4}{3}d \cdot \frac{N-n}{n})}$, for the said uniform or greatest velocity.

And, by comparing this form with that in art. 6 of the general theorems in page 379, it will appear that its greatest velocity, is equal to the velocity generated by the accelerating force $\frac{N-n}{n}$, in describing the space $\frac{4}{3}d$, or equal to the velocity generated by gravity in freely describing the space $\frac{N-n}{n} \times \frac{4}{3}d$. If $N = 2n$, or the specific gravity of the globe be double that of the fluid, then $\frac{N-n}{n} = 1 =$ the natural force of gravity; and then the globe will attain its greatest velocity in describing $\frac{4}{3}d$ or $\frac{4}{3}$ of its diameter.—It is further evident, that if the body be very small, it will very soon acquire its greatest velocity, whatever its density may be.

Exam. If a leaden ball, of 1 inch diameter, descend in water, and in air of the same density as at the earth's surface, the three specific gravities being as $11\frac{1}{3}$, and 1, and $\frac{3}{2500}$ Then $v = \sqrt{4 \cdot 16\frac{1}{12} \cdot \frac{4}{36} \cdot 10\frac{1}{3}} = \frac{1}{9}\sqrt{31 \cdot 193} = 8{\cdot}5944$ feet, is the greatest velocity per second the ball can acquire by descending in water. And $v = \sqrt{4 \cdot \frac{193}{12} \cdot \frac{4}{36} \cdot \frac{34}{3} \cdot \frac{2500}{3}}$ nearly $= \frac{50}{9}\sqrt{\frac{34 \cdot 193}{3}} = 259{\cdot}82$ is the greatest velocity it can acquire in air.

But if the globe were only $\frac{1}{100}$ of an inch diameter, the greatest velocities it could acquire, would be only $\frac{1}{10}$ of these, namely $\frac{86}{100}$ of a foot in water, and 26 feet nearly in air. And if the ball were still further diminished, the greatest velocity would also be diminished, and that in the subduplicate ratio of the diameter of the ball.

PROBLEM XXI.

To determine the Relations of Velocity, Space, and Time, of a Ball moving in a Fluid, in which it is projected with a Given Velocity.

1. Let

1. Let a = the first velocity of projection, x the space described in any time t, and v the velocity then. Now, by art. 4 of the last problem, the accelerative force $f = \frac{3nv^2}{16g\text{N}d}$, where n is the density of the fluid, N that of the ball, and d its diameter. Therefore the general equation $v\dot{v} = 2gf\dot{s}$ becomes $v\dot{v} =$ - - - - - - - - - - - - - - - - $\frac{-3nv^2}{8\text{N}d}\dot{x}$; and hence $\frac{\dot{v}}{v} = \frac{-3n}{8\text{N}d}\dot{x} = -b\dot{x}$, putting b for $\frac{3n}{8\text{N}d}$. The correct fluent of this, is log. a − log v or log. $\frac{a}{v} = bx$. Or, putting $c = 2{\cdot}718281828$, the number whose hyp. log. is 1, then is $\frac{a}{v} = c^{bx}$, and the velocity $v = \frac{a}{c^{bx}} = ac^{-bx}$.

2. The velocity v at any time being the c^{-bx} part of the first velocity, therefore the velocity lost in any time, will be the $1 - c^{-bx}$ part, or the $\frac{c^{bx}-1}{c^{bx}}$ part of the first velocity.

EXAMPLES.

Exam. 1. If a globe be projected, with any velocity, in a medium of the same density with itself, and it describe a space equal to $3d$ or 3 of its diameters. Then $x = 3d$, and $b = \frac{3n}{8\text{N}d} = \frac{3}{8d}$ therefore $bx = \frac{9}{8}$, and $\frac{c^{bx}-1}{c^{bx}} = \frac{2{\cdot}08}{3{\cdot}08}$ is the velocity lost, or nearly $\frac{2}{3}$ of the projectile velocity.

Exam. 2. If an iron ball of 2 inches diameter were projected with a velocity of 1200 feet per second; to find the velocity lost after moving through any space, as suppose 500 feet of air: we should have $d = \frac{2}{12} = \frac{1}{6}$, $a = 1200$, $x = 500$, $\text{N} = 7\frac{1}{3}$, $n = {\cdot}0012$; and therefore $bx =$ - - $\frac{3nx}{8\text{N}d} = \frac{3 \,.\, 12 \,.\, 500 \,.\, 3 \,.\, 6}{8 \,.\, 22 \,.\, 10000} = \frac{81}{440}$, and $v = \frac{1200}{c^{\frac{81}{440}}} = 998$ feet per second: having lost 202 feet, or nearly $\frac{1}{6}$ of its first velocity.

Exam. 3. If the earth revolved about the sun, in a medium as dense as the atmosphere near the earth's surface; and it were required to find the quantity of motion lost in a

year.

year. Then, since the earth's mean density is about $4\frac{1}{2}$, and its distance from the sun 12000 of its diameters, we have $24000 \times 3.1416 = 75398$ diameters $= x$, and $bx =$ - - - $\frac{3 . 75398 . 12 . 2}{8 . 10000 . 9} = 7.5398$; hence $\frac{c^{bx} - 1}{c^{bx}} = \frac{1880}{1881}$ parts are lost of the first motion in the space of a year, and only the $\frac{1}{1881}$ part remains.

Exam. 4. If it be required to determine the distance moved, x, when the globe has lost any part of its motion, as suppose $\frac{1}{2}$, and the density of the globe and fluid equal; The general equation gives $x = \frac{1}{b} \times \log. \frac{a}{v} = \frac{8d}{3} \times \log.$ of $2 = 1.8483925d$. So that the globe loses half its motion before it has described twice its diameter.

3. To find the time t; we have $\dot{t} = \frac{\dot{s}}{v} = \frac{\dot{x}}{v} = \frac{c^{bx}\dot{x}}{a}$. Now, to find the fluent of this, put $z = c^{bx}$; then is $bx =$ log. z, and $b\dot{x} = \frac{\dot{z}}{z}$, or $\dot{x} = \frac{\dot{z}}{bz}$; conseq. $\dot{t}$ or $\frac{c^{bx}\dot{x}}{a} = \frac{z\dot{x}}{a} = \frac{\dot{z}}{ab}$ and hence $t = \frac{z}{ab} = \frac{c^{bx}}{ab}$. But as t and x vanish together, and when $x = 0$, the quantity $\frac{c^{bx}}{ab}$ is $= \frac{1}{ab}$; therefore, by correction, $t = \frac{c^{bx} - 1}{ab} = \frac{1}{bv} - \frac{1}{ba} = \frac{1}{b} \left(\frac{1}{v} - \frac{1}{a}\right)$ the time sought; where $b = \frac{3n}{8Nd}$, and $v = \frac{a}{c^{bx}}$ the velocity.

Exam. If an iron ball of 2 inches diameter were projected in the air with a velocity of 1200 feet per second; and it were required to determine in what time it would pass over 500 yards or 1500 feet, and what would be its velocity at the end of that time: We should have, as in exam. 2 above,

$b = \frac{3 . 12 . 3 . 6}{8 . 12 . 10000} = \frac{1}{2716}$, and $bx = \frac{1500}{2716} = \frac{375}{679}$; hence $\frac{1}{b} = \frac{2716}{1}$, and $\frac{1}{a} = \frac{1}{1200}$, and $\frac{1}{v} = \frac{c^{bx}}{a} = \frac{1.7372}{1200} = \frac{1}{690}$ nearly. Consequently $v = 690$ is the velocity; and $t = \frac{1}{b}\left(\frac{1}{v} - \frac{1}{a}\right) = 2716 \times \left(\frac{1}{690} - \frac{1}{1200}\right) = 1\frac{31}{46}$ seconds, is the time required, or $1''$ and $\frac{2}{3}$ nearly.

PROBLEM

PROBLEM XXII.

To determine the Relations of Space, Time, and Velocity, when a Globe descends, by its own Weight, in a Fluid.

The foregoing notation remaining, viz. d = diameter, N and n the density of the ball and fluid, and v, s, t, the velocity, space, and time, in motion; we have $\frac{1}{6}pd^3$ = the magnitude of the ball, and $\frac{1}{6}pd^3$ $(N - n)$ = its weight in the fluid, also $m = \frac{pnd^2v^2}{32g}$ = its resistance from the fluid; consequently $\frac{1}{6}pd^3$ $(N - n) - \frac{pnd^2v^2}{32g}$ is the motive force by which the ball is urged; which being divided by $\frac{1}{6}Nd^3$, the quantity of matter moved, gives $f = 1 - \frac{n}{N} - \frac{3nv^2}{16gNd}$ for the accelerative force.

2. Hence $v\dot{v} = 2gf\dot{s}$, and $\dot{s} = \frac{v\dot{v}}{2gf} = \frac{Nv\dot{v}}{2g\ (N - n) - \frac{3n}{8d}v^2}$ $= \frac{1}{b} \times \frac{v\dot{v}}{a - v^2}$, putting $b = \frac{3n}{8Nd}$, and $\frac{1}{a} = \frac{3n}{2g\ .\ 8d\ (N - n)}$, or $ab = 2g$ nearly; the fluent of which is s = - - - - - - $\frac{1}{2b} \times$ log. of $\frac{a}{a - v^2}$, an expression for the space s, in terms of the velocity v. That is, when s and v begin, or are equal to nothing, both together.

But if the body commence motion in the fluid with a certain given velocity c, or enter the fluid with that velocity, like as when the body, after falling in empty space from a certain height, falls into a fluid like water; then the correct fluent will be $s = \frac{1}{2b} \times$ hyp. log. of $\frac{a - c^2}{a - v^2}$.

3. But now, to determine v in terms of s, put c = 2·718281828; then, since the log. of $\frac{a}{a - v^2} = 2bs$, therefore $\frac{a}{a - v^2} = c^{2bs}$, or $\frac{a - v^2}{a} = c^{-2bs}$; hence v = - - - - $\sqrt{a - ac^{-2bs}}$ is the velocity sought.

 4. The

4. The greatest velocity is to be found, as in art. 5 of prob. 20, by making f or $1 - \frac{n}{N} - \frac{3nv^2}{16gNd} = 0$, which gives $v = \sqrt{}(2g \,.\, 8d \,.\, \frac{N-n}{3n}) = \sqrt{a}$. The same value of v is obtained by making the fluxion of v^2, or of $a - ac^{-2bs}$, $= 0$. And the same value of v is also obtained by making s infinite, for then $c^{-2bs} = 0$. But this velocity $\sqrt{a}$ cannot be attained in any finite time, and it only denotes the velocity to which the general value of v or $\sqrt{\cdot\, a - ac^{-2bs}}$ continually approaches. It is evident however, that it will approximate towards it the faster, the greater b is, or the less d is; and that, the diameters being very small, the bodies descend by nearly uniform velocities, which are direct in the subduplicate ratio of the diameters. See also art. 5, prob. 20, for other observations on this head.

5. To find the time t. Now $\dot{t} = \frac{\dot{s}}{v} = \sqrt{\frac{1}{a}} \times \frac{\dot{s}}{\sqrt{1 - c^{-2bs}}}$. Then, to find the fluent of this fluxion, put $z = \sqrt{1 - c^{-2bs}} = \frac{v}{\sqrt{a}}$, or $z^2 = 1 - c^{-2bs}$; hence $z\dot{z} = b\dot{s}c^{-2bs}$, and $\dot{s} = \frac{z\dot{z}}{bc^{-2bs}} = \frac{1}{b} \cdot \frac{z\dot{z}}{1 - z^2}$, consequently $\dot{t} = \frac{1}{b\sqrt{a}} \cdot \frac{\dot{z}}{1 - z^2}$, - - - and therefore the fluent is $t = \frac{1}{2b\sqrt{a}} \times \log. \frac{1+z}{1-z} = \frac{1}{2b\sqrt{a}} \times \log. \frac{1 + \sqrt{1 - c^{-2bs}}}{1 - \sqrt{1 - c^{-2bs}}} = \frac{1}{2b\sqrt{a}} \times \log. \frac{\sqrt{a} + v}{\sqrt{a} - v}$, which is the general expression for the time.

Exam. If it were required to determine the time and velocity, by descending in air 1000 feet, the ball being of lead, and 1 inch diameter.

Here $N = 11\frac{1}{3}$, $n = \frac{3}{2500}$, $d = \frac{1}{12}$, and $s = 1000$.

Hence $a = \frac{2 \,.\, 16\frac{1}{12} \cdot \frac{3}{16} \cdot 11\frac{1}{3}}{3 \cdot \frac{3}{2500}} = \frac{2 \,.\, 193 \,.\, 8 \,.\, 34 \,.\, 2500}{3 \,.\, 3 \,.\, 12 \,.\, 12 \,.\, 3} = \frac{193 \,.\, 34 \,.\, 50^2}{9 \,.\, 27}$, and $b = \frac{3 \cdot \frac{3}{2500}}{8 \,.\, 11\frac{1}{3} \,.\, \frac{1}{12}} = \frac{3 \,.\, 3 \,.\, 3 \,.\, 12}{8 \,.\, 34 \,.\, 2500} = \frac{9 \,.\, 9}{68 \,.\, 50^2}$; consequently $v = \sqrt{a} \times \sqrt{1 - c^{-2bs}} = \sqrt{\frac{193 \,.\, 34 \,.\, 50^2}{9 \,.\, 27}} \times \sqrt{.}(1 - c^{-\frac{81}{85}}) = 203\frac{2}{3}$ the velocity. And $t = \frac{1}{2b\sqrt{a}} \times \log.$

2 +

$$\frac{2+\sqrt{1-c^{-2bs}}}{1-\sqrt{1-c^{-2bs}}} = \sqrt{\frac{34 \,.\, 2500}{27 \,.\, 193}} \times \text{log.}\ \frac{1{\cdot}78383}{0{\cdot}21617} = 8{\cdot}5236'',$$

the time.

Note. If the globe be so light as to ascend in the fluid; it is only necessary to change the signs of the first two terms in the value of f, or the accelerating force, by which it becomes $f = \frac{n}{N} - 1 - \frac{3nv^2}{16gNd}$; and then proceeding in all respects as before.

SCHOLIUM.

To compare this theory, contained in the last four problems, with experiment, the few following numbers are here extracted from extensive tables of velocities and resistances, resulting from a course of many hundred very accurate experiments, made in the course of the year 1786.

In the first column are contained the mean uniform or greatest velocities acquired in air, by globes, hemispheres, cylinders, and cones, all of the same diameter, and the altitude of the cone nearly equal to the diameter also, when urged by the several weights expressed in avoirdupois ounces, and standing on the same line with the velocities, each in their proper column. So, in the first line, the numbers show, that, when the greatest or uniform velocity was accurately 3 feet per second, the bodies were urged by these weights, according as their different ends went foremost; namely, by ·028 oz. when the vertex of the cone went foremost; by ·064 oz. when the base of the cone went foremost; by ·027 oz. for a whole sphere; by ·050 oz. for a cylinder; by ·051 oz. for the flat side of the hemisphere; and by ·020 oz. for the round or convex side of the hemisphere. Also, at the bottom of all, are placed the mean proportions of the resistances of these figures in the nearest whole numbers. Note, the common diameter of all the figures, was 6·375, or $6\frac{3}{8}$ inches; so that the area of the circle of that diameter is just 32 square inches or $\frac{2}{9}$ of a square foot; and the altitude of the cone was $6\frac{5}{8}$ inches. Also, the diameter of the small hemisphere was $4\frac{3}{4}$ inches, and consequently the area of its base $17\frac{3}{4}$ square inches, or $\frac{1}{8}$ of a square foot nearly.

From the given dimensions of the cone, it appears, that the angle made by its side and axis, or direction of the path, is 26 degrees, very nearly.

The

The mean height of the barometer at the times of making the experiments, was nearly 30·1 inches, and of the thermometer 62°; consequently the weight of a cubic foot of air was equal to $1\frac{1}{5}$ oz. nearly in those circumstances.

Veloc. per sec.	Cone. vertex.	Cone. base.	Whole globe.	Cylinder.	Hemisphere. flat.	Hemisphere. round.	Small Hemis. flat.
feet.	oz.	oz.	oz.	oz.	oz.	oz.	oz.
3	·028	·064	·027	·050	·051	·020	·028
4	·048	·109	·047	·090	·096	·039	·048
5	·071	·162	·068	·143	·148	·063	·072
6	·098	·225	·094	·205	·211	·092	·103
7	·129	·298	·125	·278	·284	·123	·141
8	·168	·382	·162	·360	·368	·160	·184
9	·211	·478	·205	·456	·464	·199	·233
10	·260	·587	·255	·565	·573	·242	·287
11	·315	·712	·310	·688	·698	·297	·349
12	·376	·850	·370	·826	·836	·347	·418
13	·440	1·000	·435	·979	·988	·409	·492
14	·512	1·166	·505	1·145	1·154	·478	·573
15	·589	1·346	·581	1·327	1·336	·552	·661
16	·673	1·546	·663	1·526	1·538	·634	·754
17	·762	1·763	·752	1·745	1·757	·722	·853
18	·858	2·002	·848	1·986	1·998	·818	·959
19	·959	2·260	·949	2·246	2·258	·922	1·073
20	1·069	2·540	1·057	2·528	2·542	1·033	1·196
Propor. Numb.	126	291	124	285	288	119	140

From this table of resistances, several practical inferences may be drawn. As,

1. That the resistance is nearly as the surface; the resistance increasing but a very little above that proportion in the greater surfaces. Thus, by comparing together the numbers in the 6th and last columns, for the bases of the two hemispheres, the areas of which are in the proportion of $17\frac{1}{4}$ to 32, or as 5 to 9 very nearly; it appears that the numbers in those two columns, expressing the resistances, are nearly as 1 to 2, or as 5 to 10, as far as to the velocity of 12 feet; after which the resistances on the greater surface increase gradually more and more above that proportion. And the mean resistances are as 140 to 288, or as 5 to

to $10\frac{3}{4}$. This circumstance therefore agrees nearly with the theory.

2. The resistance to the same surface, is nearly as the square of the velocity; but gradually increasing more and more above that proportion, as the velocity increases. This is manifest from all the columns. And therefore this circumstance also differs but little from the theory, in small velocities.

3. When the hinder parts of bodies are of different forms, the resistances are different, though the fore parts be alike; owing to the different pressures of the air on the hinder parts. Thus, the resistance to the fore part of the cylinder, is less than that on the flat base of the hemisphere, or of the cone; because the hinder part of the cylinder is more pressed or pushed, by the following air, than those of the other two figures.

4. The resistance on the base of the hemisphere, is to that on the convex side, nearly as $2\frac{2}{5}$ to 1, instead of 2 to 1, as the theory assigns the proportion. And the experimented resistance, in each of these, is nearly $\frac{1}{4}$ part more than that which is assigned by the theory.

5. The resistance on the base of the cone is to that on the vertex, nearly as $2\frac{3}{10}$ to 1. And in the same ratio is radius to the sine of the angle of the inclination of the side of the cone, to its path or axis. So that, in this instance, the resistance is directly as the sine of the angle of incidence, the transverse section being the same, instead of the square of the sine.

6. Hence we can find the altitude of a column of air, whose pressure shall be equal to the resistance of a body, moving through it with any velocity. Thus,

Let $a =$ the area of the section of the body, similar to any of those in the table, perpendicular to the direction of motion;
$r =$ the resistance to the velocity, in the table; and
$x =$ the altitude sought, of a column of air, whose base is a, and its pressure r.

Then $ax =$ the content of the column in feet,
and $1\frac{1}{5}ax$ or $\frac{6}{5}ax$ its weight in ounces; - - - - - - -

therefore $\frac{6}{5}ax = r$, and $x = \frac{5}{6} \times \frac{r}{a}$ is the altitude sought in feet,

feet, namely, $\frac{5}{8}$ of the quotient of the resistance of any body divided by its transverse section; which is a constant quantity for all similar bodies, however different in magnitude, since the resistance r is as the section a, as was found in art. 1. When $a = \frac{2}{9}$ of a foot, as in all the figures in the foregoing table, except the small hemisphere: then, $x = \frac{5}{8} \times \frac{r}{a}$ becomes $x = \frac{15}{4} r$, where r is the resistance in the table, to the similar body.

If, for example, we take the convex side of the large hemisphere, whose resistance is ·634 oz. to a velocity of 16 feet per second, then $r = \cdot 634$, and $x = \frac{15}{4} r = 2\cdot 3775$ feet, is the altitude of the column of air whose pressure is equal to the resistance on a spherical surface, with a velocity of 16 feet. And to compare the above altitude with that which is due to the given velocity, it will be $32^2 : 16^2 :: 16 : 4$, the altitude due to the velocity 16; which is near double the altitude that is equal to the pressure. And as the altitude is proportional to the square of the velocity, therefore, in small velocities, the resistance to any spherical surface, is equal to the pressure of a column of air on its great circle, whose altitude is $\frac{19}{32}$ or ·594 of the altitude due to its velocity.

But if the cylinder be taken, whose resistance $r = 1\cdot 526$: then $x = \frac{15}{4} r = 5\cdot 72$; which exceeds the height, 4, due to the velocity in the ratio of 23 to 16 nearly. And the difference would be still greater, if the body were larger; and also if the velocity were more.

7. Also, if it be required to find with what velocity any flat surface must be moved, so as to suffer a resistance just equal to the whole pressure of the atmosphere:

The resistance on the whole circle whose area is $\frac{2}{9}$ of a foot, is ·051 oz. with the velocity of 3 feet per second; it is $\frac{1}{9}$ of ·051, or ·0056 oz. only, with a velocity of 1 foot. But $2\frac{1}{2} \times 13600 \times \frac{2}{9} = 7555\frac{5}{9}$ oz. is the whole pressure of the atmosphere. Therefore, as $\sqrt{\cdot 0056} : \sqrt{7556} :: 1 : 1162$ nearly, which is the velocity sought. Being almost equal to the velocity with which air rushes into a vacuum.

8. Hence may be inferred the great resistance suffered by military projectiles. For in the table, it appears, that a globe of $6\frac{3}{4}$ inches diameter, which is equal to the size of an iron ball weighing 36lb, moving with a velocity of only 16 feet per second, meets with a resistance equal to the pressure of $\frac{2}{3}$ of an ounce weight; and therefore, computing only according to the

square

square of the velocity, the least resistance that such a ball would meet with, when moving with a velocity of 1600 feet, would be equal to the pressure of 417lb, and that independent of the pressure of the atmosphere itself on the fore part of the ball, which would be 487lb more, as there would be no pressure from the atmosphere on the hinder part, in the case of so great a velocity as 1600 feet per second. So that the whole resistance would be more than 900lb to such a velocity.

9. Having said, in the last article, that the pressure of the atmosphere is taken entirely off the hinder part of the ball moving with a velocity of 1600 feet per second; which must happen when the ball moves faster than the particles of air can follow by rushing into the place quitted and left void by the ball, or when the ball moves faster than the air rushes into a vacuum from the pressure of the incumbent air: let us therefore inquire what this velocity is. Now the velocity with which any fluid issues, depends on its altitude above the orifice, and is indeed equal to the velocity acquired by a heavy body in falling freely through that altitude. But, supposing the height of the borometer to be 30 inches, or $2\frac{1}{2}$ feet, the height of a uniform atmosphere, all of the same density as at the earth's surface, would be $2\frac{1}{2} \times 13{\cdot}6 \times 833\frac{1}{3}$ or 28333 feet; therefore $\sqrt{16} : \sqrt{28333} :: 32 : 8\sqrt{28333} = 1346$ feet, which is the velocity sought. And therefore, with a velocity of 1600 feet per second, or any velocity above 1346 feet, the ball must continually leave a vacuum behind it, and so must sustain the whole pressure of the atmosphere on its fore part, as well as the resistance arising from the *vis inertia* of the particles of air struck by the ball.

10. On the whole, we find that the resistance of the air, as determined by the experiments, differs very widely, both in respect to its quantity on all figures, and in respect to the proportions of it on oblique surfaces, from the same as determined by the preceding theory; which is the same as that of Sir Isaac Newton, and most modern philosophers. Neither should we succeed better if we have recourse to the theory given by Professor Gravesande, or others, as similar differences and inconsistencies still occur.

We conclude therefore, that all the theories of the resistance of the air hitherto given, are very erroneous. And the preceding one is only laid down, till further experiments, on this important subject, shall enable us to deduce from them another, that shall be more consonant to the true phænomena of nature.

ON

ON THE MOTION OF MACHINES, AND THEIR MAXIMUM EFFECTS.

Art. 1. When forces acting in contrary directions, or in any such directions as produce contrary effects, are applied to machines, there is, with respect to every simple machine (and of consequence with respect to every combination of simple machines) a certain relation between the powers and the distances at which they act, which, if subsisting in any such machine when at rest, will always keep it in a state of rest, or of *statical* equilibrium; and for this reason, because the efforts of these powers, when thus related, with regard to magnitude and distance, being equal and opposite, annihilate each other, and have no tendency to change the state of the system to which they are applied. So also, if the same machine have been put into a state of *uniform* motion, whether rectilinear or rotatory, by the action of any power distinct from those we are now considering, and *these* two powers be made to act upon the machine in such motion in a similar manner to that in which they acted upon it when at rest, their simultaneous action will preserve it in that state of uniform motion, or of *dynamical* equilibrium; and this for the same reason as before, because their contrary effects destroy each other, and have therefore no tendency to change the *state* of the machine. But, if at the time a machine is in a state of balanced rest, any one of the opposite forces be increased while it continues to act at the same distance, this excess of force will disturb the statical equilibrium, and produce motion in the machine; and if the same excess of force continues to act in the same manner it will, like every constant force, produce an accelerated motion; or, if it should undergo particular modifications when the machine is in different positions, it may occasion such variations in the motion as will render it alternately accelerated and retarded. Or the different species of resistance to which a moving machine is subjected, as the rigidity of ropes, friction, resistance of the air, &c, may so modify a motion, as to change a regular or irregular variable motion into one which is uniform.

2. Hence then the motion of machines may be considered as of *three* kinds. 1. That which is gradually accelerated, which obtains commonly in the first instants of the communication. 2. That which is entirely uniform. 3. That which is alternately accelerated and retarded. Pendulum clocks, and machines which are moved by a balance, are related to the

the third class. Most other machines, a short time after their motion is commenced, fall under the second. Now though the motion of a machine is alternately accelerated and retarded, it may, notwithstanding, be measured by a uniform motion, because of the periodical and regular repetition which may exist in the acceleration and retardation. Thus the motion of a second's pendulum, considered in respect to a single oscillation, is accelerated during the first half second, and retarded during the next: but the same motion taken for many oscillations may be considered as uniform. Suppose, for example, that the extent of each oscillation is 5 inches, and that the pendulum has made 10 oscillations: its total effect will be to have run over 50 inches in 10 seconds; and, as the space described in each second is the same, we may compare the effect to that produced by a moveable which moves for 10 seconds with a velocity of 5 inches per second. We see, therefore, that the theory of machines whose motions are uniform, conduces naturally to the estimation of the effects produced by machines whose motion is alternately accelerated and retarded: so that the problems comprised in this chapter will be directed to those machines whose motions fall under the first two heads; such problems being of far the greatest utility in practice.

Defs. 1. When in a machine there is a system of forces or of powers mutually in opposition, those which produce or tend to produce a certain effect are called *movers* or *moving powers*; and those which produce or tend to produce an effect which opposes those of the moving powers, are called *resistances*. If various movers act at the same time, their equivalent (found by means of prop. 7, Motion and Forces) is called individually *the moving force*; and, in like manner, the resultant of all the resistances reduced to some one point, *the resistance*. This reduction in all cases simplifies the investigation.

2. The *impelled point* of a machine is that to which the action of the moving power may be considered as immediately applied; and the *working point* is that where the resistance arising from the work to be performed immediately acts, or to which it ought all to be reduced. Thus, in the wheel and axle, (Mechan. prop. 32), where the moving power P is to overcome the weight or resistance W, by the application of the cords to the wheel and to the axle, B is the impelled point, and A the working point.

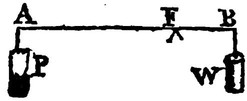
3. The *velocity of the moving power* is the same as the velocity of the impelled point; the *velocity of the resistance* the same as that of the working point.

4. The *performance* or *effect* of a machine, or the *work done*, is measured by the product of the resistance into the velocity of the working point; the *momentum of impulse* is measured by the product of the moving force into the velocity of the impelled point.

These definitions being established, we may now exhibit a few of the most useful problems, giving as much variety in their solutions as may render one or other of the methods of easy application to any other cases which may occur.

PROPOSITION I.

If R, *and* r *be the distances of the power* P, *and the weight or resistance* W, *from the fulcrum* F *of a straight lever; then will the velocity of the power and of the weight at the end of any time* t *be* $\frac{R^2P - RrW}{R^2P - r^2W}gt$, *and* $\frac{RrP - r^2W}{R^2P + r^2W}gt$, *respectively, the weight and inertia of the lever itself not being considered.*



If the effort of the power balanced that of the resistance, P would be equal to $\frac{rW}{R}$. Consequently, the difference between this value of P and its actual value, or $P - \frac{r}{R}W$, will be the force which tends to move the lever. And because this power applied to the point A accelerates the masses P and W, the mass to be substituted for W, in the point A, must be $\frac{r^2}{R^2}W$, (Mechan. prop. 50) in order that this mass at the distance R may be equally accelerated with the mass W at the distance R. Hence the power $P - \frac{r}{R}W$ will accelerate the quantity of matter $P + \frac{r^2}{R^2}W$; and the accelerating force $F = (P - \frac{r}{R}W) \div (P + \frac{r^2}{R^2}W) = \frac{PR^2 - RrW}{PR^2 + r^2W}$. But (Art. 33, Gen. Laws of Motion) $v \propto Ft$ or is $= gtF$ (g being $= 32\frac{1}{6}$ feet); which in this case $= \frac{R^2P - RrW}{R^2P + r^2W}.gt$, the velocity of P. And, because veloc. of P : veloc. of W :: R : r, therefore veloc. of $W = \frac{r}{R}$ veloc. of $P = \frac{r}{R} \times \frac{R^2P - RrW}{R^2P + r^2W}gt = \frac{RrP - r^2W}{R^2P + r^2W}.gt$.

Corol.

Corol. 1. The space described by the power in the time t, will be $= \frac{R^2P - RrW}{R^2P + r^2W} \cdot \frac{1}{2}gt^2$; the space described by W in the same time will be $= \frac{RrP - r^2W}{R^2P + r^2W} \cdot \frac{1}{2}gt^2$.

Cor. 2. If $R : r :: n : 1$, then will the force which accelerates A be $= \frac{Pn^2 - Wn}{Pn^2 + W}$.

Cor. 3. If at the same time the inertia of the moving force P be $= 0$, as in muscular action, the force accelerating A will be $= \frac{Pn^2 - Wn}{W}$.

Cor. 4. If the mass moved have no weight, but possesses inertia only, as when a body is moved along a horizontal plane, the force which accelerates A will be $= \frac{Pn^2}{Pn^2 + W}$. And either of these values may be readily introduced into the investigation.

Cor. 5. The work done in the time t, if we retain the original notation, will be $= \frac{RrP - r^2W}{R^2P + r^2W} gt \times W = \frac{RrPW - r^2W^2}{R^2P + r^2W} \cdot gt$.

Cor. 6. When the work done is to be a maximum, and we wish to know the weight when P is given, we must make the fluxion of the last expression $= 0$. Then we shall have $rR^3P^2 - 2r^2R^2PW - r^4W^2 = 0$ and $W = P \times [\sqrt{(\frac{R^4}{r^4} + \frac{R^3}{r^3})} - \frac{R^2}{r^2}]$.

Cor. 7. If $R : r :: n : 1$, the preceding expression will become $W = P \times [\sqrt{(n^4 + n^3)} - n^2]$.

Cor. 8. When the arms of the lever are equal in length, that is, when $n = 1$, then is $W = P \times (\sqrt{2} - 1) = \cdot 414214P$, or nearly $\frac{5}{12}$ of the moving force.

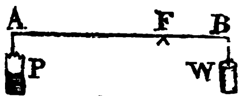
Scholium.

If we in like manner investigate the formulæ relating to motion on the axis in peritrochio, it will be seen that the expressions correspond exactly. Hence it follows, that when it is required to proportion the power and weight so as to obtain

obtain a maximum effect on the wheel and axle, (the weight of the machinery not being considered), we may adopt the conclusions of cors. 6 and 7 of this prop. And in the extreme case where the wheel and axle becomes a pulley, the expression in cor 8 may be adopted. The like conclusions may be applied to machines in general, if R and r represent the distances of the impelled and working points from the axis of motion; and if the various kinds of resistance arising from friction, stiffness of ropes, &c, be properly reduced to their equivalents at the working points, so as to be comprehended in the character W for resistance overcome.

PROPOSITION II.

Given R *and* r, *the arms of a straight lever,* M *and* m *their respective weights, and* P *the power acting at the extremity of the arm* R; *to find the weight raised at the extremity of the other arm when the effect is a maximum.*



In this case $\frac{1}{2}m$ is the weight of the shorter end reduced to B, and conseq. $\frac{mr}{2\text{R}}$ is the weight which applied at A, would balance the shorter end: therefore $\frac{mr}{2r} + \frac{r}{\text{R}}\text{W}$, would sustain both the shorter end and the weight W in equilibrio. But $\text{P} + \frac{1}{2}\text{M}$ is the power really acting at the longer end of the lever; consequently

$\text{P} + \frac{1}{2}\text{M} - (\frac{mr}{2r} + \frac{r}{\text{R}}\text{W})$, is the absolute *moving* power. Now the distance of the centre of gyration of the beam from F*

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* The distance of R, the centre of gyration, from C the centre or axis of motion, in some of the most useful cases, is as below;

In a circular wheel of uniform thickness	$\text{CR} = \text{rad.}\sqrt{\frac{1}{2}}$
In the periphery of a circle revolving about the diam.	$\text{CR} = \text{rad.}\sqrt{\frac{1}{2}}$
In the plane of a circle . . . ditto	$\text{CR} = \frac{1}{2}\,\text{rad.}$
In the surface of a sphere . . ditto	$\text{CR} = \text{rad.}\sqrt{\frac{2}{3}}$
In a solid sphere . . . . . ditto	$\text{CR} = \text{rad.}\sqrt{\frac{2}{5}}$
In a plane ring formed of circles whose radii are R, r, revolving about centre	$\text{CR} = \sqrt{\frac{\text{R}^4}{2\text{R}^2 - 2r^2}}$.
In a cone revolving about its vertex	$\text{CR} = \frac{1}{2}\sqrt{\frac{12}{5}a^2 + \frac{3}{5}r^2}$.
In a cone . . . . . . its axis	$\text{CR} = r\sqrt{\frac{3}{10}}$.
In a straight lever whose arms are R and r . . .	$\text{CR} = \sqrt{\frac{\text{R}^3 + r^3}{3(\text{R}+r)}}$.

is

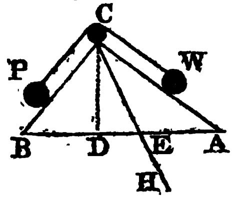
is $= \sqrt{\frac{R^3 + r^3}{3(R+r)}}$, which let be denoted by ϱ; then (Mechan. prop. 50) $\frac{\varrho^2}{R^2}.\ (M + m)$ will represent the mass equivalent to the beam or lever when reduced to the point A; while the weight equivalent to W, when referred to that point, will be $\frac{r^2}{R^2}W$. Hence, proceeding as in the last prop. we shall have $\frac{\varrho^2}{R^2}.\ (M + m) + P + \frac{r^2}{R^2}W$ for the inertia to be overcome; and $(P + \frac{1}{2}M - \frac{mr}{2R} - \frac{r}{R}W) \div \frac{\varrho^2}{R^2}(M + m) + P + \frac{r^2}{R^2}W$ = the accelerating force of P, or of W reduced to A. Multiply this by W; and, for the sake of simplifying the process, put q for $P + \frac{1}{2}M - \frac{rm}{2R}$, and n for $P + \frac{\varrho^2}{R^2}(M + m)$, then will $\dfrac{qW - \frac{rW^2}{R}}{n + \frac{r^2}{R^2}W}$ be a quantity which varies as the effect varies, and which, indeed, when multiplied by gt, denotes the effect itself. Putting the fluxion of this equal to nothing, and reducing, we at length find

$$W = \frac{R}{r}\sqrt{\left(\frac{nqR}{r} + \frac{n^2R^2}{r^2}\right)} - \frac{nR^2}{r^2}.$$

Cor. When $R = r$, and $M = m$, if we restore the values of n and q, the expression will become $W = \sqrt{(2P^2 + 2mP + \frac{4}{9}m^2)} - (P + \frac{2}{3}m)$.

PROPOSITION III.

Given the length l and angle e of elevation of an inclined plane BC; *to find the length* L *of another inclined plane* AC, *along which a given weight* W *shall be raised from the horizontal line* AB *to the point* C, *in the least time possible, by means of another given weight* P *descending along the given plane* CB: *the two weights being connected by an inextensible thread* PCW *running always parallel to the two planes.*



Here we must, as a preliminary to the solution of this proposition, deduce expressions for the motion of bodies connected by a thread, and running upon double inclined planes. Let the angle of elevation CAD be E, while e is the elevation CBD. Then at the end of the time t, P will

will have a velocity v; and gravity would impress upon it, in the instant $\dot{t}$ following, a new velocity $= g \sin e . \dot{t}$, provided the weight P were then entirely free: but, by the disposition of the system, $\dot{v}$ will be the velocity which obtains in reality. Then, estimating the spaces in the direction CP, as the body W moves with an equal velocity but in a contrary sense, it is obvious, that by applying the 3d Law of Motion, the decomposition may be made as follows. At the end of the time $t + \dot{t}$ we have, for the velocity impressed on,

$$\text{P} \ldots v + g \sin e . \dot{t}, \text{where} \begin{cases} v + \dot{v} \ldots\ldots \text{effective veloc. from C towards B.} \\ g \sin e . \dot{t} - \dot{v} \ldots\ldots \text{velocity destroyed.} \end{cases}$$

$$\text{W} . - v + g \sin \text{E} . \dot{t}, \text{where} \begin{cases} -v - \dot{v} \ldots \text{effective veloc. from C towards A.} \\ \dot{v} + g \sin \text{E} . \dot{t}, \ldots \text{velocity destroyed.} \end{cases}$$

If, therefore, gravity impresses, during the time $\dot{t}$, upon the masses P, W, the respective velocities $g \sin e . \dot{t} - \dot{v}$, and $g \sin \text{E} . \dot{t} + \dot{v}$, the system will be in equilibrio. The quantities of motion being therefore equal, it will be

$$\text{P}g \sin e . \dot{t} - \text{P}\dot{v} = \text{W}g \sin \text{E} . \dot{t} + \text{W}\dot{v}.$$

Whence the effective accelerating force is found, i. e.

$$\varphi = \frac{\dot{v}}{\dot{t}} = \frac{\text{P} \sin e - \text{W} \sin \text{E}}{\text{P} + \text{W}} \times g.$$

Thus it appears that the motion is uniformly varied, and we readily find the equations for the velocity and space from which the conditions of the motion are determined: viz,

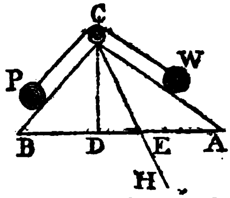
$$v = \frac{\text{P} \sin e - \text{W} \sin \text{E}}{\text{P} + \text{W}} \ldots s = \frac{\text{P} \sin e - \text{W} \sin \text{E}}{\text{P} + \text{W}} . \tfrac{1}{2}gt^2.$$

The latter of these two equations gives $t^2 = \dfrac{s(\text{P}+\text{W})}{\frac{1}{2}g(\text{P}\sin e - \text{W}\sin \text{E})}$. But in the triangle ABC it is AC : BC :: sin B : sin A, that is, $\text{L} : l :: \sin e : \sin \text{E}$; hence $\frac{1}{m}\text{L} = \sin e$, and $\frac{1}{m} l = \sin \text{E}$; m being a constant quantity always determinable from the data given. And t^2 becomes $\dfrac{s(\text{P}+\text{W})}{\frac{1}{2}g\frac{1}{m}(\text{PL} - \text{W}l)}$. Now when any quantity, as t, is a minimum, its square is manifestly a minimum: so that substituting for s its equal L, and striking out the constant factors, we have $\dfrac{\text{L}^2}{\text{PL} - \text{W}l} =$ a min. or its fluxion $\dfrac{2\text{L}\dot{\text{L}}(\text{PL} - \text{W}l) - \text{PL}^2\dot{\text{L}}}{(\text{PL} - \text{W}l)} = 0$. Here, as in all similar cases, since the fraction vanishes, its numerator must be equal to 0; consequently $2\text{PL}^2 - 2\text{W}l\text{L} - \text{PL}^2 = 0$, $\text{PL} = 2\text{W}l$, or $\text{L} : l :: 2\text{W} : \text{P}$.

Cor. 1. Since neither sin e nor sin E enters the final equation, it follows, that if the elevation of the plane BC is not given, the problem is unlimited. *Cor.*

Cor. 2. When sin $e = 1$, BC coincides with the perpendicular CD, and the power P acts with all its intensity upon the weight W. This is the case of the present problem which has commonly been considered.

Scholium.



This proposition admits of a neat *geometrical* demonstration. Thus, let CE be the plane upon which, if W were placed, it would be sustained in equilibrio by the power P on the plane CB, or the power P hanging freely in the vertical CD; then (Mechan. prop. 23) BC : CD : CE :: P : P′ : W. But W is to the force with which it tends to descend along the plane CA, as CA to CD; consequently, the weight P is to the same force in the same ratio; because either of these weights in their respective positions would sustain W on CE. Therefore the excess of P above that force (which excess is the power accelerating the motions of P and W) is to P, as CA—CE to CA; or, taking CH = CA, as EH to CA. Now, the motion being uniformly accelerated, we have $s \propto FT^2$, or $T^2 \propto \frac{S}{F}$: consequently, the square of the time in which AC is described by W, will be as AC directly, and as $\frac{EH}{AC}$ inversely; and will be least when $\frac{CA^2}{EH}$ is a *minimum*; that is, when $\frac{CE^2}{EH} + EH + 2CE$, or (because 2CE is invariable) when $\frac{CE^2}{EH} + EH$ is a minimum. Now, as, when the sum of two quantities is given, their product is a *maximum* when they are equal to each other; so it is manifest that when their product is given, their sum must be a *minimum* when they are equal. But the product of $\frac{CE^2}{EH}$ and EH is CE^2, and consequently given; therefore the sum of $\frac{EC^2}{EH}$ and EH is least when those parts are equal; that is, when $EH = CE$, or $CA = 2CE$. So that the length of the plane CA is double the length of that on which the weight W would be kept in equilibrio by P acting along CB.

When CD and CB coincide, the case becomes the same as that considered by Maclaurin, in his *View of Newton's Philosophical Discoveries*, pa. 183, 8vo. edit.

PROPOSITION

PROPOSITION IV.

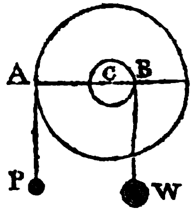
Let the given weight P *descend along* CB, *and by means of the thread* PCW *(running parallel to the planes) draw a weight* W *up the plane* AC: *it is required to find the value of* W, *when its momentum is a maximum, the lengths and positions of the planes being given. (See the preceding fig.)*

The general expression for the vel. in $v = \frac{\text{P} \sin e - \text{W} \sin \text{E}}{\text{P}+\text{W}} gt$, which, by substitut. $\frac{1}{m}\text{L}$ for sin e, and $\frac{1}{m}l$ for sin E, becomes $v = \frac{\frac{1}{m}(\text{PL} - \text{W}l)}{\text{P}+\text{W}} gt$. This mul. into W, gives $\frac{\frac{1}{m}(\text{PWL} - \text{W}^2 l)}{\text{P}+\text{W}} gt$; which, by the prop. is to be a maximum. Or, striking out the constant factors, $\frac{1}{m}$, gt, then is $\frac{\text{PWL} - \text{W}^2 l}{\text{P}+\text{W}} =$ a max. Putting this into fluxions, and reducing, we have $\text{P}^2\text{L} - 2\text{PW}l - \text{W}^2 l = 0$, or $\text{W} = \text{P}\sqrt{(\frac{\text{L}}{l} + 1)} - \text{P}$.

Cor. When the inclinations of the planes are equal, L and l are equal, and $\text{W} = \text{P}\sqrt{2} - \text{P} = \text{P} \times (\sqrt{2} - 1) = \cdot 4142\text{P}$; agreeing with the conclusion of the lever of equal arms, or the extreme case of the wheel and axle, i. e. the pulley.

PROPOSITION V.

Given the radius R *of a wheel, and the radius* r, *of its axle, the weight of both,* w, *and the distance of the centre of gyration from the axis of motion,* ρ; *also a given power* P *acting at the circumference of the wheel; to find the weight* W *raised by a cord folding about the axle, so that its momentum shall be a maximum.*



The force which absolutely impels the point A is P, while W acts in a direction contrary to P, with a force $= \frac{r\text{W}}{\text{R}}$; this therefore subducted from P, leaves $\text{P} - \frac{r\text{W}}{\text{R}} = \frac{\text{RP} - r\text{W}}{\text{R}}$, for the reduced force impelling the point A. And the inertia which resists the communication of motion to the point A will be the same as if the mass $\frac{\rho^2 w + r^2\text{W} + \text{R}^2\text{P}}{\text{R}^2}$ were concentrated in the point A (Mechan. prop. 50). If the former of these be divided by the latter, the quotient $\frac{\text{R}(\text{RP} - r^2\text{W})}{\rho^2 w + r^2\text{W} + \text{R}^2\text{P}}$ is the force accelerating A:

multiplying

multiplying this by $\frac{r}{\text{R}}$, we have $\frac{\text{R}r\text{P}-r^2\text{W}}{\varrho^2 w+r^2\text{W}+\text{R}^2\text{P}}$ for the force which accelerates the weight W in its ascent. Consequently the velocity of W will be $=\frac{\text{R}r\text{P}-r^2\text{W}}{\varrho^2 w+r^2\text{W}+\text{R}^2\text{P}}gt$; which multiplied into W gives $\frac{\text{R}r\text{PW}-r^2\text{W}^2}{\varrho^2 w+r^2\text{W}+\text{R}^2\text{P}}gt$ for the momentum. As this is to be a maximum, its fluxion will $=0$; whence we shall obtain $\text{W}=\frac{\sqrt{(\text{R}^4\text{P}^2+2\text{R}^2\text{P}\varrho^2 w+\varrho^4 w^2+\text{P}w\text{R}r\varrho^2+\text{P}^2\text{R}^3 r)}-\text{R}^2\text{P}-\varrho^2 w}{r^2}$.

Cor. 1. When $\text{R}=r$, as in the case of the single fixed pulley, then $\text{W}=\sqrt{(2\text{P}^2\text{R}^3+2\text{R}\text{P}\varrho^2 w+\frac{\varrho^4}{\text{R}}w^2+\text{P}w\text{R}\varrho^2)}-\frac{\varrho^2}{\text{R}^2}w-\text{P}$.

Cor. 2. When the pulley is a cylinder of uniform matter $\varrho^2=\frac{1}{2}\text{R}^2$, and the express. becomes $\text{W}=\sqrt{[\text{R}^3(2\text{P}^2+\frac{1}{2}\text{P}w+\frac{1}{4}w^2)]}-\frac{1}{2}w-\text{P}$.

Cor. 3. If, in the first general expression for the momentum of W, Q be put $=\text{R}^2\text{P}+\varrho^2 w$, we shall have $\frac{\text{R}r\text{PW}-r^2\text{W}^2}{\text{Q}+r^2\text{W}}$ = a maximum. Which, in fluxions and reduced, gives $\text{W}=\frac{1}{r^2}\sqrt{\text{Q}\cdot(\text{Q}+\text{R}r\text{P})}-\frac{1}{r^2}\text{Q}$.

Cor. 4. If the moving force be destitute of inertia, then will $\text{Q}=\varrho^2 w$, and W, as in the last corollary.

PROPOSITION VI.

Let a given power P *be applied to the circumference of a wheel, its radius* R, *to raise a weight* W *at its axle, whose radius is* r, *it is required to find the ratio of* R *and* r *when* W *is raised with the greatest momentum; the characters* w *and* ϱ *denoting the same as in the last proposition.*

Here we suppose r to vary in the expression for the momentum of W, $\frac{\text{WR}r\text{P}-r^2\text{W}^2}{\varrho^2 w+r^2\text{W}+\text{R}^2\text{P}}gt$. And we suppose, that by the conditions of any specified instance, we can ascertain what quantity of matter q shall make $r^2 q=\varrho^2 w$, which, in fact, may always be done as soon as we can determine ϱ. The expression for the work will then become $\frac{\text{R}r\text{PW}-r^2\text{W}^2}{\text{R}^2\text{P}+r^2(q+\text{W})}gt$. The fluxion of which being made $=0$, gives, after a little reduction, $r=\frac{\text{R}\sqrt{[\text{P}^2\text{W}^2+\text{P}^3(q+\text{W})]}-\text{PW}}{\text{P}(q+\text{W})}$.

Cor. When the inertia of the machine is evanescent, with respect to that of P + W, then is $r=\text{R}\sqrt{(1+\frac{\text{P}}{\text{W}})}-1$.

PROPOSITION VII.

In any machine whose motion accelerates, the weight will be moved with the greatest velocity, when the velocity of the power is to that of the weight, as $1 + P\sqrt{(1+\frac{P}{W})}$ *to* 1; *the inertia of the machine being disregarded.*

For any such machine may be considered as reduced to a lever, or to a wheel and axle whose radii are R and r: in which the velocity of the weight $\frac{RrP - r^2W}{R^2P + r^2W}gt$ (prop. 1) is to be a maximum, r being considered as variable. Hence then, following the usual rules, we find $PR = r(W + \sqrt{W^2 + PW})$. From which, since the velocities of the power and weight are respectively as R and r, the ratio in the proposition immediately flows.

Cor. When the weight moved is equal to the power, then is $R : r :: 1 + \sqrt{2} : 1 :: 2{\cdot}4142 : 1$ *nearly.*

PROPOSITION VIII.

If in any machine whose motion accelerates, the descent of one weight causes another to ascend, and the descending weight be given, the operation being supposed continually repeated, the effect will be greatest in a given time when the ascending weight is to the descending weight, as 1 *to* 1·618, *in the case of equal heights; and in other cases, when it is to the exact counterpoise in a ratio which is always between* 1 *to* 1½ *and* 1 *to* 2.

Let the space descended be 1, that ascended s; the descending weight 1, the ascending weight $\frac{1}{w}$: then would the equilibrium require $w = s$; and $1 - \frac{s}{w}$, will be the force acting on 1. Now the mass $\frac{1}{w}$, reduced to the point at which the mass 1 acts, will be $= \frac{1}{w}s^2 = \frac{s^2}{w}$; consequently the whole mass moved is equivalent to $1 + \frac{s^2}{w}$, and the relative force is $(1 - \frac{s}{w}) \div (1 + \frac{s^2}{w}) = \frac{w-s}{w+s^2}$. But, the space being given, the time is as the root of the accelerating force inversely, that is, as $\sqrt{\frac{w+s^2}{w-s}}$: and the whole effect in a given time, being directly as the weight raised, and inversely as the time of ascent, will be as $\frac{1}{w}\sqrt{\frac{w-s}{w+s^2}}$; which must be a maximum.

maximum. Consequently its square $\frac{w-s}{w^3+s^2w^2}$ must be a max. likewise. This latter expression, in fluxions and reduced, gives $w = \frac{s}{4}[\sqrt{(s^2+10s+9)}-a+3]$.

Here if $s = 1$, $w = \frac{1+\sqrt{5}}{2}$: but if s be diminished without limit, $w = \frac{1}{2}s$; if it be augmented without limit, then will $\sqrt{(s^2+10s+9)}$ approach indefinitely near to $s + 5$, and consequently $w = 2s$. Whence the truth of the proposition is manifest.

PROPOSITION IX.

Let φ denote the absolute effort of any moving force, when it has no velocity; and suppose it not capable of any effort when the velocity is W; *let* F *be the effort answering to the velocity* V; *then, if the force be uniform,* F *will be =* $\varphi(1-\frac{V}{W})^2$.

For it is the difference between the velocities W and V which is efficient, and the action, being constant, will vary as the square of the efficient velocity. Hence we shall have this analogy, $\varphi : F :: (W-0)^2 : (W-V)^2$: consequently, $F = \varphi(\frac{W-V}{W}{}^2) = \varphi(1-\frac{V}{W})^2$.

Though the pressure of an animal is not actually uniform during the whole time of its action, yet it is nearly so: so that in general we may adopt this hypothesis in order to approximate to the true nature of animal action. On which supposition the preceding prop. as well as the remaining one, in this chapter will apply to animal exertion.

Cor. Retaining the same notation, we have $W = \frac{V\sqrt{\varphi}}{\sqrt{\varphi}-\sqrt{F}}$. This, applied to the motion of animals, gives this theorem: *The utmost velocity with which an animal not impeded can move, is to the velocity with which it moves when impeded by a given resistance, as the square root of its absolute force, to the difference of the square roots of its absolute and efficient forces.*

PROPOSITION X.

To investigate expressions by means of which the maximum effect, in machines whose motion is uniform, may be determined.

I. It follows, from the observations made in art. 1 and the definitions in this chapter, that when a machine, whether simple or compound, is put into motion, the velocities of the impelled

impelled and working points, are inversely as the forces which are in equilibrio, when applied to those points in the direction of their motion. Consequently, if f denote the resistance when reduced to the working point, and v its velocity; while F and V denote the force acting at the impelled point, and its velocity; we shall have $FV = fv$, or introducing t the time, $FVt = fvt$. Hence, *in all working machines which have acquired a uniform motion, the performance of the machine is equal to the momentum of impulse.*

II. Let F be the effort of a force on the impelled point of a machine when it moves with the velocity V, the velocity being W when $F = 0$, and let the relative velocity $W - V = u$. Then since (prop. IX) $F = \varphi\left(\frac{W-V}{W}\right)^2$, the momentum of impulse FV will become $V\varphi\left(\frac{u}{W}\right)^2 = \varphi \cdot \frac{u^2}{W^2}(W - u)$; because $V = W - u$ Making this expression for FV a maximum, or, suppressing the constant quantities, and making $u^2(W-u)$ a max. or its flux. $= 0$, when u is variable, we find $2W = 3u$, or $u = \frac{2}{3}W$. Whence $V = W - u = W - \frac{2}{3}W = \frac{1}{3}W$.

Consequently, *when the ratio of* V *to* v *is given, by the construction of the machine, and the resistance is susceptible of variation, we must load the machine more or less till the velocity of the impelled point, is one-third of the greatest velocity of the force; then will the work done be a maximum.*

Or, the work done by an animal is greatest, when the velocity with which it moves, is one-third of the greatest velocity with which it is capable of moving when not impeded.

III. Since $F = \varphi\frac{u^2}{W^2} = \varphi\left(\frac{\frac{4}{9}W^2}{W^2}\right) = \frac{4}{9}\varphi$, in the case of the maximum, we have $FV = \frac{4}{9}\varphi V = \frac{4}{9}\varphi \cdot \frac{1}{3}W = \frac{4}{27}\varphi W$, for the momentum of impulse, or for the work done, when the machine is in its best state. *Consequently, when the resistance is a given quantity, we must make* $V : v :: 9f : 4\varphi$; *and this structure of the machine will give the maximum effect* $= \frac{4}{27}\varphi W$.

IV. If we enquire the greatest effect on the supposition that φ only is variable, we must make it infinite in the above expression for the work done, which would then become WF, or $W\frac{V}{v}f$ or $W\frac{V}{v}ft$, including the time in the formula. Hence we see, *that the sum of the agents employed to move a machine may be infinite, while the effect is finite:* for the variations of φ, which are proportional to this sum, do not influence the above expression for the effect.

Scholium.

Scholium.

The propositions now delivered contain the most material principles in the theory of machines. The manner of applying several of them is very obvious: the application of some, being less manifest, may be briefly illustrated, and the chapter concluded with two or three observations.

The last theorem may be applied to the action of men and of horses, with more accuracy than might at first be supposed. Observations have been made on men and horses drawing a lighter along a canal, and working several days together. The force exerted was measured by the curvature and weight of the track-rope, and afterwards by a spring steelyard. The product of the force thus ascertained, into the velocity per hour, was considered as the momentum. In this way the action of *men* was found to be very nearly as $(w-v)^2$: the action of horses loaded so as not to be able to trot was nearly as $(w-v)^{1\cdot7}$, or as $(w-v)^{\frac{9}{5}}$. Hence the hypothesis we have adopted may in many cases be safely assumed.

According to the best observations, the force of a man at rest is on the average about 70 pounds; and the utmost velocity with which he can walk is about 6 feet per second, taken at a medium. Hence, in our theorems, $\varphi = 70$, and $w = 6$. Consequently $\mathfrak{r} = \frac{4}{9}\varphi = 31\frac{1}{9}$ lbs. the greatest force a man can exert when in motion: and he will then move at the rate of $\frac{1}{3}w$, or 2 feet per second, or rather less than a mile and a half per hour.

The strength of a horse is generally reckoned about 6 times that of a man; that is, nearly 420 lbs. at a dead pull. His utmost walking velocity is about 10 feet per second. Therefore his maximum action will be $\frac{4}{9}$ of $420 = 186\frac{2}{3}$ lbs. and he will then move at the rate of $\frac{1}{3}$ of 10, or $3\frac{1}{3}$ feet, per second, or nearly $2\frac{1}{3}$ miles per hour. In both these instances we suppose the force to be exerted in drawing a weight along a horizontal plane; or by raising a weight by a cord running over a pulley, which makes its direction horizontal.

2. The theorems just given may serve to show, in what points of view machines ought to be considered, by those who would labour beneficially for their improvement.

The first object of the utility of machines consists in furnishing the means of *giving to the moving force the most commodious direction;* and, when it can be done, of causing its action to be applied immediately to the body to be moved. These can rarely be united: but the former can be accomplished in most instances; of which the use of the simple lever,

lever, pulley, and wheel and axle, furnish many examples. The second object gained by the use of machines, is *an accommodation of the velocity of the work to be performed, to the velocity with which alone a natural power can act.* Thus, whenever the natural power acts with a certain velocity which cannot be changed, and the work must be performed with a greater velocity, a machine is interposed moveable round a fixed support, and the distances of the impelled and working points are taken in the proportion of the two given velocities.

But the essential advantage of machines, that, in fact, which properly appertains to the *theory* of mechanics, consists in augmenting, or rather in modifying, the energy of the moving power, in such manner that it may produce effects of which it would have been otherwise incapable. Thus a man might carry up a flight of steps 20 pieces of stone, each weighing 30 pounds (one by one) in as small a time as he could (with the same labour) raise them all together by a piece of machinery, that would have the velocities of the impelled and working points as 20 to 1; and, in this case, the instrument would furnish no real advantage, except that of saving his steps. But if a large block of 20 times 30, or 600lbs. weight were to be raised to the same height, it would far surpass the utmost efforts of the man, without the intervention of some such contrivance.

The same purpose may be illustrated somewhat differently; confining the attention all along to machines whose motion is uniform. The product fv represents, during the unit of time, the effect which results from the motion of the resistance; this motion being produced in any manner whatever. If it be produced by applying the moving force immediately to the resistance, it is necessary not only that the products FV and fv should be equal; but that at the same time $F=f$, and $V=v$: if, therefore, as most frequently happens, f be greater than F, it will be absolutely impossible to put the resistance in motion by applying the moving force immediately to it. Now machines furnish the means of disposing the product FV in such a manner that it may always be equal to fv, however much the factors of FV may differ from the analogous factors in fv; and, consequently, of putting the system in motion, whatever is the excess of f over F.

Or, generally, as M. Prony remarks (Archi. Hydraul. art. 504), machines enable us to dispose the factors of FVt in such a manner, that while that product continues the same, its factors may have to each other any ratio we desire. If, for instance, time be precious, the effect must be produced in a very short

short time, and yet we should have at command a force capable of little velocity but of great effort, a machine must be found to supply the velocity necessary for the intensity of the force: if, on the contrary, the mechanist has only a weak power at his disposition, but capable of a great velocity, a machine must be adopted that will compensate, by the velocity the agent can communicate to it, for the force wanted: lastly, if the agent is capable neither of great effort, nor of great velocity, a convenient machine may still enable him to accomplish the effect desired, and make the product Fvt of force, velocity, and time, as great as is requisite. Thus, to give another example: Suppose that a man, exerting his strength immediately on a mass of 25 lbs, can raise it vertically with a velocity of 4 feet per second; the same man acting on a mass of 1000 lbs, cannot give it any vertical motion though he exerts his utmost strength, unless he has recourse to some machine. Now he is capable of producing an effect equal to $25 \times 4 \times t$: the letter t being introduced because, if the labour is continued, the value of t will not be indefinite, but comprised within assignable limits. Thus we have $25 \times 4 \times t = 1000 \times v \times t$; and consequently $v = \frac{1}{10}$ of a foot. This man may therefore with a machine, as a lever, or axis in peritrochio, cause a mass of 1000 lbs to rise $\frac{1}{10}$ of a foot, in the same time that he could raise 25 lbs 4 feet without a machine; or he may raise the greater weight as far as the less, by employing 40 times as much time.

From what has been said on the extent of the effects which may be attained by machines, it will be seen that, so long as a moving force exercises a determinate effort, with a velocity also determinate, or so long as the product of these is constant, the effect of the machine will remain the same: thus, under this point of view, supposing the preponderance of the effort of the moving power, and abstracting from inertia and friction of materials, the convenience of application, &c, all machines are equally perfect. But, from what has been shown, (props. 9, 10) a moving force may, by diminishing its velocity, augment its effort, and reciprocally. There is therefore a certain effort of the moving force, such that its product by the velocity which comports to that effort, is the greatest possible. Admitting the truth of the law assumed in the propositions just referred to, we have, when the effect is a *maximum*, $V = \frac{1}{3}W$, or $F = \frac{4}{9}\varphi$; and these two values obtaining together, their product $\frac{4}{27}\varphi W$ expresses the value of the greatest effect with respect to the unit of time. In practice it will always be adviseable to approach as nearly to these values as circumstances will admit; for it cannot be expected

expected that they can always be exactly attained. But a small variation will not be of much consequence: for, by a well-known property of those quantities which admit of a proper maximum and minimum, a value assumed at a moderate distance from either of these extremes will produce no sensible change in the effect.

If the relation of F to V followed any other law than that which we have assumed, we should find from the expression of *that law* values of F, V, &c, different from the preceding. The general method however would be nearly the same.

With respect to practice, the grand object in all cases should be to procure a *uniform motion*, because it is that from which (*cæteris paribus*) the greatest effect always results. Every irregularity in the motion wastes some of the impelling power; and it is the greatest only of the varying velocities which is equal to that which the machine would acquire if it moved uniformly throughout: for, while the motion accelerates, the impelling force is greater than what balances the resistance at that time opposed to it, and the velocity is less than what the machine would acquire if moving uniformly; and when the machine attains its greatest velocity, it attains it because the power is not then acting against the whole resistance. In both these situations therefore, the performance of the machine is less than if the power and resistance were exactly balanced; in which case it would move uniformly (art. 1.) Besides this, when the motion of a machine, and particularly a very ponderous one, is irregular, there are continual repetitions of strains and jolts which soon derange and ultimately destroy the whole structure. Every attention should therefore be paid to the removal of all causes of irregularity.

CHAPTER XII.

PRESSURE OF EARTH AND FLUIDS AGAINST WALLS AND FORTIFICATIONS, THEORY OF MAGAZINES, &C.

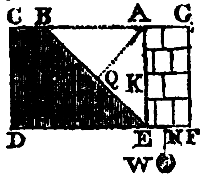
PROBLEM I.

To determine the Pressure of Earth against Walls.

When new-made earth, such as is used in forming ramparts, &c, is not supported by a wall as a facing, or by counterforts and land-ties, &c, but left to the action of its weight and the weather; the particles loosen and separate from each other,

other, and form a sloping surface, nearly regular; which plane surface is called the natural slope of the earth; and is supposed to have always the same inclination or deviation from the perpendicular, in the same kind of soil. In common earth or mould, being a mixture of all sorts thrown together, the natural slope is commonly at about half a right angle, or 45 degrees; but clay and stiff loam stands a greater angle above the horizon, while sand and light mould will only stand at a much less angle. The engineer or builder must therefore adapt his calculations accordingly.

Now, we have already given, (at prop. 45 Statics) the general theory and determination of the force with which the triangle of earth (which would slip down if not supported) presses against the wall on the most unexceptional principles, acting perpendicularly against AE at K, or $\frac{1}{3}$ of the altitude AE above the foundation at E; the expression for which force was there found to be $\frac{AE^3 \cdot AB^2}{6BE^2}m$; where m denotes the specific gravity of the earth of the triangle ABE.—It may be remarked that this was deduced from using the area only of the profile, or transverse triangular section ABE, instead of the prismatic solid of any given length, having that triangle for its base. And the same thing is done in determining the power of the wall to support the earth, viz, using only its profile or transverse section in the same plane or direction as the triangle ABE. This it is evident will produce the same result as the solids themselves, since, being both of the same given length, these have the same ratio as their transverse sections.



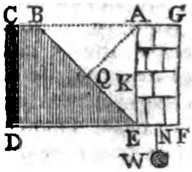
In addition to this determination we may here further observe, that this pressure ought to be diminished in proportion to the cohesion of the matter in sliding down the inclined plane BE. Now it has been found by experiments, that a body requires about one-third of its weight to move it along a plane surface. The above expression must therefore be reduced in the ratio of 3 to 2; by which means it becomes $\frac{AE^3 \cdot AB^2}{9BE^2}m$ for the true practical efficacious pressure of the earth against the wall.

Since $\frac{AB}{BE}$, which occurs in this expression of the force of the earth, is equal to the sine of the $\angle$ AEB to the radius 1, put the sine of that $\angle$ E $= e$; also put $a =$ AE the altitude of the triangle; then the above expression of the force, viz,

$\frac{\text{AE}^3 \cdot \text{AB}^2}{9\text{BK}^2} m$, becomes $\frac{1}{9}a^3e^2m$, for the perpendicular pressure of the earth against the wall. And if that angle be 45°, as is usually the case in common earth, then is $e^2 = \frac{1}{2}$, and the pressure becomes $\frac{1}{18}a^3m$.

PROBLEM II.

To determine the Thickness of Wall to support the Earth.



In the first place suppose the section of the wall to be a rectangle, or equally thick at top and bottom, and of the same height as the rampart of earth, like AEFG in the annexed figure. Conceive the weight W, proportional to the area GE, to be appended to the base directly below the centre of gravity of the figure. Now the pressure of the earth determined in the first problem, being in a direction parallel to AG, to cause the wall to overset and turn back about the point F, the effort of the wall to oppose that effect, will be the weight W drawn into FN the length of the lever by which it acts, that is W × FN, or AEFG × FN in general, whatever be the figure of the wall.

But now in case of the rectangular figure, the area GE=AE ×EF$=ax$, putting a = AE the altitude as before, and x = EF the required thickness; also in this case FN = $\frac{1}{2}$EF = $\frac{1}{2}x$, the centre of gravity being in the middle of the rectangle. Hence then $ax \times \frac{1}{2}x = \frac{1}{2}ax^2$, or rather $\frac{1}{2}ax^2n$ is the effort of the wall to prevent its being overturned, n denoting the specific gravity of the wall.

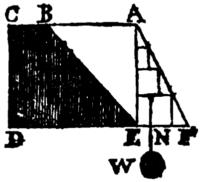
Now to make this effort a due balance to the pressure of the earth, we put the two opposing forces equal, that is $\frac{1}{2}ax^2n = \frac{1}{9}a^3e^2m$, or $\frac{1}{2}x^2n = \frac{1}{9}a^2e^2m$, an equation which gives $x = \frac{1}{3}ae\sqrt{\frac{2m}{n}}$, for the requisite thickness of the wall, just to sustain it in equilibrio.

Corol. 1. The factor ae, in this expression, is = the line AQ drawn perp. to the slope of earth BE: theref. the breadth x becomes $= \frac{1}{3}$AQ $\sqrt{\frac{2m}{n}}$, which conseq. is directly proportional to the perp. AQ.—When the angle at B is = 45°, or half a right angle, as is commonly the case, its sine e is $= \sqrt{\frac{1}{2}}$, and the breadth of the wall $x = \frac{1}{3}a\sqrt{\frac{m}{n}}$. Further, when the wall is of brick, its specific gravity is nearly the same as the

the earth, or $m = n$, and then its thickness $x = \frac{1}{3}a$, or one-third of its height.—But when the wall is of stone, of the specific gravity $2\frac{1}{2}$, that of earth being nearly 2, that is, $m = 2$, and $n = 2\frac{1}{2}$; then $\sqrt{\frac{m}{n}} = \sqrt{\frac{4}{5}} = \cdot 895$, $\frac{1}{3}$ of which is $\cdot 298$, and the breadth $x = \cdot 298a = \frac{3}{10}a$ nearly. That is, the thickness of the stone wall must be $\frac{3}{10}$ of its height.

PROBLEM III.

To determine the Thickness of the Wall at the Bottom, when its Section is a Triangle, or coming to an Edge at Top.



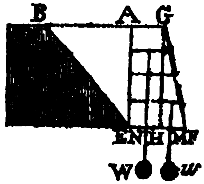
In this case, the area of the wall AEF is only half of what it was before, or only $\frac{1}{2}$AE $\times$ EF $= \frac{1}{2}ax$, and the weight $w = \frac{1}{2}axn$. But now, the centre of gravity is at only $\frac{1}{3}$ of FE from the line AE, or FN $= \frac{2}{3}$ FE $= \frac{2}{3}x$. Consequently FN $\times$ W $= \frac{2}{3}x \times \frac{1}{2}axn = \frac{1}{3}ax^2n$. This, as before, being put $=$ the pressure of the earth, gives the equation $\frac{1}{3}ax^2n = \frac{1}{9}a^3e^2m$ or, $x^2n = \frac{1}{3}a^2e^2m$, and the root x, or thickness EF $= ae\sqrt{\frac{m}{3n}} = a\sqrt{\frac{m}{6n}}$ for the slope of 45°.

Now when the wall is of brick, or $m = n$ nearly, this becomes $x = a\sqrt{\frac{1}{6}} = \cdot 408a = \frac{2}{5}a$, or $\frac{4}{10}$ of the height nearly.

But when the wall is of stone, or m to n as 2 to $2\frac{1}{2}$, then $\sqrt{\frac{m}{n}} = \sqrt{\frac{4}{5}}$, and the thickness x or $a\sqrt{\frac{m}{6n}} = a\sqrt{\frac{2}{15}} = \cdot 365a$ $= \frac{3}{8}a$ nearly, or nearly $\frac{3}{8}$ of the height.

PROBLEM IV.

To determine the Thickness of the Wall at the Top, when the Face is not Perpendicular, but Inclined as the Front of a Fortification Wall usually is.



Here GF represents the outer face of a fort, AEFG the profile of the wall, having AG the thickness at top, and EF that at the bottom. Draw GH perp. to EF; and conceive the two weights W, w, to be suspended from the centres of gravity of the rectangle AH and the triangle GHF, and to be proportional to their areas respectively. Then the two momenta of the weights W, w, acting by the levers FN, FM, must be made equal to the pressure of the earth in the direction perp. to AE.

Now

Now put the required thickness AG or EH $= x$, and the altitude AE or GH $= a$ as before. And because in such cases the slope of the wall is usually made equal to $\frac{1}{5}$ of its altitude, that is FH $= \frac{1}{5}$ AE or $\frac{1}{5}a$, the lever FM will be $\frac{2}{3}$ of $\frac{1}{5}a = \frac{2}{15}a$, and the lever FN $=$ FH $+ \frac{1}{2}$EH $= \frac{1}{5}a + \frac{1}{2}x$. But the area of GHF $=$ GH $\times \frac{1}{2}$HF $= a \times \frac{1}{10}a = \frac{1}{10}a^2 = w$, and the area AH $=$ AE $\times$ AG $= ax =$ W; these two drawn into the respective levers FM, FN, give the two momenta, $\frac{2}{15}aw = \frac{2}{15}a \times \frac{1}{10}a^2 = \frac{1}{75}a^3$, and $(\frac{1}{5}a + \frac{1}{2}x) \times ax = \frac{1}{5}a^2x + \frac{1}{2}ax^2$; theref. the sum of the two, $(\frac{1}{2}ax^2 + \frac{1}{5}a^2x + \frac{1}{75}a^3)n$ must be $= \frac{1}{18}a^3m$, or dividing by $\frac{1}{2}an$, $x^2 + \frac{2}{5}ax + \frac{2}{75}a^2 = \frac{1}{9}a^2 \times \frac{m}{n}$; now adding $\frac{1}{75}a^2$ to both sides to complete the square, the equation becomes $x^2 + \frac{2}{5}ax + \frac{1}{25}a^2 = \frac{1}{9}a^2 . \frac{m}{n} + \frac{1}{25}a^2$, the root of which is $x + \frac{1}{5}a = a\sqrt{(\frac{1}{25} + \frac{m}{9n})}$, and hence $x = a\sqrt{(\frac{1}{25} + \frac{m}{9n})} - \frac{1}{5}a$. And the base EF $= a\sqrt{(\frac{1}{25} + \frac{m}{9n})}$.

Now, for a brick wall, $m = n$ nearly, and then the breadth $x = a\sqrt{(\frac{1}{25} + \frac{1}{9})} - \frac{1}{5}a = \frac{1}{15}a\sqrt{34} - \frac{1}{5}a = \cdot 189a$, or almost $\frac{1}{5}a$ in brick walls.—But in stone walls, $\frac{m}{n} = \frac{4}{5}$, and $x = a\sqrt{(\frac{1}{25} + \frac{4}{45})} - \frac{1}{5}a = \frac{1}{15}a\sqrt{29} - \frac{1}{5}a = \cdot 159a = \frac{4}{25}a$ nearly, for the thickness AG at top, in stone walls.

In the same manner we may proceed when the slope is supposed to be any other part of the altitude, instead of $\frac{1}{5}$ as used above. Or a general solution might be given, by assuming the thickness $= \frac{1}{c}$ part of the altitude.

REMARK.

Thus then we have given all the calculations that may be necessary in determining the thickness of a wall, proper to support the rampart or body of earth, in any work. If it should be objected, that our determination gives only such a thickness of wall, as makes it an exact mechanical balance to the pressure or push of the earth, instead of giving the former a decided preponderance over the latter, as a security against any failure or accidents. To this we answer, that what has been done is sufficient to insure stability, for the following reasons and circumstances. First, it is usual to build several counterforts of masonry, behind and against the wall, at certain distances or intervals from one another; which contribute very much to strengthen the wall, and to resist the pressure of the rampart. 2dly. We have omitted to include the effect of the parapet raised above the wall; which must add somewhat, by its weight, to the force or resistance of the wall.

wall. It is true we could have brought these two auxiliaries to exact calculation, as easily as we have done for the wall itself: but we have thought it as well to leave these two appendages, thrown in as indeterminate additions, above the exact balance of the wall as before determined, to give it an assured stability. Besides these advantages in the wall itself, certain contrivances are also usually employed to diminish the pressure of the earth against it: such as land-ties and branches, laid in the earth, to diminish its force and push against the wall. For all these reasons then, we think the practice of making the wall of the thickness as assigned by our theory, may be safely depended on, and profitably adopted; as the additional circumstances, just mentioned, will sufficiently insure stability; and its expense will be less than is incurred by any former theory.

PROBLEM V.

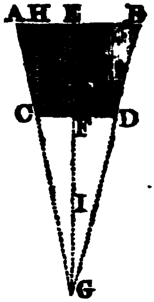
To determine the Quantity of Pressure sustained by a Dam or Sluice, made to pen up a Body of Water.

By art. 313 Hydrostatics, (in this volume) the pressure of a fluid against any upright surface, as the gate of a sluice or canal, is equal to half the weight of a column of the fluid, whose base is equal to the surface pressed, and its altitude the same as that of the surface. Or, by art. 314 of the same, the pressure is equal to the weight of a column of the fluid, whose base is equal to the surface pressed, and its altitude equal to the depth of the centre of gravity below the top or surface of the water; which comes to the same thing as the former article, when the surface pressed is a rectangle, because its centre of gravity is at half the depth.

Ex. 1. Suppose the dam or sluice be a rectangle, whose length, or breadth of the canal, is 20 feet, and the depth of water 6 feet. Here $20 \times 6 = 120$ feet, is the area of the surface pressed; and the depth of the centre of gravity being 3 feet, viz, at the middle of the rectangle; therefore $120 \times 3 = 360$ cubic feet is the content of the column of water. But each cubic foot of water weighs 1000 ounces, or $62\frac{1}{2}$ pounds; therefore $360 \times 1000 = 360000$ ounces, or 22500 pounds, or 10 tons and 100 lb, is the weight of the column of water, or the quantity of pressure on the gate or dam.

Ex. 2. Suppose the breadth of a canal at the top, or surface of the water, to be 24 feet, but at the bottom only 16 feet, the depth of water being 6 feet, as in the last example: required the pressure on a gate which, standing across the canal, dams the water up?

Here



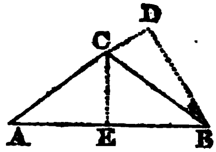
Here the gate is in form of a trapezoid, having the two parallel sides AB, CD, viz, AB = 24, and CD = 16, and depth 6 feet. Now, by mensuration, problem 3, volume 1, $\frac{1}{2}(AB + CD) \times 6, = 20 \times 6 = 120$ the area of the sluice, the same as before in the 1st example: but the centre of gravity cannot be so low down as before, because the figure is wider above and narrower below, the whole depth being the same.

Now, to determine the centre of gravity K of the trapezoid AD, produce the two sides AC, BD, till they meet in G; also draw GKE and CH perp. to AB: then AH : CH :: AE : GE, that is, $4 : 6 :: 12 : 18 = GE$; and EF being = 6, theref. FG = 12. Now, by Statics art. 229, $EF = 6 = \frac{1}{3}EG$ gives F the centre of gravity of the triangle ABG, and $FI = 4 = \frac{1}{3}FG$ gives I the centre of gravity of the triangle CDG. Then assuming K to denote the centre of AD, it will be, by art. 212 this vol. as the trap. AD : $\triangle$ CDG :: IF : FK, or $\triangle$ ABC $-$ $\triangle$ CDG : $\triangle$ CDG :: IF : FK, or by theor. 88 Geom. $GE^2 - GF^2 : GF^2 :: IF : FK$, that is $18^2 - 12^2$ to 12^2 or $3^2 - 2^2$ to 2^2 or $5 : 4 :: IF = 4 : \frac{16}{5} = 3\frac{1}{5} = FK$; and hence $EK = 6 - 3\frac{1}{5} = 2\frac{4}{5} = \frac{14}{5}$ is the distance of the centre K below the surface of the water. This drawn into 120 the area of the dam-gate, gives 336 cubic feet of water = the pressure, = 336000 ounces = 21000 pounds = 9 tons 80 lb, the quantity of pressure against the gate, as required, being a 15th part less than in the first case.

Ex. 3. Find the quantity of pressure against a dam or sluice, across a canal, which is 20 feet wide at top, 14 at bottom, and 8 feet depth of water?

PROBLEM VI.

To determine the Strongest Angle of Position of a Pair of Gates for the Lock on a Canal or River.



Let AC, BC be the two gates, meeting in the angle C, projecting out against the pressure of the water, AB being the breadth of the canal or river. Now the pressure of the water on a gate AC, is as the quantity, or as the extent or length of it, AC. And the mechanical effect of that pressure, is as the length of lever to the middle of AC, or as AC itself. On both these accounts then the pressure is as AC^2.

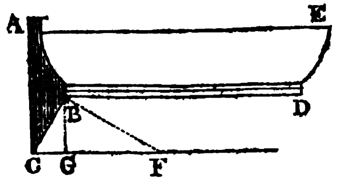
AC². Therefore the resistance or the strength of the gate must be as the resiprocal of this AC².

Now produce AC to meet BD, perp. to it, in D; and draw CE to bisect AB perpendicularly in E; then, by similar triangles, as AC : AE : : AB : AD; where, AE and AB being given lengths, AD is reciprocally as AC, or AD² reciprocally as AC²; that is, AD² is as the resistance of the gate AC. But the resistance of AC is increased by the pressure of the other gate in the direction BC. Now the force in BC is resolved into the two BD, DC; the latter of which, DC, being parallel to AC, has no effect upon it; but the former, BD, acts perpendicularly on it. Therefore the whole effective strength or resistance of the gate is as the product $AD^2 \times BD$.

If now there be put $AB = a$, and $BD = x$, then $AD^2 = AB^2 - BD^2 = a^2 - x^2$; conseq. $AD^2 \times BD = (a^2 - x^2) \times x = a^2x - x^3$ for the resistance of either gate. And, if we would have this to be the greatest, or the resistance a maximum, its fluxion must vanish, or be equal to nothing: that is, $a^2\dot{x} - 3x^2\dot{x} = 0$; hence $a^2 = 3x^2$, and $x = a\sqrt{\frac{1}{3}} = \frac{1}{3}a\sqrt{3} = \cdot 57735a$, the natural sine of 35° 16′: that is, the strongest position for the lock gates, is when they make the angle A or B = 35° 16′, or the complemental angle ACE or BCE = 54° 44′, or the whole salient angle ACB = 109° 28′.

Scholium.

Allied to this problem, are several other cases in mechanics, such as, the action of the water on the rudder of a ship, in sailing, to turn the ship about, to alter her course; and the action of the wind on a ship's sails, to impel her forward; also the action of water on the wheels of water-mills, and of the air on the sails of wind-mills, to cause them to turn round.



Thus, for instance, let ABC be the rudder of a ship ABDE, sailing in the direction BD, the rudder placed in the oblique position BC, and consequently striking the water in the direction CF, parallel to BD. Draw BF perp. to BC, and BG perp. to CF. Then the sine of the angle of incidence, of the direction of the stroke of the rudder against the water, will be BF, to the radius CF; therefore the force of the water against the rudder will be as BF², by art. 3, Mot. of bod. in Flui. this vol. But the force BF resolves into the two BG, GF, of which the latter is parallel to the ship's motion, and therefore has

has no effect to change it; but the former BG, being perp. to the ship's motion, is the only part of the force to turn the ship about and change her course. But BF : BG : : CF : CB, therefore CF : CB : : BF^2 : $\frac{\text{BC . BF}^2}{\text{CF}}$ the force upon the rudder to turn the ship about.

Now put $a = \text{CF}$, $x = \text{BC}$; then $\text{BF}^2 = a^2 - x^2$, and the force $\frac{\text{BC . BF}^2}{\text{CF}} = \frac{x(a^2-x^2)}{a} = \frac{a^2x-x^3}{a}$; and, to have this a maximum, its flux. must be made to vanish, that is, $a^2\dot{x} - 3x^2\dot{x} = 0$; and hence $x = a\sqrt{\frac{1}{3}} = \text{BC} =$ the natural sine of 35° 16′ = angle F; therefore the complemental angle C = 54° 44′ as before, for the obliquity of the rudder, when it is most efficacious.

The case will be also the same with respect to the wind acting on the sails of a wind-mill, or of a ship, viz, that the sails must be set so as to make an angle of 54° 44′ with the direction of the wind; at least at the beginning of the motion, or nearly so when the velocity of the sail is but small in comparison with that of the wind; but when the former is pretty considerable in respect of the latter, then the angle ought to be proportionally greater, to have the best effect, as shown in Maclaurin's Fluxions, pa. 734, &c.

A consideration somewhat related to the same also, is the greatest effect produced on a mill-wheel, by a stream of water striking upon its sails or float-boards. The proper way in this case seems to be, to consider the whole of the water as acting on the wheel, but striking it only with the relative velocity, or the velocity with which the water overtakes and strikes upon the wheel in motion, or the difference between the velocities of the wheel and the stream. This then is the power or force of the water; which multiplied by the velocity of the wheel, the product of the two, viz, of the relative velocity and the absolute velocity of the wheel, that is $(\text{V}-v)\,v = \text{V}v - v^2$, will be the effect of the wheel; where V denotes the given velocity of the water, and v the required velocity of the wheel. Now, to make the effect $\text{V}v - v^2$ a maximum, or the greatest, its fluxion must vanish, that is $\text{V}\dot{v} - 2v\dot{v} = 0$, hence $v = \frac{1}{2}\text{V}$; or the velocity of the wheel will be equal to half the velocity of the stream, when the effect is the greatest; and this agrees best with experiments.

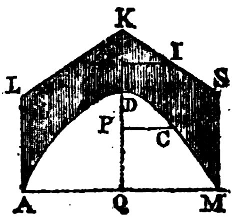
A former way of resolving this problem was, to consider the water as striking the wheel with a force as the square of the relative velocity, and this multiplied by the velocity of the wheel, to give the effect; that is, $(\text{V}-v)^2v =$ the effect. Now the flux. of this product is $(\text{V}-v)^2\dot{v} - (\text{V}-v) \times 2v\dot{v} = 0$; hence

hence $V - v = 2v$, or $V = 3v$, and $v = \frac{1}{3}V$, or the velocity of the wheel equal only to $\frac{1}{3}$ of the velocity of the water.

PROBLEM VII.

To determine the Form and Dimensions of Gunpowder Magazines.

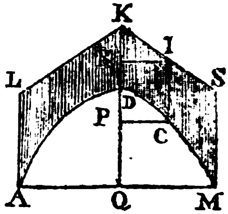
In the practice of engineering, with respect to the erection of powder magazines, the exterior shape is usually made like the roof of a house, having two sloping sides, forming two inclined planes, to throw off the rain, and meeting in an angle or ridge at the top; while the interior represents a vault, more or less extended, as the occasion may require; and the shape, or transverse section, in the form of some arch, both for strength and commodious room, for placing the powder barrels. It has been usual to make this interior curve a semicircle. But, against this shape, for such a purpose, I must enter my decided protest; as it is an arch the farthest of any from being in equilibrium in itself, and the weakest of any, by being unavoidably much thinner in one part than in others. Besides it is constantly found, that after the centering of semicircular arches is struck, and removed, they settle at the crown, and rise up at the flanks, even with a straight horizontal form at top, and still much more so in powder magazines with a sloping roof; which effects are exactly what might be expected from a contemplation of the true theory of arches. Now this shrinking of the arches must be attended with other additional bad effects, by breaking the texture of the cement, after it has been in some degree dried, and also by opening the joints of the voussoirs at one end. Instead of the circular arch therefore, we shall in this place give an investigation, founded on the true principles of equilibrium, of the only just form of the interior, which is properly adapted to the usual sloped roof.



For this purpose, put a = DK the thickness of the arch at the top, x = any absciss DP of the required arch ADCM, u = KR the corresponding absciss of the given exterior line KI and y = PC = RI their equal ordinates. Then by the principles of arches, in my tracts on that subject, it is found that CI or $w = a + x - u = Q \times \frac{\dot{y}\ddot{x} - \dot{x}\ddot{y}}{\dot{y}^3}$, or $= Q \times \frac{\ddot{x}}{\dot{y}^2}$, supposing $\dot{y}$ a constant quantity, and where Q is some certain quantity to be determined hereafter. But KR or u is $= ty$, if t be put to denote

 the

the tangent of the given angle of elevation KIR, to radius 1; and then the equation is $w = a + x - ty = \frac{Q\ddot{x}}{\dot{y}^2}$.



Now, the fluxion of the equation $w = a + x - ty$, is $\dot{w} = \dot{x} - t\dot{y}$, and the 2d fluxion is $\ddot{w} = \ddot{x}$; therefore the foregoing general equation becomes $w = \frac{Q\ddot{w}}{\dot{y}^2}$; and hence $w\dot{w}$ $\frac{Q\dot{w}\ddot{w}}{\dot{y}^2}$, the fluent of which gives $w^2 = \frac{Q\dot{w}^2}{\dot{y}^2}$: but at D the value of w is $= a$, and $\dot{w} = 0$, the curve at D being parallel to KI; therefore the correct fluent is $w^2 - a^2 = \frac{Q\dot{w}^2}{\dot{y}^2}$. Hence then $\dot{y}^2 = \frac{Q\dot{w}^2}{w^2 - a^2}$, or $\dot{y} = \frac{\dot{w}\sqrt{Q}}{\sqrt{(w^2 - a^2)}}$; the correct fluent of which gives $y = \sqrt{Q} \times$ hyp. log. of $\frac{w + \sqrt{(w^2 - a^2)}}{a}$.

Now, to determine the value of Q, we are to consider that when the vertical line CI is in the position AL or MN, then $w =$ CI becomes $=$ AL or MN $=$ the given quantity c suppose, and $y =$ AQ or QM $= b$ suppose, in which position the last equation becomes $b = \sqrt{Q} \times$ hyp log. $\frac{c + \sqrt{(c^2 - a^2)}}{a}$; and hence it is found that the value of the constant quantity $\sqrt{Q}$, is $\frac{b}{\text{h.l.}\,c. + \sqrt{(c^2 - a^2)}}$; which being substituted for it, in the above general value of y, that value becomes

$$y = b \times \frac{\text{log. of } \frac{w + \sqrt{(w^2 - a^2)}}{a}}{\text{log. of } \frac{c + \sqrt{(c^2 - a^2)}}{a}} = b \times \frac{\text{log. of } w + \sqrt{(w^2 - a^2)} - \text{log. } a}{\text{log. of } c + \sqrt{(c^2 - a^2)} - \text{log. } a}$$

from which equation the value of the ordinate PC may always be found, to every given value of the vertical CI.

But if, on the other hand, PC be given, to find CI, which will be the more convenient way, it may be found in the following manner: Put A $=$ log. of a, and C $= \frac{1}{b} \times$ log of $\frac{c + \sqrt{(c^2 - a^2)}}{a}$; then the above equation gives $cy + $ A $=$ log. of $w + \sqrt{(w^2 - a^2)}$; again, put $n =$ the number whose log. is $cy +$ A; then $n = w + \sqrt{(w^2 - a^2)}$; and hence $w = \frac{a^2 + n^2}{2n} =$ CI.

Now, for an example in numbers, in a real case of this nature,

nature, let the foregoing figure represent a transverse vertical section of a magazine arch balanced in all its parts, in which the span or width AM is 20 feet, the pitch or height DQ is 10 feet, thickness at the crown DK = 7 feet, and the angle of the ridge LKS 112° 37′, or the half of it LKD = 56° 18′$\frac{1}{2}$, the complement of which, or the elevation KIR, is 33° 41′$\frac{1}{2}$, the tangent of which is $= \frac{2}{3}$, which will therefore be the value of t in the foregoing investigation. The values of the other letters will be as follows, viz, DK$=a=7$; AQ$=b=10$; DQ$=h=10$; AL$=c = 10\frac{1}{3} = \frac{31}{3}$; A = log. of 7 = ·8450980 ; $\text{C} = \frac{1}{b} \times$ log. of $\frac{c+\sqrt{(c^2-a^2)}}{a} = \frac{1}{10}$ log. of $\frac{31+\sqrt{520}}{21} = \frac{1}{10}$ log. of 2·56207 = ·0408591 ; $\text{C}y + \text{A}$ = ·0408591y + ·8450980 = log. of n. From the general equation then, viz. $\text{CI} = w = \frac{a^2+n^2}{2n} = \frac{a^2}{2n} + \frac{1}{2}n$, by assuming y successively equal to 1, 2, 3, 4, &c, thence finding the corresponding values of $\text{C}y + \text{A}$ or ·0408591y + ·8450980, and to these, as common logs. taking out the corresponding natural numbers, which will be the values of n; then the above theorem will give the several values of w or CI, as they are here arranged in the annexed table, from which the figure of the curve is to be constructed, by thus finding so many points in it.

Val. of y or CP.	Val. of w or CI
1	7·0309
2	7·1243
3	7·2806
4	7·5015
5	7·7888
6	8·1452
7	8·5737
8	9·0781
9	9·6628
10	10·3333

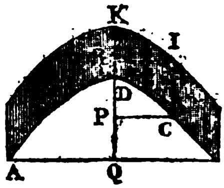
Otherwise. Instead of making n the number of the log. $\text{C}y + \text{A}$, if we put $m =$ the natural number of the log. $\text{C}y$ only; then $m = \frac{w+\sqrt{(w^2-a^2)}}{a}$, and $am - w = \sqrt{(w^2 - a^2)}$, or by squaring, &c, $a^2m^2 - 2amw + w^2 = w^2 - a^2$, and hence $w = \frac{m^2+1}{2m} \times a$: to which the numbers being applied, the very same conclusions result as in the foregoing calculation and table.

PROBLEM VIII.

To construct Powder Magazines with a Parabolical Arch,

It has been shown, in my tract on the Principles of Arches of Bridges, that a parabolic arch is an arch of equilibration, when its extrados, or form of its exterior covering, is the very same parabola as the lower or inside curve. Hence then a parabolic arch, both for the inside and outer form, will be very

very proper for the structure of a powder magazine. For, the inside parabolic shape will be very convenient as to room for stowage: 2dly, the exterior parabola, every where parallel to the inner one, will be proper enough to carry off the rain water: 3dly, the structure will be in perfect equilibrium: and 4thly, the parabolic curve is easily constructed, and the structure erected.



Put, as before, $a = $ KD, $h = $ DQ, $b = $ AQ, $x = $ DP, and $y = $ PC or RI. Then, by the nature of the parabola ADC, $b^2 : y^2 :: h : x = \frac{hy^2}{b^2}$; hence $\dot{x} = \frac{2hy\dot{y}}{bb}$, and $\ddot{x} = \frac{2h\dot{y}^2}{bb}$, by making $\dot{y}$ constant. Then CI $= \frac{\ddot{x}}{\dot{y}^2} \times$ Q is $= \frac{2hQ}{bb} =$ a constant quantity $= a$, what it is at the vertax; that is, CI is every where equal to KD.

Consequently KR is $=$ DP; and since RI is $=$ PC, it is evident that KI is the same parabolic curve with DC, and may be placed any height above it, always producing an arch of equilibration, and very commodious for powder magazines.

CHAPTER XIII.

THEORY AND PRACTICE OF GUNNERY.

In the Doctrine of Motion, Forces, &c, have been given several particulars relating to this subject. Thus, in props. 19, 20, 21, 22, is given all that relates to the parabolic theory of projectiles, that is, the mathematical principles which would take place and regulate such projects, if they were not impeded and disturbed in their motions by the air in which they move. But, from the enormous resistance of that medium, it happens, that many military projectiles, especially the smaller balls discharged with the higher velocities, do not range so far as a 20th part of what they would naturally do in empty space! That theory therefore can only be useful in some few cases, such as in the slower kind of motions, not above the velocities of 2, 3, or 400 feet per second, when the path of the projectile differs but little perhaps from the curve of a parabola.

Again, at art. 104, &c of same doctrine, are given several other practical rules and calculutions, depending partly on the fore-going

going parabolic theory, and partly on the results of certain experiments performed with cannon balls.

Again, in prop. 58, Statics, are delivered the theory and calculations of a beautiful military experiment, invented by Mr. Robins, for determining the true degree of velocity with which balls are projected from guns, with any charges of powder. The idea of this experiment, is simply, that the ball is discharged into a very large but moveable block of wood, whose small velocity, in consequence of that blow, can be easily observed and accurately measured. Then, from this small velocity, thus obtained, the great one of the ball is immediately derived by this simple proportion, viz. as the weight of the ball, is to the sum of the weights of the ball and the block, so is the observed velocity of the last, to a 4th proportional, which is the velocity of the ball sought.—It is evident that this simple mode of experiment will be the source of numerous useful principles, as results derived from the experiments thus made, with all lengths and sizes of guns, with all kinds and sizes of balls and other shot, and with all the various sorts and quantities of gunpowder; in short, the experiment will supply answers to all enquiries in projectiles, excepting the extent of their ranges; for it will even determine the resistance of the air, by causing the ball to strike the block of wood at different distances from the gun, thus showing the velocity lost by passing through those different spaces of air; all which circumstances are partly shown in my 4to vol. of Tracts published in 1786, and which will be completed in my new volumes of miscellaneous tracts now printing.

Lastly, in prob. 17, Prac. Ex. on Forces, some results of the same kind of experiment are successfully applied to determine the curious circumstances of the first force or elasticity of the air resulting from fired gunpowder, and the velocity with which it expands itself. These are circumstances which have never before been determined with any precision. Mr. Robins, and other authors, it may be said, have only guessed at, rather than determined them. That ingenious philosopher, by a simple experiment, truly showed that by the firing of a parcel of gunpowder, a quantity of elastic air was disengaged, which, when confined in the space only occupied by the powder before it was fired, was found to be near 250 times stronger than the weight or elasticity of the common atmospheric air. He then heated the same parcel of air to the degree of red hot iron, and found it in that temperature to be about 4 times as strong as before; whence he inferred, that the first strength of the inflamed fluid, must be nearly 1000 times the pressure of the

the atmosphere. But this was merely guessing at the degree of heat in the inflamed fluid, and consequently of its first strength, both which n fact are found to be much greater. It is true that this assumed degree of strength accorded pretty well with that author's experiments; but this seeming agreement, it may easily be shown, could only be owing to the inaccuracy of his own further experiments; and, in fact, with far better opportunities than fell to the lot of Mr. Robins, we have shown that inflamed gunpowder is about double the strength that he has assigned to it, and that it expands itself with the velocity of about 5000 feet per second.

Fully sensible of the importance of experiments of this kind, first practised by Mr. Robins with musket balls only, my endeavours for many years were directed to the prosecution of the same, on a larger scale, with cannon balls; and I having had the honour to be called on to give my assistance at several courses of such experiments, carried on at Woolwich by the ingenious officers of the Royal Artillery there, under the auspices of the Masters General of the Ordnance, I have assiduously attended them for many years. The first of these courses was performed in the year 1775, being 2 years after my establishment in the Royal Academy at that place: and in the Philos. Trans. for the year 1778 I gave an account of these experiments, with deductions, in a memoir, which was honoured with the Royal Society's gold medal of that year. In conclusion, from the whole, the following important deductions were fairly drawn and stated, viz.

1st, It is made evident by these experiments, that gunpowder fires almost instantaneously. *2dly*, The velocities communicated to shot of the same weight, with different charges of powder, are nearly as the square roots of those charges. *3dly*, And when shot of different weights are fired with the same charge of powder, the velocities communicated to them, are nearly in the inverse ratio of the square roots of their weights. *4thly*, So that, in general, shot which are of different weights, and impelled by the firing of different charges of powder, acquire velocities which are directly as the square roots of the charges of powder, and inversely as the square roots of the weights of the shot. *5thly*, It would therefore be a great improvement in artillery, occasionally to make use of shot of a long shape, or of heavier matter, as lead; for thus the momentum of a shot, when discharged with the same charge of powder, would be increased in the ratio of the square root of the weight of the shot; which would both augment proportionally the force of the blow with which

which it would strike, and the extent of the range to which it would go. *6thly*, It would also be an improvement, to diminish the windage; since by this means, one third or more of the quantity of powder might be saved. *7thly*, When the improvements mentioned in the last two articles are considered as both taking place, it appears that about half the quantity of powder might be saved. But, important as this saving may be, it appears to be still exceeded by that of the guns: for thus a small gun may be made to have the effect and execution of another of two or three times its size in the present way, by discharging a long shot of 2 or 3 times the weight of its usual ball, or round shot; and thus a small ship might employ shot as heavy as those of the largest now use.

Finally, as these experiments prove the regulations with respect to the weight of powder and shot, when discharged from the same piece of ordnance; so, by making similar experiments with a gun varied in its length, by cutting off from it a certain part, before each set of trials, the effects and general rules for the different lengths of guns, may be with certainty determined by them. In short, the principles on which these experiments were made, are so fruitful in consequences, that, in conjunction with the effects of the resistance of the medium, they appear to be sufficient for answering all the inquiries of the speculative philosopher, as well as those of the practical artillerist.

Such then was the summary conclusion from the first set of experiments with cannon balls, in the year 1775, and such were the probable advantages to be derived from them. I am not aware however that any alterations were adopted from them by authority in the public service: unless we are to except the instance of carronades, a species of ordnance that was afterwards invented, and in some degree adopted in the public service; for, in this instance, the proprietors of those pieces, by availing themselves of the circumstances of large balls, and very small windage, have, with small charges of powder, and at little expense, been enabled to produce very considerable and useful effects with those light pieces.

The 2d set of these experiments extended through most part of the summer seasons of the years 1783, 1784, 1785, and some in 1786. The objects of this course were numerous and various: but the principal articles as follow: 1. The velocities with which balls are projected by equal charges of powder, from pieces of equal weight and calibre, but of different lengths. 2. The velocities with different charges of powder, the weight and length of the guns being equal. 3. The greatest velocities due to the different lengths of guns, to

to be ascertained by successively increasing the charge, till the bore should be filled, or till the velocity should decrease again. 4. The effect of varying the weight of the piece; every thing else being the same. 5. The penetrations of balls into blocks of wood. 6. The ranges and times of flight of balls; to compare them with their first velocities, for ascertaining the resistance of the medium. 7. The effect of wads; of different degrees of ramming, or compressing the charge; of different degrees of windage; of different positions of the vent; of chambers and trunnions, and every other circumstance necessary to be known for the improvement of artillery.

An ample account is given of these experiments, and the results deduced from them in my volume of Tracts published in 1786; some few circumstances only of which can be noted here. In this course, 4 brass guns were employed, very nicely bored and cast on purpose, of different lengths, but equal in all other respects, viz, in weight and bore, &c. The lengths of the bores of the guns were,

the gun n° 1,	was 15 calibres,	length of bore	28·5 inc.
. . . n° 2,	. 20 calibres,	. . . .	38·4
. . . n° 3,	. 30 calibres,	. . . .	57·7
. . . n° 4,	. 40 calibres,	. . . .	80·2.

the calibre of each being $2\frac{1}{50}$ inches, and the medium weight of the balls 16 oz. 13 drams.

The mediums of all the experimented velocities of the balls, with which they struck the pendulous block of wood, placed at the distance of 32 feet from the muzzle of the gun, for several charges of powder, were as in the following table,

Table of Initial Velocities.

Powder.	The Guns.			
oz.	No. 1.	No. 2.	No. 3.	No. 4.
2	780	835	920	970
4	1100	1180	1300	1370
6	1340	1445	1590	1680
8	1430	1580	1790	1940
12	1436	1640	.	.
14	.	1660	.	.
16	.	.	2000	.
18	.	.	.	2200

placed in the 1st column, for all the four guns, the numbers denoting so many feet per second. Whence in general it

it appears how the velocities increase with the charges of powder, for each gun, and also how they increase as the guns are longer, with the same charge, in every instance.

By increasing the quantity of the charges continually, for each gun, it was found that the velocities continued to increase till they arrived at a certain degree, different in each gun; after which, they constantly decreased again, till the bore was quite filled with the charge. The charges of powder when the velocities arrived at their maximum or greatest state, were various, as might be expected, according to the lengths of the guns; and the weight of powder, with the length it extended in the bore, and the fractional part of the bore it occupied, are shown in the following table, of the charges for the greatest effect.

Gun, n°.	Length of the Bore	The Charge. Weight, oz.	Length. Inches.	Part of whole.
1	28·5	12	8·2	$\frac{3}{10}$
2	38·4	14	9·5	$\frac{3}{12}$
3	57·7	16	10·7	$\frac{3}{16}$
4	80·2	18	12·1	$\frac{3}{20}$

Some few experiments in this course were made to obtain the ranges and times of flight, the mediums of which are exhibited in the following table.

Guns.	Powder.	Balls. Weight.		Balls. Diam.	Elevat. gun.	Time of flight.	Range.	First veloc.
	oz.	oz.	dr.	inch.		secs.	feet.	feet.
n°2	2	16	10	1·96	45°	21·2	5109	863
do.	2	16	5	1·96	15	9·2	4130	868
do.	4	16	8	1·96	15	9·2	4660	1234
do.	8	16	12	1·96	15	14·4	6066	1644
do.	12	16	12	1·95	15	15·5	6700	1676
n°3.	8	15	8	1·96	15	10·1	5610	1938

In this table are contained the following concomitant data, determined with a tolerable degree of precision; viz, the weight of the powder, the weight and diameter of the ball, the initial or projectile velocity, the angle of elevation of the

gun, the time in seconds of the ball's flight through the air, and its range, or the distance where it fell on the horizontal plane. From which it is hoped that some aid may be derived towards ascertaining the resistance of the medium, and its effects on other elevations, &c, and so afford some means of obtaining easy rules for the cases of practical gunnery. Though the completion of this enquiry, for want of time at present, must be referred to another work, where we may have an opportunity of describing another more extended course of experiments on this subject, which have never yet been given to the public.

Another subject of enquiry in the foregoing experiments, was, how far the balls would penetrate into solid blocks of elm wood, fired in the direction of the fibres. The annexed tablet shows the results of a few of the trials that were made with the gun n° 2, with the most frequent charges of 2, 4, and 8 ounces of powder; and the mediums of the penetrations, as placed in the last line, are found to be 7, 15, and 20 inches, with those charges. These penetrations are nearly as the numbers

Penetrations of Balls into solid Elm wood.

Powder 2	4	8 oz.
7	16·6	18·9
	13·5	21·2
		18·1
		20·8
		20·5
Means 7	15	20

2, 4, 6, or 1, 2, 3; but the charges of powder are as
2, 4, 8, or 1, 2, 4; so that the penetrations are proportional to the charges as far as to 4 ounces, but in a less ratio at 8 ounces; whereas, by the theory of penetrations, the depths ought to be proportional to the charges, or, which is the same thing, as the squares of the velocities. So that it seems the resisting force of the wood is not uniformly or constantly the same, but that it increases a little with the increased velocity of the ball. This may probably be occasioned by the greater quantity of fibres driven before the ball; which may thus increase the spring and resistance of the wood, and prevent the ball from penetrating so deep as it otherwise might do.

From a general inspection of this second course of these experiments, it appears that all the deductions and observations made on the former course, are here corroborated and strengthened, respecting the velocities and weights of the balls, and charges of powder, &c. It further appears also that the velocity of the ball increases with the increase of charge

charge only to a certain point, which is peculiar to each gun, where it is greatest; and that by further increasing the charge, the velocity gradually diminishes, till the bore is quite full of powder. That this charge for the greatest velocity is greater as the gun is longer, but yet not greater in so high a proportion as the length of the gun is; so that the part of the bore filled with powder, bears a less proportion to the whole bore in the long guns, than it does in the shorter ones; the part which is filled being indeed nearly in the inverse ratio of the square root of the empty part.

It appears that the velocity, with equal charges, always increases as the gun is longer; though the increase in velocity is but very small in comparison to the increase in length; the velocities being in a ratio somewhat less than that of the square roots of the length of the bore, but greater than that of the cube roots of the same, and is indeed nearly in the middle ratio between the two.

It appears, from the table of ranges, that the range increases in a much lower ratio than the velocity, the gun and elevation being the same. And when this is compared with the proportion of the velocity and length of gun in the last paragraph, it is evident that we gain extremely little in the range by a great increase in the length of the gun, with the same charge of powder. In fact the range is nearly as the 5th root of the length of the bore; which is so small an increase, as to amount only to about a 7th part more range for a double length of gun.—From the same table it also appears, that the time of the ball's flight is nearly as the range; the gun and elevation being the same.

It has been found, by these experiments, that no difference is caused in the velocity, or range, by varying the weight of the gun, nor by the use of wads, nor by different degrees of ramming, nor by firing the charge of powder in different parts of it. But that a very great difference in the velocity arises from a small degree in the windage: indeed with the usual established windage only, viz, about $\frac{1}{20}$ of the calibre, no less than between $\frac{1}{5}$ and $\frac{1}{4}$ of the powder escapes and is lost: and as the balls are often smaller than the regulated size, it frequently happens that half the powder is lost by unnecessary windage.

It appears too that the resisting force of wood, to balls fired into it, is not constant: and that the depths penetrated by balls, with different velocities or charges, are nearly as the logarithms of the charges, instead of being as the charges themselves, or, which is the same thing, as the square of the velocity.—Lastly, these and most other experiments, show, that

that balls are greatly deflected from the direction in which they are projected; and that as much as 300 or 400 yards in a range of a mile, or almost ¼th of the range.

We have before adverted to a third set of experiments, of still more importance, with respect to the resistance of the medium, than any of the former; but, till the publication of those experiments, we cannot avail ourselves of all the discoveries they contain. In the mean time however we may extract from them the three following tables of resistances, for three different sizes of balls, and for velocities between 100 feet and 2000 feet per second of time.

Table I.					Table II.			Table III.		
Resistances to a ball of 1·965 *inches diameter, and* 16 *oz.* 13 *dr. weight.*					*Resistances to a ball* 2·78 *in. diam. and* 3*lb weight.*			*Resist: to a ball* 3·55 *in. diam. and* 6*lb.* 1*oz.* 8*dr. wt.*		
Vel.	Resistances.		1 Dif.	2d Dif.	Vel.	Res.	Difs.	Vel.	Res.	Difs.
feet.	lbs.	ozs.			feet.	lbs.		feet.	lbs.	
100	0·17	2¾			900	35		1200	115	
200	0·69	11	8¼	5¼	950	41	6	1250	124	9
300	1·56	25	14	6	1000	47	6	1300	133	9
400	2·81	45	20	7	1050	53	6	1350	142	9
500	4·50	72	27	8	1100	60	7	1400	152	10
600	6·69	107	35	9	1150	67	7	1450	162	10
700	9·44	151	44	10	1200	74	7	1500	172½	10½
800	12·81	205	54	12	1250	82	8	1550	184	11½
900	16·94	271	66	13	1300	91	9	1600	197	13
1000	21·88	350	79	13	1350	101	10	1650	211	14
1100	27·63	442	92	12	1400	112	11	1700	226	15
1200	34·13	546	104	11	1450	122½	10½	1750	242	16
1300	41·31	661	115	9	1500	132½	10	1800	259	17
1400	49·06	785	124	7	1550	141½	9			
1500	57·25	916	131	4	1600	150	8½			
1600	65·69	1051	135	0	1650	158	8			
1700	74·13	1186	135	—2	1700	165	7			
1800	82·44	1319	133	—5	1750	171	6			
1900	90·44	1447	128	—6	1800	176	5			
2000	98·06	1569	122							

PROBLEM I.

To determine the Resistance of the Medium against a Ball of any other size, moving with any of the Velocities given in the foregoing Tables.

The analogy among the numbers in all these tables is very remarkable and uniform, the same general laws running through

through them all. The same laws are also observable as in the table of resistances in page 412 of this volume, particularly the 1st and 2d remarks immediately following that table, viz, that the resistances increase in a higher proportion than the square of the velocities, with the same body; and that the resistances also increase in a rather higher ratio than the surfaces, with different bodies, but the same velocity. Yet this latter case, viz, the ratios of the resistances and of the surfaces, or of the squares of the diameters which is the same thing, are so nearly alike, that they may be considered as equal to each other in any calculations relating to artillery practice. For example, suppose it were required to determine what would be the resistance of the air against a 24lb ball discharged with a velocity of 2000 feet per second of time. Now, by the 1st of the foregoing tables, the ball of 1·965 inches diameter, when moving with the velocity 2000, suffered a resistance of 98lb: then since the resistances, with the same velocity, are as the surfaces; and the surfaces are as the squares of the diameters; and the diameters being 1·965 and 5·6, the squares of which are 3·86 and 31·36, therefore as 3·86 : 31·36 : : 98lb : 796lb; that is, the 24lb ball would suffer the enormous resistance of 796lb in its flight, in opposition to the direction of its motion!

And, in general, if the diameter of any proposed ball be denoted by d, and r denote the resistance in the 1st table due to the proposed velocity of the 1·965 ball; then $\frac{d^2r}{3\cdot86}$ will denote the resistance with the same velocity against the ball whose diameter is d; or it is nearly $\frac{1}{4}d^2r$, which is but the 28th part greater than the former.

PROBLEM II.

To assign a Rule for determining the Resistance due to any Indeterminate Velocity of a Given Ball.

This problem is very difficult to perform near the truth, on account of the variable ratio which the resistance bears to the velocity, increasing always more and more above that of the square of the velocity, at least to a certain extent; and indeed it appears that there is no single integral power whatever of the velocity, or no expression of the velocity in one term only, that can be proportional to the resistances throughout. It is true indeed, that such an expression can be assigned by means of a fractional power of the velocity, or rather one whose index is a mixed number, viz, $2\frac{1}{10}$ or 2·1; thus $\frac{v^{2\cdot1}}{5400}$ = the

the resistance, is a formula in one term only, which will answer to all the numbers in the first table of resistances very nearly, and consequently, by means of the ratio of the squares of the diameters of the balls, for any other balls whatever. This formula then, though serving quite well for some particular resistance, or even for constructing a complete series or table of resistances, is not proper for the use of problems in which fluxions and fluents are concerned, on account of the mixed number $2\frac{1}{10}$, in the index of the velocity v.

We must therefore have recourse to an expression in two terms, or a formula containing two integral powers of the velocity, as v^2 and v, the first and 2d powers, affected with general coefficients m and n; as $mv^2 + nv = r$ the resistance. Now, to determine the general numerical values of the coefficients m and n, we must adapt this general expression $mv^2 + nv = r$, to two particular cases of velocity, at a convenient distance from each other, in one of the foregoing tables of resistances, as the first for instance. Now, after making several trials in this way, I have found that the two velocities of 500 and 1000 answer the general purpose better than any other that has been tried. Thus then, employing these two cases, we must first make $v = 500$, and $r = 4\frac{1}{2}$lb, its correspondent resistance, and then again $v = 1000$, and $r = 21{\cdot}88$lb, the resistance belonging to it: this will give two equations, by which the general value of m and of n will be determined. Thus then the two equations being

$$500^2m + 500n = 4{\cdot}5,$$

$$\text{and } 1000^2m + 1000n = 21{\cdot}88;$$

dividing the 1st by 500, and the 2d by 1000, they are $\left\{\begin{array}{l} 500m + n = {\cdot}009, \\ 1000m + n = {\cdot}02188; \end{array}\right.$

the dif. of these is . . . . . . $500m = {\cdot}01288$,

and therefore div. by 500, gives $m = {\cdot}00002576$;

hence $n = {\cdot}009 - 500m = {\cdot}009 - {\cdot}01288 = -{\cdot}00388 = n$.

Hence then the general formula will be ${\cdot}00002576v^2 - {\cdot}00388v = r$ the resistance nearly in avoirdupois pounds, in all cases or all velocities whatever.

Now,

Veloc. or v.	Comput. resists.	Exper. resists.
100	$-$·13	·17
200	$+$·25	·69
300	1·15	1·56
400	2·57	2·81
500	4·50	4·50
600	6·94	6·69
700	9·90	9·44
800	13·38	12·81
900	17·37	16·94
1000	21·88	21·88
1100	26·90	27·63
1200	32·44	34·13
1300	38·49	41·31
1400	45·06	49·06
1500	52·14	57·25
1600	59·74	65·69
1700	67·85	74 13
1800	76·48	82·44
1900	85·62	90·44
2000	95·28	98·06

Now, to find how near to the truth this theorem comes, in every instance in the table, by substituting for v, in this formula, all the several velocities, 100, 200, 300, &c, to 2000, these give the correspondent values of r, or the resistances, as in the 2d column of the annexed table, their velocities being in the first column; and the real experimented resistances are set opposite to them in the 3d or last column of the same. By the comparison of the numbers in these two columns together, it is seen that there are no where any great difference between them, being sometimes a little in excess, and again a little in defect, by very small differences; so that, on the whole, they will nearly balance one another, in any particular instance of the range or flight of a ball, in all degrees of its velocity, from the first or greatest, to the smallest or last. Except in the first two or three numbers, at the beginning of the table, for the velocities 100, 200, 300, for which cases another theorem may be employed. Now, in these three velocities, as well as in all that are smaller, down to nothing, the theorem $\cdot 00001725v^2 = r$ the resistance, will very well serve, as it brings out for the first three resistances ·17, and ·69, and 1·55½, differing in the last only by a very small fraction.

Corol. 1. The foregoing rule $\cdot 00002576v^2 - \cdot 00388v = r$, denotes the resistance for the ball in the first table, whose diameter is 1·965, the square of which is 3·86, or almost 4; hence to adapt it to a ball of any other diameter d, we have only to alter the former in proportion to the squares of the diameters, by which it becomes $\frac{dd}{3\cdot 86}(\cdot 00002576v^2 - \cdot 00388v)$ $= (\cdot 00000667v^2 - \cdot 001v)d^2 = (\cdot 00000\tfrac{2}{3}v^2 - \cdot 001v)d^2$, which is the resistance for the ball whose diameter is d, with the velocity v.

Corol. 2. And, in a similar manner, to adapt the theorem $\cdot 00001725v^2 = r$, for the smaller velocities, to any other size of

of ball, we must multiply it by $\frac{dd}{3 \cdot 86}$, the ratio of the surfaces, by which it becomes $\cdot 00000447 d^2 v^2 = r$.

We shall soon take occasion to make some applications in the use of the foregoing formulas, after considering the effects of such velocities in the cases of nonresistances.

PROBLEM III.

To determine the Height to which a Ball will rise, when fired from a cannon Perpendicularly Upwards with a Given Velocity, in a Nonresisting Medium, or supposing no Resistance in the Air.

By art 73, Motion and Forces, this vol. it appears that any body projected upwards, with a given velocity, will ascend to the height due to the velocity, or the height from which it must naturally fall to acquire that velocity; and the spaces fallen being as the square of the velocities; also 16 feet being the space due to the velocity 32; therefore the space due to any proposed velocity v, will be found thus, as $32^2 : 16 :: v^2 : s$ the space, or as $64 : 1 :: v^2 : \frac{1}{64}v^2 = s$ the space, or the height to which the velocity v will cause the body to rise independent of the air's resistance.

Exam. For example, if the first or projectile velocity, be 2000 feet per second, being nearly the greatest experimented velocity, then the rule $\frac{1}{64}v^2 = s$ becomes $\frac{1}{64} \times 2000^2 = 62500$ feet $= 11\frac{4}{5}$ miles: that is, any body, projected with the velocity 2000 feet, would ascend nearly 12 miles in height, without resistance.

Corol. Because, by art. 88 Projectiles this vol. the greatest range is just double the height due to the projectile velocity, therefore the range, at an elevation of 45°, with the velocity in the last example, would be $23\frac{2}{3}$ miles, in a nonresisting medium. We shall now see what the effects will be with the resistance of the air.

PROBLEM IV.

To determine the Height to which a Ball projected Upwards, as in the last problem, will ascend, being Resisted by the Atmosphere.

Putting x to denote any variable and increasing height ascended by the ball; v its variable and decreasing velocity there; d the diameter of the ball, its weight being w; $m = \cdot 00000\frac{1}{3}$, and $n = \cdot 001$, the co-efficients of the two terms denoting the law of the air's resistance. Then $(mv^2 - nv)d^2$, by cor. 1 to prob.

prob. 2, will be the resistance of the air against the ball in avoirdupois pounds; to which if the weight of the ball be added, then $(mv^2 - nv)d^2 + w$ will be the whole resistance to the ball's motion; this divided by w, the weight of the ball in motion, gives $\frac{(mv^2 - nv)d^2 + w}{w} = \frac{mv^2 - nv}{w}d^2 + 1 = f$ the retarding force. Hence the general formula $v\dot{v} = 2gf\dot{x}$ (theor. 10 pa. 379 this volume.) becomes $-v\dot{v} = 2g\dot{x} \times \frac{(mv^2 - nv)d^2 + w}{w}$ making $\dot{v}$ negative because v is decreasing, where $g = 16$ ft.; and hence

$$\dot{x} = -\frac{w}{2g} \times \frac{v\dot{v}}{(mv^2 - nv)d^2 + w} = \frac{-w}{2gmd^2} \times \frac{v\dot{v}}{v^2 - \frac{n}{m}v + \frac{w}{md^2}}.$$

Now, for the easier finding the fluent of this, assume $v - \frac{n}{2m} = z$; then $v = z + \frac{n}{2m}$, and $v^2 = z^2 + \frac{n}{m}z + \frac{n^2}{4m^2}$ and $v\dot{v} = z\dot{z} + \frac{n}{2m}\dot{z}$, and $v^2 - \frac{n}{m}v + \frac{n^2}{4m^2} = z^2$, and $v^2 - \frac{n}{m}v = z^2 - \frac{n^2}{4m^2}$; these being substituted in the above value of $\dot{x}$, it becomes $\dot{x} =$

$$\frac{-w}{2gmd^2} \times \frac{z\dot{z} + \frac{n}{2m}\dot{z}}{z^2 - \frac{n^2}{4m^2} + \frac{w}{md^2}} = \frac{-w}{2gmd^2} \times \frac{z\dot{z} + p\dot{z}}{z^2 + \frac{w}{md^2} - p^2} = \frac{-w}{2gmd^2} \times \frac{z\dot{z} + p\dot{z}}{z^2 + q^2},$$

putting $p = \frac{n}{2m}$, and $q^2 = \frac{w}{md^2} - p^2$, or $p^2 + q^2 = \frac{w}{md^2}$.

Then the general fluents, taken by the 8th and 11th forms of the table of Fluents, give $x = \frac{-w}{2gmd^2} \times [\frac{1}{2} \log. (z^2 + q^2) + \frac{p}{q^2} \times$ arc to rad. q and tan. $z] = \frac{-w}{2gmd^2} \times [\frac{1}{2} \log. (v^2 - \frac{n}{m}v + \frac{w}{md^2}) + \frac{p}{q^2} \times$ arc to rad. q and tang. $v - p]$. But, at the beginning of the motion, when the first velocity is v for instance, and the space x is $= 0$, this fluent becomes

$0 = \frac{-w}{2gmd^2} \times [\frac{1}{2} \log. (\text{v}^2 - \frac{n}{m}\text{v} + \frac{w}{md^2}) + \frac{p}{q^2} \times$ arc radius q tan. $\text{v} - p]$. Hence by substraction, and taking $v = 0$ for the end of the motion, the correct fluent becomes

$x = \frac{-w}{2gmd^2} \times [\frac{1}{2} \log. (\text{v}^2 - \frac{n}{m}\text{v} + \frac{w}{md^2}) - \frac{1}{2} \log \frac{w}{md^2} + \frac{p}{q^2} \times$ (arc tan. $\text{v} - p$ — arc tan. $- p$ to rad q)].

But as part of this fluent, denoted by $\frac{p}{q^2} \times$ the dif. of the two arcs to tans. $\text{v} - p$ and $- p$, is always very small in comparison

parison with the other preceding terms, they may be omitted without material error in any practical instance ; and then the fluent is $x = \frac{w}{4gmd^2} \times$ hyp. log. $\frac{v^2 - \frac{n}{m}v + \frac{w}{md^2}}{\frac{w}{md^2}}$, for the utmost height to which the ball will ascend, when its motion ceases, and is stopped, partly by its own gravity, but chiefly by the resistance of the air.

But now, for the numerical value of the general coefficient $\frac{w}{4gmd^2}$, and the term $\frac{w}{md^2}$; because the mass of the ball to the diameter d, is $\cdot5236d^3$, if its specific gravity be s, its weight will be $\cdot5236sd^3 = w$; therefore $\frac{w}{d^2} = \cdot5236sd$, and $\frac{w}{md^2} = 78540sd$, this divided by $4g$ or 64 it gives $\frac{w}{4gmd^2} = 1227\cdot2sd$ for the value of the general coefficient, to any diameter d and specific gravity s. And if we further suppose the ball to be cast iron, the specific gravity, or weight of one cubic inch of which is $\cdot26855$, it becomes $330d$, for that coefficient; also $78540sd = 21090d = \frac{w}{md^2}$, and $\frac{n}{m} = 150$. And hence the foregoing fluent becomes $330d \times$ hyp. log. $\frac{v^2 - 150v + 21090d}{21090d}$ or $760d \times$ com. log. $\frac{v^2 - 150v + 21090d}{21090d}$ changing the hyperbolic for the common logs. And this is a general expression for the altitude in feet, ascended by any iron ball, whose diameter is d inches, discharged with any velocity v feet. So that, substituting any values of d and v, the particular heights will be given to which the balls will ascend, which it is evident will be nearly in proportion to the diameter d.

Exam. 1. Suppose the ball be that belonging to the first table of resistances, its weight being 16 oz. 13 dr. or 1·05 lb, and its diameter 1·965 inches, when discharged with the velocity 2000 feet, being nearly the greatest charge for any iron ball. The calculation being made with these values of d and v, the height ascended is found to be 2920 feet, or little more than half a mile ; though found to be almost 12 miles without the air's resistance. And thus the height may be found for any other diameter and velocity.

Exam. 2. Again, for the 24 lb ball, with the same velocity 2000, its diameter being $5\cdot6 = d$. Here $760d = 4256$; and $\frac{v^2 - 150v + 21090d}{21090d} = \frac{38181}{1181}$, the log. of which is 1·50958 ;

theref.

theref. $1\cdot50958 \times 4256 = 6424 = x$ the height, being a little more than a mile.

We may now examine what will be the height ascended, considering the resistance always as the square of the velocity.

PROBLEM V.

To determine the Height ascended by a Ball projected as in the two foregoing problems; supposing the Resistance of the Air to be as the Square of the Velocity.

Here it will be proper to commence with selecting some experimented resistance corresponding to a medium kind of velocity between the first or greatest velocity and nothing, from which to compute the other general resistances, by considering them as the squares of the velocities. It is proper to assume a near medium velocity and its resistance, because, if we assume or commence with the greatest, or the velocity of projection, and compute from it downwards, the resistances will be every where too great, and the altitude ascended much less than just; and, on the other hand, if we assume or commence with a small resistance, and compute from it all the others upwards, they will be much too little, and the computed altitude far too great. But, commencing with a medium degree, as for instance that which has a resistance about the half of the first or greatest resistance, or rather a little more, and computing from that, then all those computed resistances above that, will be rather too little, but all those below it too great; by which it will happen, that the defect of the one side will be compensated by the excess on the other, and the final conclusion must be near the truth.

Thus then, if we wish to determine, in this way, the altitude ascended by the ball employed in the 1st table of resistances, when projected with 2000 feet velocity; we perceive by the table, that to the velocity 2000 corresponds the resistance 98 lb; the half of this is 49, to which resistance corresponds the velocity 1400 in the table, and the next greater velocity 1500, with its resistance $57\frac{1}{4}$, which will be properest to be employed here. Hence then, for any other velocity v, in general, it will be, according to the law of the squares of the velocities, as $1500^2 : v^2 :: 57\frac{1}{4} : \frac{57\frac{1}{4}v^2}{1500^2} = \cdot000025\frac{4}{9}v^2 = av^2$, putting $a = \cdot000025\frac{4}{9}$, which will denote the air's resistance for any velocity v, very nearly, counting from 2000.

Now let x denote the altitude ascended when the velocity is v, and w the weight of the ball: then, as above, av^2, is the resistance

resistance from the air, hence $av^2 + w$ is the whole resisting force, and $\frac{av^2 + w}{w} = f$ the retarding force;

$$\text{therefore } -\mathrm{v}\dot{v} = 2gf\dot{x} = \frac{av^2 + w}{w} \times 2g\dot{x};$$

$$\text{and hence } \dot{x} = \frac{-w}{2g} \times \frac{v\dot{v}}{av^2 + w} = \frac{-w}{2ga} \times \frac{v\dot{v}}{v^2 + \frac{w}{a}};$$

the fluent of which, by form 8, is $\frac{-w}{4ga} \times$ h. log. $(v^2 + \frac{w}{a})$; which when $x = 0$, and $v = \mathrm{v}$ the first or projectile velocity, becomes $0 = \frac{-w}{4ga} \times$ h. l. $(\mathrm{v}^2 + \frac{w}{a})$; theref. by subtracting the correct fluent is $x = \frac{w}{4ga} \times$ h. l. $\frac{a\mathrm{v}^2 + w}{av^2 + w}$, the height x when the velocity is reduced to v; and when $v = 0$, or the velocity is quite exhausted, this becomes $\frac{w}{4ga} \times$h. l. $\frac{a\mathrm{v}^2 + w}{w}$ for the whole height to which the ball will ascend.

Ex. 1. The values of the letters being $w = 1{\cdot}05$lb, $4g = 64$, $a = {\cdot}0000254$, the last expression becomes 645 × hyp. log. $\frac{\mathrm{v}^2 + 41266}{41266}$, or 1484 × com. log. $\frac{\mathrm{v}^2 + 41266}{41266}$. And here the first velocity v being 2000, the same expression 1484 × log. $\frac{\mathrm{v}^2 + 41266}{41266}$ becomes 1484 × log. of 97·93 = 2955 for the height ascended, on this hypothesis; which was 2920 by the former problem, being nearly the same.

Ex. 2. Supposing the same ball to be projected with the velocity of only 1500 feet. Then taking 1100 velocity, whose tabular resistance is 27·6, being next above the half of that for 1500. Hence, as $1100^2 : v^2 :: 27{\cdot}6 : {\cdot}00002375v^2 = av^2$. This value of a substituted in the theorem $\frac{w}{4ga} \times$ h. l. $\frac{a\mathrm{v}^2 + w}{w}$ also 1500 for v, and 1·05 for w, it brings out $x = 2728$ for the height in this case, being but a little above the ratio of the square roots of the velocities 2000 and 1500, as that ratio would give only 2560.

Ex. 3. To find the height ascended by the first ball, projected with 860 feet velocity. Here taking 600, whose resistance 6·69 is a near medium; then as $600^2 : 6{\cdot}69 :: 1 : {\cdot}0000186 = a$. Hence $\frac{w}{64a} \times$ h. l. $\frac{a\mathrm{v}^2 + w}{w} = 2334$ the height; which is less than half the range (5100) at 45° elevation, but more than half the range (4100) at 15° elevation, art. 105 of Mot. and Forces, being indeed nearly a medium between the two.

Ex.

Ex. 4. With the same ball, and 1640 velocity. Assume 1200, whose resistance 34·13 is nearly a medium. Then as $1200^2 : 34{\cdot}13 :: 1 : {\cdot}0000237 = a$. Hence $\frac{w}{64a} \times$ h. l. $\frac{av^2+w}{w} = 2854$; again less than half the range (6000) by experiment in this vol. even with 15° elevation.

Ex. 5. For any other ball, whose diameter is d, and its weight w, the resistance of the air being $\frac{ad^2v^2}{3{\cdot}86} = \frac{d^2v^2}{150000} = bd^2v^2$ putting $b = \frac{1}{150000}$, the retarding force will be $\frac{bd^2v^2+w}{w}$. thence $-v\dot{v} = 2g\dot{x} \times \frac{bd^2v^2+w}{w}$, and $\dot{x} = \frac{-w}{2g} \times \frac{v\dot{v}}{bd^2v^2+w}$, and the cor. flu. $x = \frac{w}{4gbd^2} \times$ h. l. $\frac{bd^2V^2+w}{bd^2v^2+w} = \frac{w}{4gbd^2} \times$ h. l. $\frac{bd^2V^2+w}{w}$ for the whole height when $v = 0$. Now if the ball be a 24 pounder, whose diameter is 5·6, and its square 31·36; then $bd^2 = \frac{62{\cdot}72}{300000} = {\cdot}0002091$, and $\frac{w}{4gbd^2} = \frac{24}{64bd^2} = \frac{3}{8bd^2} = 1794$; and $bd^2V^2 = 836$, and $\frac{bd^2V^2+w}{w} = \frac{836+24}{24} = \frac{860}{24} = \frac{215}{6}$; therefore $x = 1794 \times$ h. l. $\frac{215}{6} = 1794 \times 3{\cdot}57888 = 6420$; being more than double the height of that of the small ball, or a little more than a mile, and very nearly the same as in the 2d example to prob. 4.

PROBLEM VI.

To determine the Time of the Ball's ascending to the Height determined in the last prob. by the same Projectile Velocity as there given.

By that prob. $\dot{x} = \frac{-w}{2ga} \times \frac{v\dot{v}}{v^2 + \frac{w}{a}}$, ther. $\dot{t} = \frac{\dot{x}}{v} = \frac{-w}{2ga} \times \frac{\dot{v}}{v^2 + \frac{w}{a}}$; the fluent of which, by form 11, is $\frac{-w}{2ga}\sqrt{\frac{a}{w}} \times$ arc to radius 1 tang. $\frac{v}{\sqrt{\frac{w}{a}}} = \frac{-1}{2g}\sqrt{\frac{w}{a}} \times$ arc tan. $\frac{v}{\sqrt{\frac{w}{a}}}$; or by correction $t = \frac{1}{2g}\sqrt{\frac{w}{a}} \times$ (arc tang. $\frac{V}{\sqrt{\frac{w}{a}}}$ − arc tan. $\frac{v}{\sqrt{\frac{w}{a}}}$), the time in general when the first velocity V is reduced to v. And when $v = 0$, or the velocity ceases, this becomes $t = \frac{1}{2g}\sqrt{\frac{w}{a}} \times$ arc to tang. $\frac{V}{\sqrt{\frac{w}{a}}}$ for the time of the whole ascent.

Now,

Now, as in the last prob. $v = 2000$, $w = 1\cdot05$, $a = \cdot000025\frac{4}{9}$ $= \frac{229}{9000000}$. Hence $\frac{w}{a} = 41266$, and $\sqrt{\frac{w}{a}} = 203\cdot14$, and $\frac{v}{\sqrt{\frac{w}{a}}} = 98\cdot445$ the tangent, to which corresponds the arc of $89^\circ\ 25'$, whose length is $1\cdot5606$; then $\frac{1}{2g} \times 203\cdot14 \times 1\cdot5606 = \frac{203\cdot14 \times 1\cdot5606}{32} = 9''\cdot91$, the whole time of ascent.

Remark. The time of *freely* ascending to the same height 2955 feet, that is, without the air's resistance, would be $\sqrt{\frac{2955}{16}} = \frac{1}{4}\sqrt{2955} = 13''\cdot59$; and the time of *freely* ascending, commencing with the same velocity 2000, would be $\frac{v}{2g} = \frac{2000}{32} = 62''\frac{1}{2} = 1'2''\frac{1}{2}$.

PROBLEM VII.

To determine the same as in prob. V, *taking into the account the Decrease of Density in the Air as the Ball ascends in the Atmosphere.*

In the preceding problems, relating to the height and time of balls ascending in the atmosphere, the decrease of density in the upper parts of it has been neglected, the whole height ascended by the ball being supposed in air of the same density as at the earth's surface. But it is well known that the atmosphere must and does decrease in density upwards, in a very rapid degree; so much so indeed, as to decrease in geometrical progression, at altitudes which rise only in arithmetical progression: by which it happens, that the altitudes ascended are proportional only to the logarithms of the decrease of density there. Hence it results, that the balls must be always less and less resisted in their ascent, with the same velocity, and that they must consequently rise to greater heights before they stop. It is now therefore to be considered what may be the difference resulting from this circumstance.

Now, the nature and measure of this decreasing density, of ascents in the atmosphere, has been explained and determined in prop. 76, Pneumatics. It is there shown, that if D denote the air's density at the earth's surface, and d its density at any altitude a, or x; then is $x = 63551 \times$ log. of $\frac{D}{d}$ in feet, when the temperature of the air is 55°; and $60000 \times \log.\frac{D}{d}$ for the temperature of freezing cold; we

we may therefore assume for the medium $x = 62000 \times \log. \frac{D}{d}$ for a mean degree between the two.

But to get an expression for the density d, in terms of x out of logarithms, without which it could not be introduced into the measure of the ball's resistance, in a manageable form, we find in the first place, by a neat approximate expression for the natural number to the log. of a ratio $\frac{D}{d}$, whose terms do not greatly differ, invented by Dr. Halley, and explained in the Introduction to our Logarithms, p. 110, that $\frac{n - \frac{1}{2}l}{n + \frac{1}{2}l} \times D$ nearly, is the number answering to the log. l of the ratio $\frac{D}{d}$, where n denotes the modulus ·43429448 &c of the common logarithms. But, we before found that $x = 62000 \times \log.$ of $\frac{D}{d}$, or $\frac{x}{62000}$ is the log. of $\frac{D}{d}$, which log. was denoted by l in the expression just above, for the number whose log. is l or $\frac{x}{62000}$; substituting therefore $\frac{x}{62000}$ for l, in the expression $\frac{n - \frac{1}{2}l}{n + \frac{1}{2}l} \times D$, it gives the natural number $\frac{n - \frac{x}{124000}}{n + \frac{x}{124000}} \times D = d$, or $\frac{124000n - x}{124000n + x} = d$, the density of the air at the altitude x, putting $D = 1$ the density at the surface. Now put $124000n$ or nearly $54000 = c$; then $\frac{c - x}{c + x}$ will be the density of the air at any general height x.

But, in the 5th prob. it appears that av^2 donotes the resistance to the velocity v, or at the height, x for the density of air the same as at the surface, which is too great in the ratio of $c + x$ to $c - x$; therefore $av^2 \times \frac{c - x}{c + x}$ will be the resistance at the height x, to the velocity v, where $a =$ ·0000025$\frac{4}{5}$. To this adding w, the weight of the ball, gives $av^2 \times \frac{c - x}{c + x} + w$ for the whole resistance, both from the air and the ball's mass; conseq. $\frac{av^2}{w} \times \frac{c - x}{c + x} + \frac{w}{w}$ will denote the accelerating force of the ball. Or, if we include the small part $\frac{w}{w}$ or 1, within the factor $\frac{c - x}{c + x}$, which will make no sensible difference in the result, but be a great deal simpler in

in the process, then is $\frac{av^2+w}{w} \times \frac{c-x}{c+x} = f$ the accelerating force. Conseq. $-v\dot{v} = 2gf\dot{x} = 2g\dot{x} \times \frac{c-x}{c+x} \times \frac{av^2+w}{w}$, and hence $\frac{c-x}{c+x}\dot{x} = \frac{w}{2g} \times \frac{-w\dot{v}}{av^2+w}$, or by division, $-\dot{x} + \frac{2c}{c+x}\dot{x} = \frac{w}{32a} \times \frac{-v\dot{v}}{v^2+\frac{w}{a}}$.

Now the fluent of the first side of this equation is evidently $-x + 2c \times$ h. l. $(c+x)$; and the fluent of the latter side, the same as in prob. 5, is $\frac{-w}{64a} \times$ h. l. $(v^2 + \frac{w}{a})$; therefore the general fluential equa. is $-x + 2c \times$ h. l. $(c+x) = \frac{-w}{64a} \times$ h. l. $(v^2 + \frac{w}{a})$. But, when $x = 0$, and $v = \text{v}$ the initial velocity, this becomes $0 + 2c \times$ h. l. $c = \frac{-w}{64a} \times$ h. l. $(\text{v}^2 + \frac{w}{a})$; theref. by subtraction, the correct fluents are $-x + 2c \times$ h. l. $\frac{c+x}{c} = \frac{w}{64a} \times$ h. l. $\frac{a\text{v}^2+w}{av^2+w}$, when the first velocity v is diminished to any less one v; and when it is quite extinct, the state of the fluents becomes $-x + 2c \times$ h. l. $\frac{c+x}{c} = \frac{w}{64a} \times$ h. l. $\frac{a\text{v}^2+w}{w}$ for the greatest height x ascended.

Here, in the quantity h. l. $\frac{c+x}{c}$, the term x is always small in respect of the other term c; therefore, by the nature of logarithms, the h. l. of $\frac{c+x}{c}$ is nearly $= \frac{x}{c+\frac{1}{2}x}$ or $\frac{2x}{2c+x}$; theref. the above fluents become $-x + \frac{4cx}{2c+x} = \frac{2cx-x^2}{2c+x} = \frac{2c-x}{2c+x}x = \frac{w}{64a} \times$ h. l. $\frac{a\text{v}^2+w}{w}$. Now the latter side of this equation is the same value for x as was found in the 5th problem, which therefore put $= b$; then the value of x will be easily found from the formula $\frac{2c-x}{2c+x}x = b$, by a quadratic equation. Or, still easier, and sufficiently near the truth, by substituting b for x in the numerator and the denominator of $\frac{2c-x}{2c+x}$, then $\frac{2c-b}{2c+b}x = b$, and hence $x = \frac{2c+b}{2c-b}b$, or by proportion, as $2c-b : 2c+b :: b : x$; that is, only increase the value of x, found by prob. 5, in the ratio of $2c-b$ to $2c+b$.

Now, in the first example to that prob. the value of x or b was

b was there found $= 2955$; and $2c$ being $= 108000$, theref. $2c - b = 105045$, and $2c + b = 110955$, then as $105045 : 110955 :: 2955 : 3121 =$ the value of the height x in this case, being only 166 feet, or $\frac{1}{18}$th part more than before.

Also, for the 2d example to the 5th prob. where x was $= 6420$; therefore as $2c - b : 2c + b$ or as $105045 : 110955 :: 6420 : 6780$ the height ascended in this example, being also the 18th part more than before. And so on, for any other examples; the value of $2c$ being the constant number 108000.

PROBLEM VIII.

To determine the Time of a Ball's Ascending, considering the Decreasing Density of the Air as in the last prob.

The fluxion of the time is $\dot{t} = \frac{\dot{x}}{v}$. But the general equation of the fluxions of the space x and velocity v, in the last prob. was $\frac{c-x}{c+x}\dot{x} = \frac{w}{32} \times \frac{-v\dot{v}}{av^2 + w}$; ther. $\dot{x} = \frac{w}{32} \times \frac{c+x}{c-x} \times \frac{-v\dot{v}}{av^2 + w}$; hence $\dot{t} = \frac{\dot{x}}{v} = \frac{w}{32} \times \frac{c+x}{c-x} \times \frac{-\dot{v}}{av^2 + w}$. But x, which is always small in respect of c, is nearly $= b$ as determined in the last problem; theref. $\frac{c+b}{c-b}$ may be substituted for $\frac{c+x}{c-x}$ without sensible error; and then $\dot{t}$ becomes $= \frac{w}{32} \times \frac{c+b}{c-b} \times \frac{-\dot{v}}{av^2 + w}$. Now, this fluxion being to that in prob. 6, in the constant ratio of $c - b$ to $c + b$, their fluents will be also in the same constant ratio. But, by the last prob. $c = 54000$, and $b = 2955$ for the first example in prob. 5; therefore $c - b = 51045$, and $c + b = 56955$, also, the time in problem 6 was $9''\cdot91$; therefore as $51045 : 56955 :: 9''\cdot91 : 11''\cdot04$ for the time in this case, being $1''\cdot13$ more than the former, or nearly the 9th part more; which is nearly the double, or as the square of the difference, in the last prob. in the height ascended.

PROBLEM IX.

To determine the circumstances of Space, Time, and Velocity, of a Ball Descending through the Atmosphere by its own Weight.

It is here meant that the balls are at least as heavy as cast iron, and therefore their loss of weight in the air insensible; and that their motion commences by their own gravity from a state of rest. The first object of enquiry may be, the utmost degree of velocity any such ball acquires by thus descending. Now it is manifest that the ball's motion is commenced, and uniformly increased, by its own weight, which is its constant urging force, being always the same, and producing an equal

increase of velocity in equal times, excepting for the diminution of motion by the air's resistance. It is also evident that this resistance, beginning from nothing, continually increases, in some ratio, with the increasing velocity of the ball. Now, as the urging force is constantly the same, and the resisting force always increasing, it must happen that the latter will at length become equal to the former: when this happens, there can afterwards be no further acceleration of the motion, the impelling force and the resistance being equal, and the ball must ever after descend with a uniform motion. It follows therefore that, to answer the first enquiry, we have only to determine when or what velocity of the ball will cause a resistance just equal to its own weight.

Now, by inspecting the tables of resistances preceding prob 1, particularly the 1st of the three tables, the weight of the ball being 1·05 lb, we perceive that the resistance increases in the 2d column, till 0·69 opposite to 200 velocity, and 1·56 answering to 300 velocity, between which two the proposed resistance 1·05, and the correspondent velocity, fall. But, in two velocities not greatly different, the resistances are very nearly proportional to the squares of the velocities. Therefore, having given the velocity 200 answering to the resistance 0·59, to find the velocity answering to the resistance 1·05, we must say, as $0·69 : 1·05 :: 200^2 : v^2 = 60870$, theref. $v = \surd 60870 = 246$, is the greatest velocity this ball can acquire; after which it will descend with that velocity uniformly, or at least with a velocity nearly approaching to 246.

The same greatest or uniform velocity will also be directly found from the rule $·00001725v^2 = r$, near the end of problem 2, where r is the resistance to the velocity v, by making $1·05 = r$; for then $v^2 = \frac{1·05}{·00001725} = 60870$, the same value for v^2 as before.

But now, for any other weight of ball; as the weights of the balls increase as the cubes of their diameters, and their resistances, being as the surfaces, increase only as the squares of the same, which is one power less; and the resistances being also in this case, as the squares of the velocities, we must therefore increase the squares of the velocity in the ratio of the diameters of the balls; that is, as $1·965 : d :: 246^2 : \frac{246^2}{1·965}d = v^2$, and hence $v = 246\sqrt{\frac{d}{1·965}} = 175\frac{1}{2}\sqrt{d}$.

If we take here the 3lb ball, belonging to the 2d table of resistances, whose diameter d is $=2·80$; then $\surd 2·80 = 1·673$, and $175\frac{1}{2} \times 1·67 = 294$, is the greatest or uniform velocity, with which the 3lb ball will descend. And if we take the

6lb

6lb ball, whose diameter is 3·53 inches, as in the 3d table of resistances: then $\sqrt{3 \cdot 53} = 1 \cdot 88$, and $175\frac{1}{2} \times 1 \cdot 88 = 330$, being the greatest velocity that can be acquired by the 6lb ball, and with which it will afterwards uniformly descend. For a 9lb ball, whose diameter is 4·04, the velocity will be $175\frac{1}{2} \times 2 \cdot 01 = 353$. And so on for any other size of iron ball, as in the following table. Where the first column contains the weight of the balls in lbs; the 2d their diameters in inches; the 3d their velocities to which they nearly approach, as a limit, and therefore called their terminal or last velocities, with which they afterward descend uniformly; and the 4th or last column the heights due to those velocities, or the heights from which the balls must descend in vacuo to acquire them.

Wt. lbs.	Diam. inch.	Term. Veloc. feet.	Height due to v, feet,
1	1·94	244	930
2	2·45	275	1182
3	2·80	294	1260
4	3·08	308	1482
6	3·53	330	1701
9	4·04	353	1958
12	4·45	370	2139
18	5·09	396	2450
24	5·60	415	2691
32	6·17	436	2970
36	6·41	444	3080
42	6·75	456	3249

But it is manifest that the balls can never attain exactly to these velocities in any finite time or descent, being only the limits to which they continually approach, without ever really reaching, though they arrive very nearly at them in a short space of time; as will appear by the following calculation.

To obtain general expressions for the space descended, and the time of the descent, in terms of the velocity v: put $x =$ any space descended, $t =$ its time, and v the velocity acquired, the weight of the ball $w = 1 \cdot 05$ lb. Now, by the theorem near the end of prob. 2, which is the proper rule for this case, the velocity being small, $\cdot 00001725v^2 = cv^2$ is the resistance due to the velocity v; theref. $w - cv^2$ is the impelling force, and $\frac{w - cv^2}{w} = f$ the accelerating force; conseq. $v\dot{v}$ or $2gf\dot{x} = 2g\dot{x} \times \frac{w - cv^2}{w}$, and $\dot{x} = \frac{w}{2g} \times \frac{v\dot{v}}{w - cv^2}$, the correct fluent of which, by the 8th form, is $x = \frac{w}{4gc} \times$ h. l. $\frac{w}{w - cv^2}$ the general value of the space x descended.

Here it appears that the denominator $w - cv^2$ decreases as v increases; conseq. the whole value of x, the descent, increases with v, till it becomes infinite, when the resistance cv^2

cv^2 is $= w$ the weight of the ball, when the motion becomes uniform, as before remarked. We may however easily assign the value of x a little before the velocity becomes uniform, or before cv^2 becomes $= w$. Thus, when $cv^2 = w$, then $v = 246$, as found in the beginning of this problem. Assume therefore v a little less than that greatest velocity, as for instance 240: then this value of v substituted in the general formula for x above deduced, gives $x = 2781$ feet, a little before the motion becomes uniform, or when the velocity has arrived at 240, its maximum being 246.

In like manner is the space to be computed that will be due to any other velocity less than the greatest or terminal velocity. On the contrary, to find the velocity due to any proposed space x, from the formula $x = \frac{w}{4gc} \times$ h. l. $\frac{w}{w - cv^2}$. Here x is given, to find v. First then $\frac{4gcx}{w} =$ h. l. $\frac{w}{w - cv^2}$; take therefore the number to the hyp. log. of $\frac{4gcx}{w}$, which number call N; then $N = \frac{w}{w - cv^2}$; conseq. $Nw - Ncv^2 = w$, and $Nw - w = Ncv^2$, and $v = \sqrt{\frac{N-1}{Nc}w}$, a general theorem for the value of v due to any distance x. Suppose, for instance, x is 1000. Now $4g$ being $= 64$, $w = 1{\cdot}05$, and $c = {\cdot}00001725$; theref. $\frac{4gcx}{w} = 1{\cdot}0514$, and the natural number belonging to this, considered as an hyp. log. is $2{\cdot}8617 = N$; hence then $v = \sqrt{\frac{N-1}{Nc}w} = 199$, is the velocity due to the space 1000, or when the ball has descended 1000 feet.

Again, for the time t of descent: here $\dot{t} = \frac{\dot{x}}{v}$; but $\dot{x} = \frac{w}{2g} \times \frac{v\dot{v}}{w - cv^2}$, as found above, theref. $\dot{t} = \frac{w}{2g} \times \frac{\dot{v}}{w - cv^2}$, the fluent of which is $\frac{1}{4g}\sqrt{\frac{w}{c}} \times$ h. l. $\frac{\sqrt{\frac{w}{c}} + v}{\sqrt{\frac{w}{c}} - v}$, the general value of the time t for any value of the velocity v; which value of t evidently increases as the denominator $\sqrt{\frac{w}{c}} - v$ decreases, or as the velocity v increases; and consequently the time is infinite when that denominator vanishes, which is

is when $v = \sqrt{\frac{w}{c}}$, or $cv^2 = w$, the resistance equal to the ball's weight, being the same case as when the space x becomes infinite, as above remarked. But, like as was done for the distance x as above, we may here also find the value of t corresponding to any value of v, less than its maximum 246, and consequently to any value of x, as when v is 240 for instance, or $x = 2781$, as determined above. Now, by substituting 240 for v, in the general formula

$t = \frac{1}{4g}\sqrt{\frac{w}{c}} \times \text{h.l.}\, \frac{\sqrt{\frac{w}{c}} + v}{\sqrt{\frac{w}{c}} - v}$, it brings out $t = 16''{\cdot}575$; so that it would be nearly $16\frac{1}{2}$ seconds when the velocity arrives at 240, or a little less than the maximum or uniform degree, viz, 246, or when the space descended is 2781 feet.

Also, to determine the time corresponding to the same, or when the descent is 1000 feet, or the velocity 199: find the value of $\frac{1}{4g}\sqrt{\frac{w}{c}} = \frac{1}{64}\sqrt{\frac{1{\cdot}05}{{\cdot}00001725}} = \frac{246}{64} = \frac{123}{32}$. Then

$\frac{\sqrt{\frac{w}{c}} + v}{\sqrt{\frac{w}{c}} - v} = \frac{246+199}{246-199} = \frac{445}{47}$; the hyp. log. of which is 2·2479.

Hence $2{\cdot}2479 \times \frac{123}{32} = 8''{\cdot}64$, the time of descending 1000 feet, or when the velocity is 199.

See other speculations on this problem, in Prob. 22, Projectiles, as determined from theory, viz, without using the experimented resistance of the air.

PROBLEM X.

To determine the Circumstances of the Motion of a Ball projected Horizontally in the Air; abstracted from its Vertical Descent by its Gravitation.

Putting d for the diameter, and w the weight of the ball, V the velocity of projection, and v the velocity of the ball after having moved through the space x. Then by corol. 1 to prob. 2, if the velocity is considerable, such as usual in practice, the resistance of the ball, moving with the velocity v, is $(mv^2 - nv)d^2$, and therefore $\frac{mv^2 - nv}{w}d^2$ is the retardive force f; hence the common formula $v\dot{v} = 2gf\dot{x}$, is $-v\dot{v} = 32\dot{x} \times \frac{mv^2 - nv}{w}d^2$, and theref. $\dot{x} = \frac{w}{32d^2} \times \frac{-v\dot{v}}{mv^2 - nv} = \frac{w}{32d^2} \times \frac{-\dot{v}}{mv - n} = \frac{w}{32d^2 m} \times \frac{-\dot{v}}{v - \frac{n}{m}}$, the fluent of which is obviously

$\frac{w}{32md^2} \times -$ hyp. log. of $v - \frac{n}{m}$, and by the correction by the first velocity v, it becomes $x = \frac{w}{32md^2} \times$ h. log. $\frac{\text{v} - \frac{n}{m}}{v - \frac{n}{m}}$, the general formula for the distance passed over in terms of the velocity.

Now, for an application, let it be required first, to determine in what space a 24lb ball will have its velocity reduced from 1780 feet to 1500, that is, losing 280 feet of its first velocity. Here, $d = 5{\cdot}6$, $w = 24$, $\text{v} = 1780$, and $v = 1500$; also $\frac{n}{m} = 150$. Hence $\frac{w}{16md^2} = 3587{\cdot}4$, then $x = 3587{\cdot}4 \times$ h. l. $\frac{\text{v} - 150}{v - 150} = 3587{\cdot}4 \times$ h. l. $\frac{1630}{1350} = 3587{\cdot}4 \times$ h. l. $\frac{163}{135} =$ 676 feet, the space passed over when the ball has lost 280 feet of its motion.

Again, to find with what velocity the same ball will move, after having described 1000 feet in its flight. The above theorem is x or $1000 = 3587{\cdot}4 \times$ h. l. $\frac{\text{v} - 150}{v - 150} = 3587{\cdot}4 \times$ h. l. $\frac{1630}{v - 150}$, or $\frac{10000}{35874} =$ h. l. $\frac{1630}{v - 150}$; but the number to the hyp. log. $\frac{10000}{35874}$ is $1{\cdot}7416 = \text{N}$ suppose; then $\text{N} = \frac{1630}{v - 150}$, and $\text{N}v - 150\text{N} = 1630$, or $\text{N}v = 1630 + 150\text{N}$, and $v = \frac{1630}{\text{N}} - 150 = 936 - 150 = 786$, the velocity when the ball has moved 1000 feet.

Next, to find a theor. for the time of describing any space, or destroying any velocity: Here $\dot{t} = \frac{\dot{x}}{v} = \frac{w}{32md^2} \times \frac{-v^{-1}\dot{v}}{v - \frac{n}{m}}$ the fluent of which, by the 9th form, is $t = \frac{w}{32md^2} \times \frac{m}{n} \times$ h. l. $\frac{v}{v - \frac{n}{m}} = \frac{w}{32nd^2} \times$ h. l. $\frac{v}{v - \frac{n}{m}}$, and by correction $t = \frac{w}{32nd^2} \times$ (h. l. $\frac{v}{v - \frac{n}{m}} -$ h. l. $\frac{\text{v}}{v - \frac{n}{m}}) = \frac{w}{32nd^2} \times$ hyp. log. $\frac{\text{v} - 150}{v - 150} \cdot \frac{v}{\text{v}}$, putting v for the first velocity, and 150 for $\frac{n}{m}$ its value, as before.

Now, to take for an example the same 24lb ball, and its projected

projected velocity 1780, as before; let it be required to find in what time this velocity will be reduced to 786. Here then $V = 1780$, $v = 786$, $w = 24$, $d = 5\cdot6$, $d^2 = 31\cdot36$, $n = \cdot001$; hence $\frac{w}{32nd^2} = \frac{750}{31\cdot36} = 23\cdot916$; and $\frac{V-150}{v-150} \cdot \frac{v}{V} = \frac{1630}{636} \times \frac{786}{1780} = \frac{21353}{18868}$, the hyp. log. of which is $\cdot1099$; then $31\cdot36 \times \cdot1099 = 2''\cdot628$, the time required.

For another example, let it be required to find when the velocity will be reduced to 1000, or 780 destroyed. Here $v = 1000$, and all the other quantities as before. Then $\frac{V-150}{v-150} \times \frac{v}{V} = \frac{1630}{850} \times \frac{1000}{1780} = \frac{1630}{1513}$, the hyp. log. of which is $\cdot07449$; theref. $31\cdot36 \times \cdot07449 = 1''\cdot78$, is the time sought.

On the other hand, if it be required to find what will be the velocity after the ball has been in motion during any given time, as suppose 2 seconds, we must reverse the calculation thus: $t = 2''$ being $= \frac{w}{32nd^2} \times$ h. l. $\frac{V-150}{v-150} \cdot \frac{v}{V} = 23\cdot916 \times$ h. l. $\frac{V-150}{v-150} \cdot \frac{v}{V}$; theref. $\frac{2}{23\cdot916} = \cdot083626$ is the hyp. log. of $\frac{V-150}{v-150} \cdot \frac{v}{V}$, the number answering to which is $1\cdot08725 = N$ suppose, that is, $N = \frac{V-150}{v-150} \cdot \frac{v}{V}$. Hence $NVv - 150\,NV = Vv - 150v$, and $v = \frac{150NV}{150 + NV - V} = \frac{290290}{305\cdot305} = 951$, the velocity at the end of 2 seconds.

The foregoing calculations serve only for the higher velocities, such as exceed 200 or 300 feet per second of time. But, for those that are below 300, the rule is simpler, as the resistance is then, by cor. 2 prob. 2, $\cdot00000447d^2v^2 = cd^2v^2$, where d denotes the diameter of any ball. Hence then, employing the same notation as before, $\frac{cd^2v^2}{w} = f$, and $-v\dot{v} = 32f\dot{x} = 32\dot{x} \times \frac{cd^2v^2}{w}$; theref. $\dot{x} = \frac{w}{32cd^2} \times \frac{-\dot{v}}{v}$, the correct fluent of which is $x = \frac{w}{32cd^2} \times$ h. l. $\frac{V}{v}$.

Now, for an example, suppose the first velocity to be $300 = V$, and the last $v = 100$, for a 24lb ball. Then $w = 24$, $d = 5\cdot6$, $d^2 = 31\cdot36$, $c = \cdot00000447$; therefore $\frac{w}{32cd^2} = \frac{3}{125\cdot44c} = 5350$; and $\frac{V}{v} = \frac{300}{100} = 3$, the hyp. log. of which is $1\cdot0986$; theref. $1\cdot0986 \times 5350 = 5878 = x$, is the distance.—If the first velocity be only $200 = V$; then

V

$\frac{V}{v} = 2$, the hyp. log. of which is ·69315, therefore ·69315 $\times$ 5350 = 3708 = x, the distance.

And conversely, to find what velocity will remain after passing over any space, as 4000 feet, the first velocity being $V = 200$. Here the hyp. log. of $\frac{V}{v}$ is $\frac{x}{5350} = \frac{4000}{5350} = \frac{400}{535} = \frac{80}{107} =$ ·74766, the natural number of which is 2·1120, that is, $2{\cdot}112 = \frac{V}{v}$; therefore $v = \frac{V}{2{\cdot}112} = \frac{200}{2{\cdot}112} = 94{\cdot}7$, the velocity.

Again, for the time t: since $\dot{x} = \frac{w}{32cd^2} \times \frac{-\dot{v}}{v}$, therefore $\dot{t} = \frac{\dot{x}}{v} = \frac{w}{32cd^2} \times \frac{-\dot{v}}{v^2}$, the correct fluent of which is $t = \frac{w}{32cd^2} \times (\frac{1}{v} - \frac{1}{V}) = \frac{w}{32cd^2} \times \frac{V-v}{Vv}$.—So, for example, if $V = 300$, and $v = 100$; then $\frac{V-v}{Vv} = \frac{200}{30000} = \frac{2}{300}$; then $\frac{w}{32cd^2}$ or $5350 \times \frac{2}{300} = 35''\frac{2}{3} = t$, the time of reducing the 300 velocity to 100, or of passing over the space 5878 feet.

And, reversing, to find the velocity v, answering to any given time t: Since $t = \frac{w}{32cd^2} \times (\frac{1}{v} - \frac{1}{V}) = 5350 \times (\frac{1}{v} - \frac{1}{V})$, theref. $v = \frac{5350}{5350 + tV}$. Here, if t be given $= 30''$, and $V = 300$; then $v = \frac{5350V}{5350+9000} = \frac{535}{1435} \times 300 = \frac{32100}{287} =$ 112, the velocity sought.

Corol. The same form of theorem, $x = \frac{w}{32cd^2} \times \text{h. l.} \frac{V}{v}$ as above, is brought out for small velocities, will also serve for the higher ones, if we employ the medium resistance between the two proposed velocities, as was done in prob. 5. Thus, as in the first example of this problem, where the two velocities are 1780 and 1500, the resistance due to the velocity 1700, in the first table of resistances, being 74·13, say as $1700^2 : 1780^2 :: 74{\cdot}13 : 81{\cdot}27$, the resistance due to the velocity 1780; then the mean between 81·27 and 57·25, due to 1500 velocity, is 69·26, or rather take $69\frac{1}{4}$. Again, as $\sqrt{65{\cdot}7} : \sqrt{69\frac{1}{4}} :: 1600 : 1646$, the velocity due to the medium resistance $69\frac{1}{4}$. Hence, as in prob. 5, as $1646^2 : v^2 :: 69\frac{1}{4} : {\cdot}00002565v^2 =$ suppose av^2, the resistance due to any velocity

velocity v, between 1780 and 1500, for the 1·05lb ball. And, as $1\cdot965^2 : 5\cdot6^2 :: av^2 : 8\cdot124av^2 \doteq \cdot00020838v^2 = bv^2$ suppose, the resistance due to the same velocity with the 24lb ball. Therefore $\frac{bv^2}{24} = f$, and $-v\dot{v} = 32f\dot{x} \doteq \frac{4}{3}bv^2\dot{x}$, and $\dot{x} = \frac{-3\dot{v}}{4bv}$, the correct fluent of which is $\frac{3}{4b} \times$ h. l. $\frac{V}{v} = \frac{3}{4b} \times$ h. l. $\frac{178}{150} = \frac{3}{4b} \times$ h. l. $\frac{89}{75} = 3600 \times \cdot171148 = 616$ the velocity sought.

PROBLEM XI.

To determine the Ranges of Projectiles in the Air.

To determine, by theory, the trajectory a projectile describes in the air, is one of the most difficult problems in the whole course of dynamics, even when assisted by all the experiments that have hitherto been made on this branch of physics; and is indeed much too difficult for this place, in the full extent of the problem: the consideration of it must therefore be reserved for another occasion, when the nature of the air's resistance can be more amply discussed. Even the solutions of Newton, of Bernoulli, of Euler, of Borda, &c, &c, after the most elaborate investigations, assisted by all the resources of the modern analysis, amount to no more than distant approximations, that are rendered nearly useless, even to the speculative philosopher, from the assumption of a very erroneous law of resistance in the air, and much more so to the practical artillerist, both on that account, and from the very intricate process of calculation, which is quite inapplicable to actual service. The solution of this problem requires, as an indispensable datum, the perfect determination by experiment of the nature and laws of the air's resistance at different altitudes, to balls of different sizes and densities, moving with all the usual degrees of celerity. Unfortunately however, hardly any experiments of this kind have been made, excepting those which on some occasions have been published by myself, as in my Tracts of 1786, as well as in my Dictionary, some few of which are also given in art 105 of Mot. and Forces, with some practical inferences. And though I have many more yet to publish, of the same kind, much more extensive and varied, I cannot yet undertake to pronounce that they are fully adequate to the purpose in hand.

All that can be here done then, in the solution of the present problem, besides what is delivered in this volume, is to collect together some of the best practical rules, founded

partly on theory, and partly on practice. 1. In the first place then, it may be remarked, that the initial or first velocity of a ball may be directly computed by prob. 17, page 393 of this volume; having given the dimensions of the piece, the weight of the ball, and the charge of powder. Or otherwise, the same may be made out from the table of experimented ranges and velocities in pa. 141 of this volume, by this rule, that the velocities to different balls, and different charges of powder, are as the square roots of the weights of the powder directly, and as the square roots of the weights of the balls inversely. Thus, if it be enquired, with what velocity a 24lb ball will be discharged by 8lb of powder. Now it appears in the table, that 8 ounces of powder discharge the 1lb ball with 1640 feet velocity; and because 8lb are = 128 ounces; therefore by the rule, as $\sqrt{\frac{8}{1}} : \sqrt{\frac{128}{24}} :: 1640 : 1640\sqrt{\frac{16}{24}} = 1640\sqrt{\frac{2}{3}} = 1339$, the velocity sought. Or otherwise, by rule 1, p. 142 of this vol. as $\sqrt{24} : \sqrt{16} :: 1600 : 1306$, the same velocity nearly. But when the charges bear the same ratio to one another as the weight of the balls, that is when the pieces are said to be alike charged, then the velocities will be equal. Thus, the 1lb ball by the 2 oz charge, being the 8th part of the weight, and the 24lb ball, with 3lb of powder, its 8th part also, will have the same velocity, viz, 860 feet. In like manner, the 1230 tabular velocity, answering to 4 oz of powder, the 4th part of the ball, will equally belong to the 24lb ball with 6lb of powder, being its 4th part, and the tabular velocity 1640, answering to the 8oz charge, which is ½ the weight of ball, will equally belong to the 24lb ball with 12lb of powder, being also the ½ of its weight.

2. By prob. 9 will be found what is called the *terminal velocity*, that is, the greatest velocity a ball can acquire by descending in the air; indeed a table is there given of the several terminal velocities belonging to the different balls, with the heights, in an annexed column, due to those velocities in vacuo, that is the heights from which a body must fall in vacuo, to acquire those velocities.

3. Given the initial velocity, to find the elevation of the piece to have the greatest range, and the extent of that range. These will be found by means of the annexed table, altered

from

from Professor Robison's in the Encyclopædia Britannica, and founded on an approximation of Sir I. Newton's. The numbers in the first column, multiplied by the terminal velocity of the ball, give the initial velocity; and the numbers in the last column, being multiplied by the height, give the greatest ranges; the middle column showing the elevations to produce those ranges.

Table of Elevations giving the Greatest Range.

Initial vel div. by *v*.	Elevation.	Range div. by *a*.
0·6910	44° 0′	0·3914
0·9445	43 15	0·5850
1·1980	42 30	0·7787
1·4515	41 45	0·9724
1·7050	41 0	1·1661
1·9585	40 15	1·3598
2·2120	39 30	1·5535
2·4655	38 45	1·7472
2·7190	38 0	1·9409
2 9725	37 15	2·1346
3·2260	36 30	2·3283
3·4795	35 45	2·5220
3·7330	35 0	2·7157
3·9865	34 15	2·9094
4·2400	33 30	3·1031
4·4935	32 45	3·2968
4·7470	32 0	3·4905
5·0000	31 15	3·6842

To use this table then, divide the given initial velocity by the terminal velocity peculiar to the ball, found in the table in prob. 9, and look for the quotient in the first column here annexed. Against this, in the 2d column will be found the elevation to give the greatest range; and the number in the 3d column multiplied by a, the altitude due to the terminal velocity, also found in the table in problem 9, will give the range, nearly.

Ex. 1. Let it be required to find the greatest range of a 24lb ball, when discharged with 1640 feet velocity, and the corresponding angle to produce that range. By the table in prob. 9, the terminal velocity of the 24lb ball is 415, and its producing altitude 2691: hence $\frac{1640}{415} = 3{\cdot}95$, nearly equal to 3·9865 in the 1st column of our table, to which corresponds the angle 34° 15′, being the elevation to produce the greatest range; and the corresponding number 2·9094, in the 3d column, multiplied by 2691′, gives 7829 feet, for the greatest range, being nearly a mile and a half.

Exam. 2. In like manner, the same ball discharged with the velocity 860 feet, will have for its greatest range 3891 feet, or nearly ¾ of a mile, and the elevation producing it 39° 55′.

These examples, and indeed the whole table in the 9th problem,

problem, are only adapted to the use of cannon balls. But it is not usual, and indeed not easily practicable, to discharge cannon shot at such elevations, in the British service, that practice being the peculiar office of mortar shells. On this account then it will be necessary to make out a table of terminal velocities, and altitudes due to them, for the different sizes of such shells. The several kinds of these in present use, are denominated, from the diameters of their mortar bores in inches, being the five following, viz, the 4·6, the 5·8, the 8, the 10, and the 13 inch mortars, as in the first column of the following table. But the outer diameters of the shells are somewhat smaller, to leave a little room or space as windage, as contained in the 2d column.

Table of dimensions, &c, of Mortar Shells.

Diam. of Mortar.	Diam. of Shells.	Weight of Shells filled.	Weight of equal solid.	Ratio of shell to solid.	Terminal velocity.	Alt. due to veloc.
inch.	inch.	lbs.	lbs.		feet.	feet.
4·6	4·53	9	$12\frac{3}{4}$	1·42	314	1541
5·8	5·72	18	$25\frac{1}{2}$	1·42	352	1936
8	7·90	47	67	1·42	414	2678
10	9·84	$91\frac{1}{2}$	130	1·42	462	3335
13	12·80	201	286	1·42	527	4340

The 3d column contains the weight of each shell when the hollow part is filled with powder: the diameter of the hollow is usually $\frac{7}{10}$ of that of the mortar: the weight of the shells empty and when filled, with other circumstances, may be seen at quest. 53, in Mensuration, end of vol. 1. On account of the vacuity of the shell being filled only with gunpowder, the weight of the whole so filled, and contained in column 3, is much less than the weight of the same size of solid iron, and the corresponding weights of such equal solid balls are contained in col. 4. The ratio of these weights, or the latter divided by the former, occupies the 5th column.

Now because the loaded or filled shells are of less specific gravity, or less heavy, than the equal solid iron balls, in the ratio of 1 to 1·42, as in column 5, the former will have less power or force to oppose the resistance of the air, in that same proportion, and the terminal or greatest velocity, as determined in the 9th prob. will be correspondently less. Therefore, instead of the rule there given, viz, $175·5\sqrt{d}$, for that velocity, the rule must now be $175·5\sqrt{\frac{d}{1·42}} = 147·3\sqrt{d} = v$, the

the diameter of the shell being d; that is, the terminal velocities will be all less in the ratio of 147·3 to 175·5. Now, computing these several velocities by this rule, to all the different diameters, they are found as placed in the 6th col.; and in the 7th or last column are set the altitude which would produce these velocities in vacuo, as computed from this theorem $\frac{vv}{64}$.

Having now obtained these terminal velocities, and their producing altitudes, for the shells, we can, from them and the former table of ranges and elevations, easily compute the greatest range, and the corresponding angle of elevation, for any mortar and shell, in the same way as was done for the balls in this problem. Thus, for example, to find the greatest range and elevation, for the 13 inch shell, when projected with the velocity of 2000 feet per second, being nearly the greatest velocity that balls can be discharged with. Now, by the method before used $\frac{2000}{527} = 3·796$; opposite to this, found in the first column of the table of ranges, corresponds 34°49′ for the elevation in the 2d column, and the number 2·764 in the 3d column; this multiplied by the altitude 4340, gives 11995 feet, or more than $2\frac{1}{4}$ miles, for the greatest range.

This however is much short of the distance which it is said the French have lately thrown some shells at the siege of Cadiz, viz, 3 miles, which it seems has been effected by means of a peculiar piece of ordnance, and by loading or filling the cavity of the shell with lead, to render it heavier, and thus make it fitter to overcome the resistance of the air. Let us then examine what will be the greatest range of our 13 inch shell, if its usual cavity be quite filled with lead when discharged, with the projectile velocity of 2000 feet.

Now the diameter of the cavity, being about $\frac{7}{10}$ of that of the mortar 13, will be nearly 9 inches. And the weight of a globe of lead of this diameter is 139·3lb; which added to 187·8, the weight of the shell empty, gives 327lb, the whole weight of the shell when the cavity is filled with lead, which was found 286 when supposed all of solid iron, their ratio or quotient is ·8783. Then, as before, the theorem will be $175·5\sqrt{\frac{d}{·8783}} = 187·3\sqrt{d}$ for the terminal velocity; which, when $d = 12·8$, becomes 670 for the terminal velocity; therefore its producing altitude is $\frac{670^2}{64} = 7014$. Then, by the same method as before, $\frac{2000}{670} = 2·985$; which number found

found in the first column of the table of ranges, the opposite number in the 2d col. is 37° 15′ for the elevation of the piece, and in the 3d column 2·14, multiplied by 7014, gives 15010 feet, or nearly 3 miles. So that our 13 inch shells, discharged at an elevation of about 37¼ degrees, would range nearly the distance mentioned by the French, when filled with lead, if they can be projected with so much as 2000 feet velocity, or upwards. This however it is thought cannot possibly be effected by our mortars; and that it is therefore probable the French, to give such a velocity to those shells, must have contrived some new kind of large cannon on the occasion.

4. Having shown in the preceding articles and problems, how, from our theory of the air's resistance, can be found, first the initial or projectile velocity of shot and shells; 2dly, the terminal velocity, or the greatest velocity a ball can acquire by descending by its own weight in the air; 3dly, the height a ball will ascend to in the air, being projected vertically with a given velocity, also the time of that ascent; 4thly, the *greatest* horizontal ranges of given shot, projected with a given velocity; as also the particular angle of elevation of the piece, to produce that greatest range. It remains then now to enquire, what laws and regulations can be given respecting the ranges, and times of flight, of projects made at other angles of elevation.

Relating to this enquiry, the Encyclopædia Britannica mentions the two following rules: 1st. "Balls of equal density, projected with the same elevation, and with velocities which are as the square roots of their diameters, will describe similar curves. This is evident, because, in this case, the resistance will be in the ratio of their quantities of motion; therefore all the homologous lines of the motion will be in the proportion of the diameters." But though this may be nearly correct, yet it can hardly ever be of any use in practice, since it is usual and proper to project small balls, not with a less, but with a greater velocity, than the larger ones. 2dly, the other rule is, "If the initial velocities of balls, projected with the same elevation, be in the *inverse* subduplicate ratio of the whole resistances, the ranges, and all the homologous lines in their track, will be inversely as those resistances." This rule will come to the same thing, as having the initial velocities in the inverse ratio of the diameters, as distant perhaps from fitness as the former. Two tables are next given in the same place, for the comparison of ranges and projectile velocities, the numbers in which appear to be much wide of the truth, as depending on very erroneous effects of the resistance. Most of the accompanying remarks, however,

however, are very ingenious, judicious, and philosophical, and very justly recommending the making and recording of good experiments on the ranges and times of flight of projects, of various sizes, made with different velocities, and at various angles of elevation.

Besides the above, we find rules laid down by Mr. Robins and Mr. Simpson, for computing the circumstances relating to projectiles as affected by the resistance of the air. Those of the former respectable author, in his ingenious Tracts on Gunnery, being founded on a quantity which he calls *r*, (answering to our letter *a* in the foregoing pages), I find to be almost uniformly double of what it ought to be, owing to his improper measures of the air's resistance; and therefore the conclusions derived by means of those rules must needs be very erroneous. Those of the very ingenious Mr. Simpson, contained in his Select Exercises, being partly founded on experiment, may bring out conclusions in some of the cases not very incorrect; while some of them, particularly those relating to the impetus and the time of flight, must be very wide of the truth. We must therefore refer the student, for more satisfaction, to our rules and examples before given in pa. 142 this vol. &c, especially for the circumstances of different ranges and elevations, &c, after having determined, as above, those for the greatest ranges, founded on the real measure of the resistances.

CHAPTER

CHAPTER XIV.

PROMISCUOUS PROBLEMS, AS EXERCISES IN MECHANICS, STATICS, DYNAMICS, HYDROSTATICS, HYDRAULICS, PROJECTILES, &c. &c.

PROBLEM I.

Let AB *and* AC *be two inclined planes, whose common altitude* AD *is given = 64 feet; and their lengths such, that a heavy body is 2 seconds of time longer in descending through* AB *than through* AC, *by the force of gravity; and if two balls, the one weighing 3 and the other 2lb, be connected by a thread and laid on the planes, the thread sliding freely over the vertex* A, *they will mutually sustain each other. Quere the lengths of the two planes.*

The lengths of planes of the same height being as the times of descent down them (art 133 this vol.), and also as the weights of bodies mutually sustaining each other on them (art. 122), therefore the times must be as the weights; hence as 1, the difference of the weights, is to 2 sec. the diff. of times, :: $\left\{\begin{matrix} 3 : 6 \text{ sec.} \\ 2 : 4 \text{ sec.} \end{matrix}\right\}$ the times of descending down the two planes. And as $\sqrt{16} : \sqrt{64} :: 1$ sec. : 2 sec. the time of descent down the perpendicular height (art 70,). Then, by the laws of descents (art. 132), as 2 sec. : 64 feet $\left\{\begin{matrix} 6 \text{ sec.} : 192 \\ 4 \text{ sec.} : 128 \end{matrix}\right\}$ feet, the lengths of the planes.

Note. In this solution we have considered 16 feet as the space freely descended by bodies in the 1st second of time, and 32 feet as the velocity acquired in that time, omitting the fractions $\frac{1}{12}$ and $\frac{1}{6}$, to render the numeral calculations simpler, as was done in the preceding chapter on projectiles, and as we shall do also in solving the following questions, wherever such numbers occur.

Another Solution by means of Algebra.

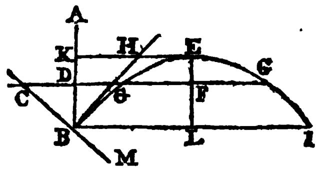
Put x = the time of descent down the less plane; then will $x + 2$ be that of the greater, by the question. Now the weights being as the lengths of the planes, and these again as the times, therefore as $2 : 3 :: x : x + 2$; hence

$2x +$

$2x + 4 = 3x$, and $x = 4$ sec. Then the lengths of the planes are found as in the last proportion of the former solution.

PROBLEM 2.

If an elastic ball fall from the height of 50 feet above the plane of the horizon, and impinge on the hard surface of a plane inclined to it in an angle of 15 degrees; it is required to find what part of the plane it must strike, so that after reflection, it may fall on the horizontal plane, at the greatest distance possible beyond the bottom of the inclined plane?



Here it is manifest that the ball must strike the oblique plane continued on a point somewhere below the horizontal plane; for otherwise there could be no maximum. Therefore let BC be the inclined plane, CDG the horizontal one, B the point on which the ball impinges after falling from the point A, BEGI the parabolic path, E its vertex, BH a tangent at B, being the direction in which the ball is reflected; and the other lines as are evident in the figure. Now, by the laws of reflection, the angle of incidence ABC, is equal to the angle of reflection HBM, and therefore this latter, as well as the former, is equal to the complement of the $\angle$ C the inclination of the two planes; but the part IBM is $= \angle$ C, therefore the angle of projection HBI is = the comp. of double the $\angle$ C, and being the comp. of HBK, theref. $\angle$ HBK $= 2 \angle$ C. Now, put $a = 50 =$ AD the height above the horizontal line, $t =$ tang. $\angle$ DBC or 75° the complement of the plane's inclination, $\tau =$ tang. HBI or $\angle$ H $= 60°$ the comp. of $2\angle$ C, $s =$ sine of $2\angle$ HBI $= 120°$ the double elevation, or = sine of $4\angle$ C; also $x =$ AB the impetus or height fallen through. Then,

$$\text{BI} = 4\text{KH} = 2sx, \text{ by the projectiles prop. 21,}$$

$$\text{and} \left\{ \begin{array}{l} \text{BK} = \tau \times \text{KH} = \tfrac{1}{2}s\tau x \\ \text{CD} = t \times \text{BD} = t\,(x-a). \end{array} \right\} \text{ by trigonometry;}$$

also, $\text{KD} = \text{BK} - \text{BD} = \frac{1}{2}s\tau x - x + a$, and $\text{KE} = \frac{1}{2}\text{BI} = sx$; then, by the parabola, $\sqrt{\text{BK}} : \sqrt{\text{DK}} :: \text{KE} : \text{FG} = \text{KE} \times$ $\sqrt{\frac{\text{KD}}{\text{KD}}} = \sqrt{\frac{\tau s^2x^2 - 2sx^2 + 2asx}{\tau}} = \sqrt{[\frac{2s}{\tau}ax - (\frac{2s}{\tau} - s^2)x^2]} =$ $2b\sqrt{(ax - b^2x^2)}$, putting $b =$ sine of $2\angle$ C = sine of 30°. Hence $\text{CG} = \text{CD} + \text{DF} \pm \text{FG} = tx - ta + sx \pm 2b\sqrt{(ax - b^2x^2)}$ a maximum, the fluxion of which made = 0, and the equation reduced, gives $x = \frac{a}{2b^2} \times (1 \pm \sqrt{\frac{n^2}{(n^2+4b^4)}}$, where $n = s$

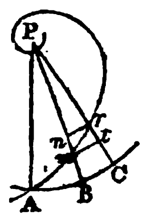
$+t$, and the double sign $\pm$ answers to the two roots or values of x, or to the two points G, G, where the parabolic path cuts the horizontal line CG, the one in ascending and the other in descending.

Now, in the present case, when the $\angle$C $= 15°$, $t =$ tang. $75° = 2 + \sqrt{3}$, $\tau =$ tan. $60° = \sqrt{3}$, $s =$ sin. $60° = \frac{1}{2}\sqrt{3}$, $b =$ sin. $30° = \frac{1}{2}$, $n = s + t = 2 + \frac{3}{2}\sqrt{3}$; then $\frac{a}{2b^2} = 2a = 100$, and $\frac{n^2}{n^2+4b^4} = \frac{n^2}{n^2+\frac{1}{4}} = \frac{41+6\sqrt{3}}{52}$; theref. $x = \frac{a}{2b^2} \times (1 \pm \sqrt{\frac{n^2}{n^2+4b^4}})$ $= 100 \times (1 \pm \frac{1}{2}\sqrt{\frac{41+6\sqrt{3}}{13}}) = 100 \times (1 \pm \cdot 99414) = 199 \cdot 414$ or $\cdot 586$; but the former must be taken. Hence the body must strike the inclined plane at 149·414 feet below the horizontal line; and its path after reflection will cut the said line in two points; or it will touch it when $x = \frac{a}{bb}$. Hence also the greatest distance CG required is 826·9915 feet.

Corol. If it were required to find CG or $tx - ta + sx \pm 2b\sqrt{(ax - b^2x_2)} = g$ a given quantity, this equation would give the value of x by solving a quadratic.

PROBLEM 3.

Suppose a ship to sail from the Orkney Islands, in latitude 59° 3′ north, on a N. N. E. *course, at the rate of* 10 *miles an hour; it is required to determine how long it will be before she arrives at the pole, the distance she will have sailed, and the difference of longitude she will have made when she arrives there?*



Let ABC represent part of the equator; P the pole; AmrP a loxodromic or rhumb line, or the path of the ship continued to the equator; PB, PC, any two meridians indefinitely near each other; nr, or mt, the part of a parallel of latitude intercepted between them.

Put c for the cosine, and t for the tangent of the course, or angle nmr to the radius r; Am, any variable part of the rhumb from the equator, $= v$; the latitude Bm $= w$; its sine x, and cosine y; and AB, the dif. of longitude from A, $= z$. Then, since the elementary triangle mnr may be considered as a right-angled plane triangle, it is, as rad. $r : c =$ sin. $\angle\, mrn :: \dot{v} = mr : \dot{w} = mn :: \dot{v} : \dot{w}$; theref. $c\dot{v} = r\dot{w}$, or $\dot{v} = \frac{r\dot{w}}{c} = \frac{s\dot{w}}{r}$, by putting s for the secant of the $\angle nmr$ the ship's course. In like manner,

ner, if W be any other latitude, and V its corresponding length of the rhumb; then $V = \frac{rW}{c}$; and hence $V - v = r \times \frac{W - w}{c}$, or $D = \frac{rd}{c}$, by putting $D = V - v$ the distance, and $d = W - w$ the dif. of latitude; which is the common rule.

The same is evident without fluxions: for since the $\angle\, mrn$ is the same in whatever point of the path AmrP the point m is taken, each indefinitely small particle of AmrP, must be to the corresponding indefinitely small part of Bm, in the constant ratio of radius to the cosine of the course; and therefore the whole lines, or any corresponding parts of them, must be in the same ratio also, as above determined. In the same manner it is proved that radius : sine of the course : : distance : the departure.

Again, as radius $r : t =$ tang. $nmr : : \dot{w} = mn : nr$ or mt, and as $r : y : :$ PB : P$m : : \dot{z} =$ BC : mt; hence, as the extremes of these proportions are the same, the rectangles of the means must be equal, viz, $y\dot{z} = t\dot{w} = \frac{tr\dot{x}}{y}$ because $\dot{w} = \frac{r\dot{x}}{y}$ by the property of the circle; theref. $\dot{z} = \frac{tr\dot{x}}{y^2} = \frac{tr\dot{x}}{r^2 - x^2}$; the general fluents of these are $z = t \times$ hyp. log. $\sqrt{\frac{r+x}{r-x}} + c$; which corrected by supposing $z = 0$ when $x = a$, are $z = t \times$ (hyp. log. $\sqrt{\frac{r+x}{r-x}}$ − hyp. log. $\sqrt{\frac{r+a}{r-a}}$); but $r \times$ (hyp. log. $\sqrt{\frac{r+x}{r-x}}$ − hyp. log. $\sqrt{\frac{r+a}{r-a}}$) is the meridional parts of the dif. of the latitudes whose sines are x and a, which call b; then is $z = \frac{tb}{r}$, the same as it is by Mercator's sailing.

Further, putting $m = 2{\cdot}71828$ the number whose hyp. log. is 1, and $n = \frac{2z}{t}$; then, when z begins at A, $m^n = \frac{r+x}{r-x}$ and theref. $x = r \times \frac{m^n - 1}{m^n + 1} = r - \frac{2r}{m^n + 1}$: hence it appears that as m^n, or rather n or z increases (since m is constant), that x approximates to an equality with r, because $\frac{2r}{m^n + 1}$ decreases or converges to 0, which is its limit; consequently r is the limit or ultimate value of x: but when $x = r$, the ship will be at the pole; theref. the pole must be the limit, or evanescent state, of the rhumb or course: so that the ship may be said to arrive at the pole after making an infinite number of revolutions round it; for the above expression $\frac{2r}{m^n + 1}$ vanishes

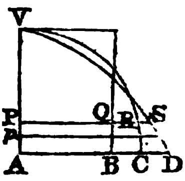
ishes when n, and consequently z, is infinite, in which case x is $= r$.

Now, from the equation $D = \frac{rd}{c} = \frac{sd}{r}$, it is found, that, when $d = 30° 57'$ the comp. of the given lat. $59° 3'$, and $c =$ sine of $67° 30'$ the comp. of the course, D will be $= 2010$ geographical miles, the required ultimate distance; which, at the rate of 10 miles an hour, will be passed over in 201 hours or $8\frac{3}{8}$ days. The dif. of long. is shown above to be infinite. When the ship has made one revolution, she will be but about a yard from the pole, considering her as a point.

When the ship has arrived infinitely near the pole, she will go round in the manner of a top, with an infinite velocity; which at once accounts for this paradox, viz, that though she make an infinite number of revolutions round the pole, yet her distance run will have an ultimate and definite value, as above determined: for it is evident that however great the number of revolutions of a top may be, the space passed over by its pivot or bottom point, while it continues on or nearly on the same point, must be infinitely small, or less than a certain assignable quantity.

PROBLEM 4.

A current of water is discharged by three equal openings or sluices, in the following shapes: the first a rectangle, the second a semicircle, and the third a parabola, having their altitudes equal, and their bases in the same horizontal line, and the water level with the tops of the arches: on this supposition it is required to show what may be the proportion of the quantities discharged by these sluices.



Let VB be half the parallelogram, AVC half the semicircle, and AVD half the parabola, that is, the halves of the respective sluices or gates. Put $a =$ AV the common altitude, and $c = \cdot 7854$: then is ca^2 the area of each of the figures; also $ca =$ AB, $a =$ AC, and $\frac{3}{2}ca =$ AD; also put $x =$ VP any variable depth, and $\dot{x} =$ Pp. Then, the water discharged, at any depth x, being as the velocity and aperture, and the velocity being in all the figures as $\sqrt{x}$, therefore $\dot{x}\sqrt{x} \times$ PQ, and $\dot{x}\sqrt{x} \times$ PR, and $\dot{x}\sqrt{x} \times$ PS, or $cax^{\frac{1}{2}}\dot{x}$, and $x\dot{x}\sqrt{(2a-x)}$, and $\frac{3}{2}c\sqrt{a} \times x\dot{x}$, are proportional to the fluxions of the quantity of water discharged by the said figures or sluices respectively; the correct fluents of which, when $x = a$, are $\frac{2}{3}ca^{\frac{5}{2}}$, and $\frac{2}{15}a^{\frac{5}{2}}(8\sqrt{2}-7)$, and $\frac{3}{4}ca^{\frac{5}{2}}$, the 2d fluent being found by art. 60 pa. 336 of this vol. Hence the quantities of

of water discharged by the rectangle, the semicircle, and the parabola, are respectively as $\frac{2}{3}c$, and $\frac{2}{15}(8\sqrt{2}-7)$, and $\frac{3}{4}c$, or as 1, and $\frac{2}{5c}(8\sqrt{2}-7)$, and $\frac{9}{8}$, or as 1, and 1·09847, and $1\frac{1}{8}$.

PROBLEM 5.

The initial velocity of a 24lb ball of cast iron, which is projected in a direction perpendicular to the horizon, being supposed 1200 feet per second; and that the resistance of the medium is constantly as the square of the velocity, and everywhere of the same density: required the time of flight, and the height to which it will ascend.

Answer. By problems 5 and 6, of the last chapter, the ascent will be found = 5337 feet, and the time of the ascent 28 seconds.

PROBLEM 6.

To determine the same as in the last question; supposing the density of the atmosphere to decrease in ascending after the usual way?

Ans. By probs. 7 and 8, the height will be 5614 feet, and the time 34 seconds.

PROBLEM 7.

It is required to find the diameter of a circular parachute, by means of which a man of 150lb weight may descend on the earth, from a balloon at a height in the air, with the velocity of only 10 feet in a second of time, being the velocity acquired by a body freely descending through a space of only 1 foot 6¾ inches, or of a man jumping down from a height of 18¾ inches: the parachute being made of such materials and thickness, that a circle of it of 50 feet diameter, weighs only 150lb, and so in proportion more or less according to the area of the circle.

If a falling body descend with a uniform velocity, it must necessarily meet with a resistance, from the medium it descends in, equal to the whole weight that descends. Let x denote the diameter of the parachute, and $a = {\cdot}7854$; then ax^2 will be its area, and as $50^2 : x^2 :: 150 : \frac{3}{50}x^2$ the weight of the same, to which adding 150lb, the man's weight, the sum $\frac{3}{50}x^2 + 150$ will be the whole descending weight. Again, in the table of resistances (in the scholium to prop. 22, *Mot. of bod. in Fluids*), we find that a circle of $\frac{2}{9}$ of a square foot area, moving with 10 feet velocity, meets with a resistance of ·57 ounces = ·0475 lb; and the resistances, with the same velocity, being

being as the surfaces, therefore, as $\frac{2}{9} : \cdot 0475 :: ax^2 : \cdot 21375ax^2 = \cdot 16788x^2$ the resistance of the air to the parachute, to which the descending weight must be equal; that is, $\cdot 16788x^2 = \frac{3}{50}x^2 + 150$; hence $\cdot 10788x^2 = 150$, or $x^2 = 1390 \cdot 5$, and hence $x = 37\frac{2}{7}$ feet, the diameter of the parachute required.

PROBLEM 8.

To determine the effects of Pile-Engines.

The form of the pile-engine, as used by the ancients, is not known. Many have been invented and described by the moderns. Among all these, that appears to be the best which was invented by Vauloue, as described by Desaguliers, and was used at piling the foundations at building Westminster Bridge. Its chief properties are, that the ram or weight be raised with the least expense of force, or with the fewest men; that it fall freely from its greatest height; and that, having fallen, it is presently laid hold of by the forceps, and so raised up to its height again. By which means, in the shortest time, and with the fewest men, or the least force, the most piles can be driven to the greatest depth.

Belidor has given some theory as to the effect of the pile-engine, but it appears to be founded on an erroneous principle: he deduces it from the laws of the collision of bodies. But who does not perceive that the rules of collision suppose a free motion and a non-resisting medium? It cannot therefore be applied in the present case, where a very great resistance is opposed to the pile by the ground. We shall therefore here endeavour to explain another theory of this machine.

Since the percussion of the weight acts on the pile during the whole time the pile is penetrating and sinking in the earth, by each blow of the ram, during which time its whole force is spent; it is manifest that the effect of the blow is of that nature which requires the force of the blow to be estimated by the square of the velocity. But the square of the velocity acquired by the fall of the ram, is as the height it falls from; therefore the force of any blow will be as the height fallen through. But it is also more or less in proportion to the weight of the ram; consequently the effect or force of each blow must be directly in the compound ratio of both, viz, as aw, where w denotes the weight, and a the altitude it falls from; or it will be simply as the altitude a, when the weight w is constant.

Again, the force of the blow is opposed by the mass of the pile, and by the consistence of the earth penetrated by the point

point of the pile, and also by the friction of the earth against the surface or sides of the pile that have penetrated below the surface. Consequently the effect of the blow, or the depth penetrated by the pile, will be inversely in the compound ratio of these three, viz, inversely as mtf, where m denotes the mass of the pile, t the tenacity or cohesion of the earth, and f the friction of the surface penetrated in the earth. But, in the same soil and with the same pile, m and t are both constant, in which case the depth of penetration will be inversely only as f the friction. On all accounts then the penetration will be as $\frac{aw}{mtf}$, or simply as $\frac{a}{f}$ only, for the same weight and pile and soil.

To determine the depth sunk by the pile at each stroke of the ram.

After a few strokes, so as to give the pile a little hold in the ground, to make it stand firmly, the blows of the ram may be considered as commencing, and causing the pile to sink a little at every stroke, by which small successive sinkings of the pile, the space the ram falls through will be successively increased by these small accessions, and the force of the successive blows proportionally increased. But these, on the other hand, are resisted and opposed by the friction of the part of the pile which has been sunk before, and which also sinks at each stroke; and as the quantities of these rubbing surfaces increase in a greater ratio to each other, than the heights fallen through, that is, the resisting forces increasing faster than the impelling forces, it is manifest that the depths successively sunk by the blows must gradually decrease by little and little every time; which is also found to be quite conformable to experience. Thus then the successive sinkings will proceed gradually diminishing, till they become so small as to be almost imperceptible.

Now it was found above that $\frac{a}{f}$ is as the penetration by any blow of the ram, by the same pile in the same soil, that is, as the height fallen directly, and as the resistance or friction in the earth inversely. Let A denote any other and greater height, by an after stroke, and F its friction; also P the penetration by the former blow, and p that by the latter, which must be the smaller: then, by the foregoing principle, $\frac{a}{f} : \frac{A}{F} :: P : p$; hence $a : A :: fP : Fp$, which is a general theorem.

But

But now, with respect to the quantity of friction from any blow, though it be not known from experiment that the friction is exactly proportional to the rubbing surface, there is great reason to believe that it must be at least very nearly so: there is also equal reason to conclude that the effect or resistance from that rubbing surface must be nearly or exactly as the length of space it moves over, that is by the penetration of the pile by any blow. Now, if d denote the depth of the pile in the ground before any new blow is struck by the ram, and b the depth or penetration produced by the blow, then the length of the rubbing surface will be $d + \frac{1}{2}b$; for, the length of the rubbing surface is only d at the beginning of the motion, and it is $d + b$ at the end of it, the medium of the two, or $d + \frac{1}{2}b$, is therefore the due length of the surface, and the space or depth it moves over is b; therefore the whole resistance from the friction is $(d + \frac{1}{2}b)\,b$. If D then denote any other depth of the pile in the earth, and b' the next penetration, then $(D + \frac{1}{2}b')\,b'$ will be its friction. Substituting now b for P, and b' for p, also $d + \frac{1}{2}b$ for f, and $D + \frac{1}{2}b'$ for F, in the general theorem $a : A :: fP : Fp$, it becomes $a : A :: (d + \frac{1}{2}b)b : (D + \frac{1}{2}b')b'$, for the general relation between the heights fallen and the resistance and penetration.

This theorem will very conveniently give the series of effects, or successive sinkings of the piles, by the blows of the ram. Thus, after the pile has been properly fixed, or indeed driven to any depth in the earth, denoted by d, then to give a blow, the ram falls from the height $a + d$, and thereby sinks the pile the space b suppose; hence, for the next stroke, the fall will be $a + d + b = A$ in the theorem above, and $D + \frac{1}{2}b' = d + b + \frac{1}{2}b'$, the next penetration or sinking being b'; theref. $a + d : a + d + b :: (d + \frac{1}{2}b)b : (d + b + \frac{1}{2}b')b'$, a proportion which gives the quadratic equa. $b'^2 + 2b'(d+b) = \frac{a+d+b}{a+d} \times (2d + b)b$, the root of which is $b' = -(d+b) + \sqrt{\left[(d+b)^2 + \frac{a+d+b}{a+d} \times (2d+b)b\right]} = \frac{a+d+b}{a+d} \times \frac{d + \frac{1}{2}b}{d+b}b$ nearly, or indeed $= \frac{d + \frac{1}{2}b}{d+b}b$ nearly, because b is small in comparison with $a + d$.

Now, for an example in numbers, suppose $a = 5$ feet $= 60$ inches, $d = 10$, $b = 3$, that is $a = 60$ the height of the ram above the top of the pile before this enters the ground; $d = 10$, after being fixed in the ground; and $b = 3$ the sinking by the next blow: then $\frac{d + \frac{1}{2}b}{d+b}b = \frac{11\cdot5}{13} \times 3 = 2\cdot65 = b$, the

the 2d stroke. Next, substituting $d + b$ for d, and b' for b, the same theorem gives 2·48 for the next sinking, or the next value of b'. And so on continually, by which means the series of the successive corresponding values of the letters will be as in the margin, the last column showing the several successive sinkings of the pile by the repeated strokes of the ram.

Specimen of the Series of the Successive values of d, b, b'.

d	b	b'
10	3	2·65
13	2·65	2·48
15·65	2·49	2·32
18·14	2·32	2·19
20·46	2·19	2·08
&c.		

Scholium. Thus then it appears, that the effect of any operation of pile-driving may be determined. It is manifest also that the greater a is, or the higher the top of the machine is where the ram falls from, above the top of the pile at first, the greater will be every stroke of the ram, and consequently the fewer the strokes requisite to drive the pile to the requisite depth. But then every stroke will take a longer time, as the ram will be both longer in falling and longer in raising: so that it may be a question whether on the whole the business may be effected in the less time by a greater height of the machine, or whether there be any limit to the height, so as to produce the greatest effect in a given time.

To answer this question, let x denote the indeterminate height from which any weight w is to fall, z the time of raising it after a fall, which time is supposed to be as the height x to which it is raised, also m the given time of producing a proposed effect; then $\frac{1}{4}\sqrt{x}$ = the time of the weight falling; therefore $\frac{1}{4}\sqrt{x} + z$ = the whole time of one stroke; conseq. $\frac{m}{\frac{1}{4}\sqrt{x}+z}$ or $\frac{4m}{\sqrt{x}+4z}$ is the number of strokes made in the given time m, and hence $\frac{4mxw}{\sqrt{x}+4z}$ = the whole force or effect in the time m. Now this effect or fraction increases continually as x increases, because the numerator increases faster than the denominator, since the former increases as x, while in the latter though the one term z increases as x, yet the other term $\sqrt{x}$ only increases as the root of x. So that, on the whole, it appears that the effect, in any given time, increases more and more as the height is increased.

PROBLEM 9.

To determine how far a man, who pushes with the force of 100lb. can force a sponge into a piece of ordnance, whose diameter is 5 inches, and length ten feet, when the barometer stands at 30 inches: the vent, or touch-hole, being stopped, and the sponge having no windage, that is, fitting the bore quite close?

A column of quicksilver 30 inches high, and 5 in diameter, is $5^2 \times 30 \times \cdot 7854 = 589\cdot 05$ inches; which, at 8·102 oz each inch, weighs 4772·48 oz or 298·28lb, which is the pressure of the atmosphere alone, being equal to the elasticity of the air in its natural state; to this adding the 100lb, gives 398·28lb, the whole external pressure. Then, as the spaces which a quantity of air possesses, under different pressures, are in the reciprocal ratio of those pressures, it will be, as 398·28 : 298·28 : : 10 feet or 120 inches : 90 inches nearly, the space occupied by the air; theref. $120 - 90 = 30$ inches, is the distance sought.

PROBLEM 10.

To assign the Cause of the Deflection of Military Projectiles.

It having been surmized that, in the practice of artillery, the deflexion of the shot in its flight, to the right or left, from the line or direction the gun is laid in, chiefly arises from the motion of the gun during the time the shot is passing out of the piece; it is required to determine what space an 18 pounder will recoil or fly back, while the shot is passing out of the gun; supposing its weight to be 4800lb, that of the carriage 2400lb, the quantity of powder 9lb, the length of the cylinder 108 inches, that of the charge 13 inches, and the diameter of the bore 5·13 inches; supposing also that the resistance from the friction between the platform and carriage is equal to 3600lb?

It is well known that confined gunpowder, when fired, immediately changes in a great measure into an elastic air, which endeavours to expand in all directions. Now, in the question, the action of this fluid is exerted equally on the bottom of the bore of the gun and on the ball, during the passage of the latter through the cylinder; the two bodies therefore move in opposite directions, with velocities which are at all times in the inverse ratio of the quantities of matter moved. Now let x be the space through which the gun recoils; then, as the charge occupies 13 inches of the barrel, and the semidiameter of the barrel is 2·565, the space moved through

through by the ball when it quits the piece, is $108 - 13 - 2{\cdot}565 - x = 92{\cdot}435 - x$: and as the elastic fluid expands in both directions, the quantity which advances towards the muzzle, is to that which retreats from it, as $92{\cdot}435 - x$ to x: conseq. $\frac{8x}{92{\cdot}435}$ and $\frac{92{\cdot}435-x}{92{\cdot}435} \times 8$ are the quantities of the powder which move, the former with the gun, and the latter with the ball; besides these, the weight of ball that moves forwards being 18lb, and of the weights and resistance backwards $4800 + 2400 + 3600 = 10800$lb, hence the whole weights moved in the two directions are $10800 + \frac{8x}{92{\cdot}435}$ and $18 + \frac{92{\cdot}435 - x}{92{\cdot}435} \times 8$, or $\frac{998298 + 8x}{92{\cdot}435}$ and $\frac{2403{\cdot}31 - 8x}{92{\cdot}435}$, or as the numerators of these only. But when the time and moving force are given, or the same, then the spaces are inversely as the quantities of matter; therefore $x : 92{\cdot}435 - x :: 2403{\cdot}31 - 8x : 998298 + 8x$, or by composition, $x : 92{\cdot}435 :: 2403{\cdot}31 - 8x : 1000701{\cdot}31$, and by div. $x : 1 :: 2403{\cdot}31 - 8x : 10826$, theref. $10826x = 2403{\cdot}31 - 8x$, or $10834x = 2403{\cdot}31$, and hence $x = {\cdot}2218$ inch $= \frac{2}{9}$ of an inch nearly, or the recoil of the gun is less than a quarter of an inch.

Hence it may be concluded, that so small a recoil, straight backwards, can have no effect in causing the ball to deviate from the pointed line of direction: and that it is very probable we are to seek for the cause of this effect in the ball striking or rubbing against the sides of the bore, in its passage through it, especially near the exit at the muzzle; by which it must happen, that if the ball strike against the right side, the ball will deviate to the left; if it strike on the left side, it must deviate to the right; if it strike against the under side, it must throw the ball upwards, and make it to range farther; but if it strike against the upper side, it must beat the ball downwards; and cause a shorter range: all which irregularities are found to take place, especially in guns that have much windage, or which have the balls too small for the bore.

PROBLEM 11.

A ball of lead, of 4 inches diameter, is dropped from the top of a tower, of 65 yards high, and falls into a cistern full of water at the bottom of the tower, of $20\frac{1}{4}$ yards deep: it is required to determine the times of falling, both to the surface and to the bottom of the water.

The fall in air is 195 feet, and in water $60\frac{3}{4}$ feet. By the common rules of descent, as $\sqrt{16} : \sqrt{195} :: 1'' : \frac{1}{4}\sqrt{195} =$

3·49

3·49 seconds, the time of descending in air. And as $\surd\ 16 : \surd\ 195 :: 32 : 8\surd\ 195 = 111\cdot71$ feet, the velocity at the end of that time, or with which the ball enters the water.

Again, by prob. 22 of this vol. art. 2, the space $s = \frac{1}{2b} \times$ hyp. log. of $\frac{a-e^2}{a-v^2}$, or rather $\frac{1}{2b} \times$ hyp. log. of $\frac{e^2-a}{v^2-a}$(the velocity being decreasing and e^2 greater than a) $= \frac{m}{2b} \times$ com. log. of $\frac{e^2-a}{v^2-a}$, where $N = 11325$ the density of lead, $n = 1000$ that of water, $a = \frac{256d(N-n)}{3n}$, $b = \frac{3n}{8dN}$, $e = 111\cdot71$ the velocity at entering the water, and v the velocity at any time afterwards, also d the diameter of the ball $= 4$ inches, and $m = 2\cdot302585$ the hyp. log. of 10.

Hence then $N = 11325$, $n = 1000$, $N - n = 10325$, $d = \frac{4}{12} = \frac{1}{3}$; then $a = \frac{256d(N-n)}{3n} = \frac{256 \,.\, 10325}{9000} = 293\frac{2}{3}$, and $b = \frac{3n}{8dN} = \frac{9n}{8N} = \frac{9000}{90600} = \frac{15}{151} = \frac{1}{10}$ nearly. Also $e = 111\cdot71$; therefore $s = 60\frac{3}{4} = \frac{m}{2b} \times$ log. of $\frac{e^2-a}{v^2-a} = 5m \times$ log. $\frac{e^2-a}{v^2-a}$. This theorem will give s when v is given, and by reverting it will give v in terms of s in the following manner.

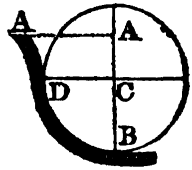
Dividing by $5m$, gives $\frac{s}{5m} =$ log. of $\frac{e^2-a}{v^2-a} = ns$, by putting $n = \frac{1}{5m}$; therefore, the natural number is $10^{ns} = \frac{e^2-a}{v^2-a}$; hence $v^2 - a = \frac{e^2-a}{10^{ns}}$, and $v = \surd(a + \frac{e^2-a}{10^{ns}})$, which, by substituting the numbers above mentioned for the letters, gives $v = 17\cdot134$ for the last velocity, when the space $s = 60\frac{3}{4}$, or when the ball arrives at the bottom of the water.

But now to find the time of passing through the water, putting $t =$ any time in motion, and s and v the corresponding space and velocity, the general theorem for variable forces gives $\dot{t} = \frac{\dot{s}}{v}$. But the above general value of s being $\frac{1}{2b} \times$ hyp. log. $\frac{e^2-a}{v^2-a}$ or $5 \times$ hyp. log. $\frac{e^2-a}{v^2-a}$, therefore its fluxion $\dot{s} = \frac{-10v\dot{v}}{v^2-a}$, conseq. $\dot{t}$ or $\frac{\dot{s}}{v} = \frac{-10\dot{v}}{v^2-a}$, the correct fluent of which is $\frac{5}{\surd a} \times$ hyp. log. $(\frac{e-\surd a}{e+\surd a} \times \frac{v+\surd a}{v-\surd a}) = t$ the time, which when $v = 17\cdot134$, or $s = 60\frac{3}{4}$, gives $2\cdot6542$ seconds, for the time of descent through the water.

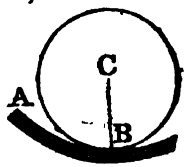
PROBLEM

PROBLEM 12.

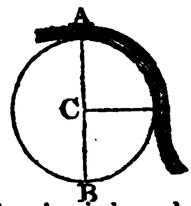
Required to determine what must be the diameter of a water-wheel, so as to receive the greatest effect from a stream of water of 12 feet fall?



In the case of an undershot wheel, put the height of the water AB = 12 feet = a, and the radius BC or CD of the wheel = x, the water falling perpendicularly on the extremity of the radius CD at D. Then AC or AD = cx, and the velocity due to this height, or with which the water strikes the wheel at D, will be as $\sqrt{(a-x)}$, and the effect on the wheel being as the velocity and as the length of the lever CD, will be denoted by $x\sqrt{(a-x)}$ or $\sqrt{(ax^2 - x^3)}$, which therefore must be a maximum, or its square $ax^2 - x^3$ a maximum. In fluxions, $2ax\dot{x}-3x^2\dot{x} = 0$; and hence $x = \frac{2}{3}a = 8$ feet, the radius.



But if the water be considered as conducted so as to strike on the bottom of the wheel, as in the annexed figure, it will then strike the wheel with its greatest velocity, and there can be no limit to the size of the wheel, since the greater the radius or lever BC, the greater will be the effect.



In the case of an overshot wheel, $a - 2x$ will be the fall of water, $\sqrt{(a-2x)}$ as the velocity, and $x\sqrt{(a-2x)}$ or $\sqrt{(ax^2 - 2x^3)}$ the effect, then $ax^2 - 2x^3$ is a maximum, and $2ax\dot{x} - 6x^2\dot{x} = 0$; hence $x = \frac{1}{3}a = 4$ feet is the radius of the wheel.

But all these calculations are to be considered as independent of the resistance of the wheel, and of the weight of the water in the buckets of it.

PROBLEM 13.

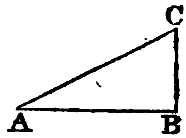
What angle must a projectile make with the plane of the horizon, discharged with a given velocity. v, so as to describe in its flight a parabola including the greatest area possible?

By the set of theorems (in art. 92 Projectiles) for any proposed angle, there can be assigned expressions for the horizontal range and the greatest height the projectile rises to, that is the base and axis of the parabolic trajectory. Thus, putting s and c for the sine and cosine of the angle of elevation;

tion; then, by the first line of those theorems, the velocity being v, the horizontal range R is $= \frac{1}{16}scv^2$; and, by the 4th or last line of theorems, the greatest height H is $= \frac{1}{64}s^2v^2$. But, by the parabola, $\frac{2}{3}$ of the product of the base or range and the height is the area, which is now required to be the greatest possible. Therefore $R \times H = \frac{1}{16}scv^2 \times \frac{1}{64}s^2v^2$ must be a maximum, or, rejecting the constant factors, s^3c a maximum. But the cosine c, of the angle whose sine is s, is $\sqrt{(1-s^2)}$; therefore $s^3c = s^3\sqrt{(1-s^2)} = \sqrt{(s^6-s^8)}$ is the maximum, or its square $s^6 - s^8$ a maximum. In fluxions $6s^5\dot{s} - 8s^7\dot{s} = 0 = 3 - 4s^2$; hence $4s^2 = 3$, or $s^2 = \frac{3}{4}$, and $s = \frac{1}{2}\sqrt{3} = \cdot8660254$, the sine of 60°, which is the angle of elevation to produce a parabolic trajectory of the greatest area.

PROBLEM 14.

Suppose a cannon were discharged at the point A; *it is required to determine how high in the air the point* C *must be raised above the horizontal line* AB, *so that a person at* C *letting fall a leaden bullet at the moment of the cannon's explosion, it may arrive at* B *at the same instant as he hears the report of the cannon, but not till* $\frac{1}{10}$*th of a second after the sound arrives at* B: *supposing the velocity of sound to be* 1140 *feet per second, and that the bullet falls freely without any resistance from the air?*

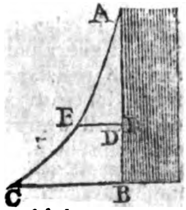


Let x denote the time in which the sound passes to C; then will $x - \frac{1}{10}$ be the time in passing to B, and x the time also the bullet is falling through CB. Then, by uniform motion, $1140x = $ AC, and $1140x - 114 = $ AB, also by descents of gravity, $1^2 : x^2 :: 16 : 16x^2 = $ BC. Then, by right-angled triangles, $AC^2 - BC^2 = AB^2$, that is $1140^2x^2 - 16^2x^4 = 1140^2x^2 - 224 \times 1140x + 114^2$, hence $224 \times 1140x - 16^2x^4 = 114^2$, or $1015\cdot3x - x^4 = 50\cdot77$, the root of which equa. is $x = 10\cdot03$ seconds, or nearly 10 seconds; conseq. $BC = 16x^2 = 1610$ feet nearly, the height required.

PROBLEM 15.

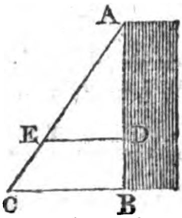
Required the quantity, in cubic feet, of light earth, necessary to form a bank on the side of a canal, which will just support a pressure of water 5 feet deep, and 300 feet long. And what will the carriage of the earth cost, at the rate of 1 shilling per ton?

This



This question may be considered as relating either to water sustained by a solid wall, or by a bank of loose earth. In the former case, let ABC denote the wall, sustaining the pressure of the water behind it. Put the whole altitude AB $=a$, the base BC or thickness at bottom $=b$, any variable depth AD $=x$, and the thickness there DE $=y$. Now the effect which any number of particles of the fluid pressing at D have to break the wall at B, or to overturn it there, is as the number of particles AD or x, and as the lever BD $= a - x$; therefore the fluxion of the effect of all the forces is $(a-x)x\dot{x} = ax\dot{x} - x^2\dot{x}$, the fluent of which is $\frac{1}{2}ax^2 - \frac{1}{3}x^3$, which, when $x=a$, is $\frac{1}{6}a^3$ for the whole effect to break or overturn the wall at B; and the effects of the pressure to break at B and D will be as AB^3 and AD^3. But the strength of the wall at D, to resist the fracture there, like the lateral strength of timber, is as the square of the thickness, DE^2. Hence the curve line AEC, bounding the back of the wall, so as to be every where equally strong, is of such a nature, that x^3 is always proportional to y^2, or y as $x^{\frac{3}{2}}$, and is therefore what is called the semicubical parabola.

Now, to find the area ABC, or content of the wall bounded by this convex curve, the general fluxion of all are as $y\dot{x}$ becomes $x^{\frac{3}{2}}\dot{x}$, the fluent of which is $\frac{2}{5}x^{\frac{5}{2}} = \frac{2}{5}xx^{\frac{3}{2}} = \frac{2}{5}xy$, that is $\frac{2}{5}$ of the rectangle AB × BC; and is therefore less than the triangle ABC, of the same base and height, in the proportion of $\frac{2}{5}$ to $\frac{1}{2}$, or of 4 to 5.



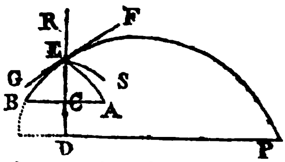
But in the case of a bank of made earth, it would not stand with that concave form of outside, if it were necessary, but would dispose itself in a straight line AC, forming a triangular bank ABC. And even if this were not the case naturally, it would be proper to make it such by art; because now neither is the bank to be broken as with the effect of the lever, or overturned about the pivot or point C, nor does it resist the fracture by the effect of a lever, as before; but, on the contrary, every point is attempted to be pushed horizontally outwards, by the horizontal pressure of the water, and it is resisted by the weight or resistance of the earth at any part, DE. Here then, by hydrostatics, the pressure of the water against any point D, is as the depth AD; and, in the triangle of earth ADE, the resisting quantity in DE is as DE, which

which is also proportional to AD by similar triangles. So that, at every point D in the depth, the pressure of the water and the resistance of the soil, by means of this triangular form, increase in the same proportion, and the water and the earth will everywhere mutually balance each other, if at any one point, as B, the thickness BC of earth be taken such as to balance the pressure of the water at B, and then the straight line AC be drawn, to determine the outer shape of the earth. All the earth that is afterwards placed against the side AC, for a convenient breadth at top for a walking path, &c, will also give the whole a sufficient security.

But now to adapt these principles to the numeral calculation proposed in the question; the pressure of water against the point B being denoted by the side AB = 5 feet, and the weight of water being to earth as 1000 to 1984, therefore as 1984 : 1000 : : 5 : 2·52 = BC, the thickness of earth which will just balance the pressure of the water there; hence the area of the triangle ABC = $\frac{1}{2}$AB × BC = $2\frac{1}{2}$ × 2·52 = 6·3; this mult. by the length 300, gives 1890 cubic feet for the quantity of earth in the bank; and this multiplied by 1984 ounces, the weight of 1 cubic foot, gives, for the weight of it, 3749760 ounces = 234360lbs = 104·625 tons; the expense of which, at 1 shilling the ton, is 5*l.* 4*s.* $7\frac{1}{2}$*d.*

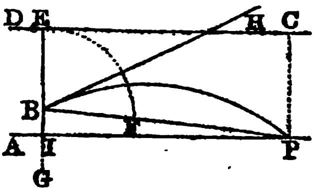
PROBLEM 16.

A person standing at the distance of 20 feet from the bottom of a wall, which is supposed perfectly smooth and hard, desires to know in what direction he must throw an elastic ball against it, with a velocity of 80 feet per second, so that, after reflection from the wall, it may fall at the greatest distance possible from the bottom, on the horizontal plane, which is $2\frac{1}{2}$ feet below the hand discharging the ball?



In the annexed figure let DR be the wall against which the ball is thrown, from the point A, in such a direction, that it shall describe the parabolic curve AE before striking the wall, and afterwards be so reflected as to describe the curve EP. Now if ES be the tangent at the point E, to the curve AE described before the reflection, and EF the tangent at the same point to the curve which the ball will describe after reflection, then will the angle REF be = CES; and if the curve PE be produced, so as to have GF for its tangent, it will meet AC produced in B, making BC = AC, and the curve AE will be similar

similar and equal to the portion BE of the parabola BEP, but turned the contrary way. Conceiving either the two curves AE and EP, or the continued curve BEP, to be described by a projectile in its motion, it is manifest that, whether the greater portion of the curve be described before or after the ball reaches the wall DB, will depend on its initial velocity, and on the distance AC or BC, and on the angle of projection. The problem then is now reduced to this, viz, To find the angle at which a ball shall be projected from B, with a given impetus, so that the distance DP, at which it falls, from the given point D on the plane DP, parallel to the horizon, shall be a maximum.



Now this problem may be constructed in the following manner: From any point E in the horizontal line DC, let fall the indefinite perp. EG, on which set off EB = the impetus corresponding to the given velocity, and BI = 2½ the distance of the horizontal plane below the point of projection; also, through I draw AP parallel to DC. From the point B set off BP = BE + BI, and bisect the angle EBP by the line BH: then will BH be the required direction of the ball, and IP the maximum range on the plane AP.

For, since the ball moves from the point B with the velocity acquired by falling through EB, it is manifest, from p. 136 this vol. that DC is the directrix of the parabola described by the ball. And since both B and P are points in the curve, each of them must, from the nature of the parabola, be as far from the forces as it is from the directrix; therefore B and P will be the greatest distance from each other when the focus F is directly between them, that is, when BP = BE + CP. And when BP is a maximum, since BI is constant, it is obvious that IP is a maximum too. Also, the angle FBH being = EBH, the line BH is a tangent to the parabola at the point B, and consequently it is the direction necessary to give the range IP.

Cor. 1. When B coincides with I, IP will be = BP = BE + EI = 2EI, and the angle EBH will be 45°: as is also manifest from the common modes of investigation.

Cor. 2. When the impetus corresponding to the initial velocity of the ball is very great compared with AC or BC (fig. 1), then the part AE of the curve will very nearly coincide with its tangent, and the direction and velocity at A may be accounted the same as those at E without any sensible error.

error. In this case too the impetus BE (fig. 2) will be very great compared with BI, and consequently, B and I nearly coinciding, the angle EBH will differ but little from 45°.

Calcul. From the foregoing construction the calculation will be very easy. Thus, the first velocity being 80 feet $= v$, then (art. 92 Projectiles) $\frac{v^2}{4g} = \frac{80 \times 80}{64\frac{1}{3}} = 99{\cdot}48186 =$ BE the impetus; hence EI = FP = 101·98186, and BP = BE + EI = 201·46372. Now, in the right-angled triangle BIP, the sides BI and BP are known, hence IP = 201·4482, and the angle IBP = 89° 17′ 20″: half the suppl. of this angle is 45° 21′ 20″ = EBH. And, in fig. 1, IP − ID = 201·4482 − 10 = 191·4482 = DP; the distance the ball falls from the wall after reflection.

PROBLEM 17.

From what height above the given point A must an elastic ball be suffered to descend freely by gravity, so that, after striking the hard plane at B, it may be reflected back again to the point A, in the least time possible from the instant of dropping it?

Let C be the point required; and put AC $= x$, and AB $= a$; then $\frac{1}{4}\sqrt{\text{CB}} = \frac{1}{4}\sqrt{(a+x)}$ is the time in CB, and $\frac{1}{4}\sqrt{\text{CA}} = \frac{1}{4}\sqrt{x}$ is the time in CA; therefore $\frac{1}{4}\sqrt{(a+x)} - \frac{1}{4}\sqrt{x}$ is the time down AB, or the time of rising from B to A again: hence the whole time of falling through CB and returning to A, is $\frac{1}{2}\sqrt{(a+x)} - \frac{1}{4}\sqrt{x}$, which must be a min. or $2\sqrt{(a+x)} - \sqrt{x}$ a minimum, in fluxions $\frac{\dot{x}}{\sqrt{(a+x)}} - \frac{\dot{x}}{2\sqrt{x}} = 0$, and hence $x = \frac{1}{3}a$, that is, AC $= \frac{1}{3}$AB.

PROBLEM 18.

Given the height of an inclined plane; required its length, so that a given power acting on a given weight, in a direction parallel to the plane, may draw it up in the least time possible.

Let a denote the height of the plane, x its length, p the power, and w the weight. Now the tendency down the plane is $= \frac{aw}{x}$, hence $p - \frac{aw}{x} =$ the motive force, and $\frac{p - \frac{aw}{x}}{p+w} = \frac{px - aw}{(p+w)x} =$ the accelerating force f; hence, by the theorems for constant forces (See Introduc. to Prac. Ex. on Forces) $t^2 = \frac{s}{gf} =$ $(p+w)$

$\frac{(p+w)x^2}{(px-aw)g}$ must be a minimum, or $\frac{x^2}{px-aw}$ a min.; in fluxions, $2(px-aw)x\dot{x} - px^2\dot{x} = 0$, or $px = 2aw$, and hence $p : w :: 2a : x ::$ double the height of the plane to its length.

PROBLEM 19.

A cylinder of oak is depressed in water till its top is just level with the surface, and then is suffered to ascend; it is required to determine the greatest altitude to which it will rise, and the time of its ascent.

Let $a =$ the length, and b the area or base of the cylinder, m the specific gravity of oak, that of water being 1, also x any variable height through which the cylinder has ascended. Then, $a - x$ being the part still immersed in the water, $(a-x) \times b \times 1 = (a-x)b$ is the force of the water upwards to raise the cylinder; and $a \times b \times m = abm$ is the weight of the cylinder opposing its ascent; therefore the efficacious force to raise the cylinder is $(a-x)b - abm$; and, the mass being abm, the accelerating force is

$$\frac{(a-x)b-abm}{abm} = \frac{a-x-am}{am} = \frac{an-x}{am} = f,$$

putting $n = 1 - m$ the difference between the specific gravities of water and oak.

Now if v denote the velocity of ascent at the same time when x space is ascended, then by the theorems for variable forces, $v\dot{v} = 32f\dot{x} = \frac{32}{am} \times (an\dot{x} - x\dot{x})$, therefore $v^2 = \frac{32}{am} \times (2anx - x^2)$, and $v = 8\sqrt{\frac{2anx-x^2}{2am}}$: but when the cylinder has acquired its greatest ascent, v and $v^2 = 0$, therefore $2anx - x^2 = 0$, and hence $x = 2an$ the part of the cylinder that rises out of the water, being $= \cdot 15a$ or $\frac{3}{20}$ of its length.

To find when the velocity is the greatest, the factor $2anx - x^2$ in the velocity must be a max. then $2an\dot{x} - 2x\dot{x} = 0$, and $x = an$, being the height above the water when the velocity is the greatest, and which it appears is just equal to the half of $2an$ above found for the greatest rise, when the upward motion ceases, and the cylinder descends again to the same depth as at first, after which it again returns ascending as before; and so on, continually playing up and down to the same highest and lowest points, like the vibrations of a pendulum, the motion ceasing in both cases in a similar manner at the extreme points, then returning, it gradually accelerates till arriving at the middle point, where it is the greatest, then gradually retarding all the way to the next extremity

extremity of the vibration, thus making all the vibrations in equal times, to the same extent between the highest and lowest points, except that, by the small tenacity and friction &c, of the water against the sides of the cylinder, it will be gradually and slowly retarded in its motion, and the extent of the vibrations decrease till at length the cylinder, like the pendulum, come to rest in the middle point of its vibrations, where it naturally floats in its quiescent state, with the part na of its length above the water.

The quantity of the greatest velocity will be found, by substituting na for x, in the general value of the velocity $8\sqrt{\frac{2anx-x^2}{2am}}$, when it becomes $8n\sqrt{\frac{a}{2m}} = \frac{4}{9}\sqrt{a}$ very nearly, the value of m being $\cdot925$, and consequently that of $n = 1 - m = \cdot075$.

To find the time t answering to any space x. Here $\dot{t} = \frac{\dot{x}}{v} = \frac{\dot{x}}{8\sqrt{\frac{2nax-x^2}{2ma}}} = \sqrt{\frac{ma}{32}} \times \frac{\dot{x}}{\sqrt{(2nax-x^2)}}$, and by the 13th form the fluent is $t = \frac{1}{8}\sqrt{2ma} \times \text{A}$, where A denotes the circular arc to radius 1 and versed sine $\frac{x}{na}$. Now at the middle of a vibration x is $= na$, and then the vers. $\frac{x}{na} = \frac{na}{na} = 1$ the radius, and A is the quadrantal arc $= 1\cdot5708$; then the flu. becomes $\frac{1}{8}\sqrt{2ma} \times 1\cdot5708 = \cdot17\sqrt{a} \times 1\cdot5708 = \cdot267\sqrt{a}$ for the time of a semivibration; hence the time of each whole vibration is $\cdot534\sqrt{a} = \frac{8}{15}\sqrt{a}$, which time therefore depends on the length of the cylinder a. To make this time $= 1$ second, a must be $= (\frac{15}{8})^2$ very nearly $= 3\frac{1}{2}$ feet or 42 inches. That is, the oaken cylinder of 42 inches length makes its vertical vibrations each in 1 second of time, or is isochronous with a common pendulum of $39\frac{1}{8}$ inches long, the extent of each vibration of the former being $6\frac{3}{10}$ inches.

PROBLEM 20.

Required to determine the quantity of matter in a sphere, the density varying as the nth power of the distance from the centre?

Let r denote the radius of the sphere, d the density at its surface, $a = 3\cdot1416$ the area of a circle whose radius is 1, and x any distance from the centre. Then $4ax^2$ will be the surface of a sphere whose radius is x, which may be considered by expansion as generating the magnitude of the solid; therefore $4ax^2\dot{x}$ will be the fluxion of the magnitude; but

as

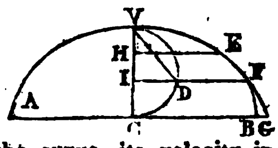
as $r^n : x^n :: d : \frac{dx^n}{r^n}$ the density at the distance x, therefore $4ax^2\dot{x} \times \frac{dx^n}{r^n} = \frac{4adx^{n+2}\dot{x}}{r^n}$ = the fluxion of the mass, the fluent of which $\frac{4adx^{n+3}}{(n+3)r^n}$, when $x = r$, is $\frac{4adr^3}{n+3}$, the quantity of the matter in the whole sphere.

Corol. 1. The magnitude of a sphere whose radius is r, being $\frac{4}{3}ar^3$, which call m; then the mass or solid content will be $\frac{3d}{n+3} \times m$, and the mean density is $\frac{3d}{n+3}$.

Corol. 2. It having been computed, from actual experiments, that the medium density of the whole mass of the earth is about 5 times the density d at the surface, we can now determine what is the exponent of the decreasing ratio of the density from the centre to the circumference, supposing it to decrease by a regular law, viz. as r^n; for then it will be $5d = \frac{3d}{n+3}$, and hence $n = -\frac{12}{5}$. So that, in this case the law of decrease is as $r^{-\frac{12}{5}}$, or as $\frac{1}{r^{\frac{12}{5}}}$, that is, inversely as the $\frac{12}{5}$ths power of the radius.

PROBLEM 21.

Required to determine where a body moving down the convex side of a cycloid, will fly off and quit the curve.



Let AVEB represent the cycloid, the properties of which may be seen at arts. 146 and 147 this vol. and VDC its generating semicircle. Let E be the point where the motion commences, whence it moves along the curve, its velocity increasing both on the curve, and also in the horizontal direction DF, till it come to such a point, F suppose, that the velocity in the latter direction is become a constant quantity, then that will be the point where it will quit the cycloid, and afterwards describe a parabola FG, because the horizontal velocity in the latter curve is always the same constant quantity, (by art. 76 Projectiles)

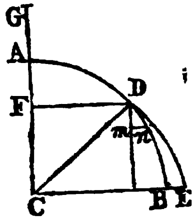
Put the diameter VC $= d$, VH $= a$, VI $= x$; then VD $= \sqrt{dx}$, and ID $= \sqrt{(dx - x^2)}$. Now the velocity in the curve at F in descending down EF, being the same as by falling through HI or $x - a$, by art. 139, will be $= 8\sqrt{(x-a)}$; but this velocity

locity in the curve at F, is to the horizontal velocity there, as VD to ID, because VD is parallel to the curve or to the tangent at F, that is $\sqrt{dx} : \sqrt{(dx-x^2)} :: 8\sqrt{(x-a)} : \frac{8\sqrt{(x-a)} \times \sqrt{(d-x)}}{\sqrt{d}}$, which is the horizontal velocity at F, where the body is supposed to have that velocity a constant quantity; therefore also $\sqrt{(x-a)} \times \sqrt{(d-x)}$, as well as $(x-a) \times (d-x) = ax + dx - ad - x^2$ is a constant quantity, and also $ax + dx - x^2$; but the fluxion of a constant quantity is equal to nothing, that is $a\dot{x} + d\dot{x} - 2x\dot{x} = 0 = a + d - 2x$, and hence $x = \frac{1}{2}a + \frac{1}{2}d =$ VI, the arithmetical mean between VH and VC.

If the motion should commence at V, then x or VI would be $= \frac{1}{2}d$, and I would be the centre of the semicircle.

PROBLEM 22.

If a body begin to move from A, *with a given velocity, along the quadrant of a circle* AB; *it is required to show at what point it will fly off from the curve.*



Let D denote the point where the body quits the circle ADB, and then describes the parabola BE. Draw the ordinate DF, and let GA be the height producing the velocity at A. Put GA $= a$, AC or CD $= r$, AF $= x$; then the velocity in the curve at D will be the same as that acquired by falling through GF or $a+x$, which is, as before, $8\sqrt{(a+x)}$; but the velocity in the curve is to the horizontal velocity as Dn to mn or as CD to CF by similar triangles, that is, as

$r : r - x :: 8\sqrt{(x+a)} : 8\sqrt{(x+a)} \times \frac{r-x}{r}$, which is to be a constant quantity where the body leaves the circle, therefore also $(r-x)\sqrt{(x+a)}$ and $(r-x)^2 \times (x+a)$ a constant quantity; the fluxion of which made to vanish, gives $x = \frac{r-2a}{3} =$ AF.

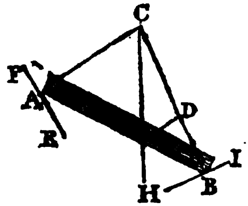
Hence, if $a = 0$, or the body only commence motion at A, then $x = \frac{1}{3}r$, or AF $= \frac{1}{3}$AC when it quits the circle at D. But if a or Ga were $= \frac{1}{2}r$ or $\frac{1}{2}$ AC, then $r - 2a = 0$, and the body would instantly quit the circle at the vertex A, and describe a parabola circumscribing it, and having the same vertex A.

PROBLEM

PROBLEM 23.

To determine the position of a bar or beam AB, *being supported in equilibrio by two chords* AC, BC, *having their two ends fixed in the beam, at* A *and* B.

By art. 210 Statics, the position will be such, that its centre of gravity G will be in the perpendicular or plumb line CG.



Corol. 1. Draw GD parallel to the cord AC. Then the triangle CGD, having its three sides in the directions of, or parallel to, the three forces, viz, the weight of the beam, and the tensions of the two cords AC, BC, these three forces will be proportional to the three sides CG, GD, CD, respectively, by art. 44; that is, CG is as the weight of the beam, GD as the tension or force of AC, and CD as the tension or force of BC.

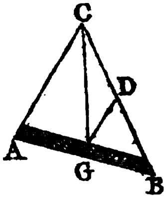
Corol. 2. If two planes EAF, HBI, perpendicular to the two cords, be substituted instead of these, the beam will be still supported by the two planes, just the same as before by the cords, because the action of the planes is in the direction perpendicular to their surface; and the pressure on the planes will be just equal to the tension or force of the respective cords. So that it is the very same thing, whether the body is sustained by the two chords AC, BC, or by the two planes EF, HI; the directions and quantities of the forces acting at A and B being the same in both cases.—Also, if the body be made to vibrate about the point C, the points A, B will describe circular arcs coinciding with the touching planes at A, B; and moving the body up and down the planes, will be just the same thing as making it vibrate by the cords; consequently the body can only rest, in either case, when the centre of gravity is in the perpendicular CG.

PROBLEM 24.

To determine the position of the beam AB, *hanging by one cord* ACB, *having its ends fastened at* A *and* B, *and sliding freely over a tack or pulley fixed at* C.

G being the centre of gravity of the beam, CG will be perpendicular to the horizon, as in the last problem. Now as the

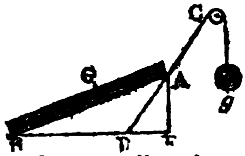
the cord ACB moves freely about the point C, the tension of the cord is the same in every part, or the same both in AC and BC. Draw GD parallel to AC : then the sides of the triangle CGD are proportional to the three forces, the weight and the tensions of the string ; that is, CD and DG are as the forces or tensions in CB and CA. But these tensions are equal ; therefore CD = DG, and conseq. the opposite angles DCG and DGC are also equal ; but the angle DGC is = the alternate angle ACG ; theref. the angle ACG = BCG ; and hence the line CG bisects the vertical angle ACB, and conseq. AC : CB : : AG : GB.



PROBLEM 25.

To determine the position of the beam AB, *moveable about the end* B, *and sustained by a given weight* g, *hanging by a cord* ACg, *going over a pulley at* C, *and fixed to the other end* A.

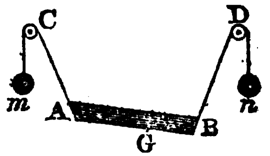
Let w = the weight of the beam, and G denote the place of its centre of gravity. Produce the direction of the cord CA to meet the horizontal line BE in D ; also let fall AE perp. to BE : then AE is the direction of the weight of the beam, and DA the direction of the weight g, the former acting at G by the lever BG, and the latter at A by the lever BA ; theref. the intensity of the former is $w \times$ BG, and that of the latter $g \times$ BA ; but these are also proportional to the sines of their angles of direction with AB, that is, of the angles BAE, and BAD ; therefore the whole intensity of the former is $w \times$ BG $\times$ sin. BAE, and of the latter it is $g \times$ BA $\times$ sin. BAD. But, since these two forces balance each other, they are equal, viz, $w \times$ BG $\times$ sin. BAE $= g \times$ BA $\times$ sin. BAD, and therefore $w : g$: : BA $\times$ sin. BAD : BG $\times$ sin. BAE, or $w \times$ BG : $g \times$ BA : : sin. BAD : sin. BAE.



PROBLEM 26.

To determine the position of the beam AB, *sustained by the given weights* m, n, *by means of the cords* ACm, BDn, *going over the fixed pulleys* C, D.

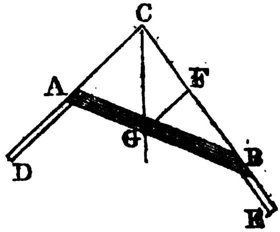
Let



Let G be the place of the centre of gravity of the beam. Now the effect of the weight m, is as m, and as the lever AG, and as the sine of the angle of direction A; and the effect of the weight n, is as n, and as the lever BG, and as the sine of the angle of direction B; but these two effects are equal, because they balance each other; that is, $m \times AG \times \sin. A = n \times BG \times \sin. B$; theref. $m \times AG : n \times BG :: \sin. B : \sin. A$.

PROBLEM 27.

To determine the position of the two posts AD *and* BE, *supporting the beam* AB, *so that the beam may rest in equilibrio.*



Through the centre of gravity G of the beam, draw CG perp. to the horizon; from any point C in which draw CAD, CBE through the extremities of the beam; then AD and BE will be the positions of the two posts or props required, so as AB may be sustained in equilibrio; because the three forces sustaining any body in such a state, must be all directed to the same point C.

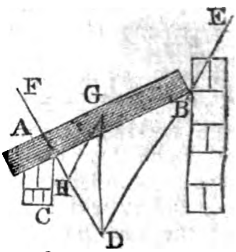
Corol. If GF be drawn parallel to CD; then the quantities of the three forces balancing the beam, will be proportional to the three sides of the triangle CGF, viz, CG as the weight of the beam, CF as the thrust or pressure in BE, and FG as the thrust or pressure in AD.

Scholium. The equilibrium may be equally maintained by the two posts or props AD, BE, as by the two cords AC, BC, or by two planes at A and B perp. to those cords.—It does not always happen that the centre of gravity is at the lowest place to which it can get, to make an equilibrium; for here when the beam AB is supported by the posts DA, EB, the centre of gravity is at the highest it can get; and being in that position, it is not disposed to move one way more than another, and therefore is as truly in equilibrio, as if the centre was at the lowest point. It is true this is only a tottering equilibrium, and any the least force will destroy it; and then, if the beam and posts be moveable about the angles A, B, D, E, which

which is all along supposed, the beam will descend till it is below the points D, E, and gain such a position as is described in prob. 26, supposing the cords fixed at C and D, in the fig. to that prob. and then G will be at the lowest point, coming there to an equilibrium again. In planes, the centre of gravity G may be either at its highest or lowest point. And there are cases, when that centre is neither at its highest nor lowest point, as may happen in the case of prob. 24.

PROBLEM 28.

Supposing the beam AB *hanging by a pin at* B, *and lying on the wall* AC; *it is required to determine the forces or pressures, at the points* A *and* B, *and their directions.*



Draw AD perp. to AB, and through G, the centre of gravity of the beam, draw GD perp. to the horizon; and join BD. Then the weight of the beam, and the two forces or pressures at A and B, will be in the directions of the three sides of the triangle ADG; or in the directions of, and proportional to, the three sides of the triangle GDH, having drawn GH parallel to BD; viz, the weight of the beam as GD, the pressure at A as HD, and the pressure B as GH, and in these directions.

For, the action of the beam is in the direction GD; and the action of the wall at A, is in the perp. AD; conseq. the stress on the pin at B must be in the direction BD, because all the three forces sustaining a body in equilibrio, must tend to the same point, as D.

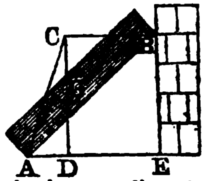
Corol. 1. If the beam were supported by a pin at A, and laid upon the wall at B; the like construction must be made at B, as has been done at A, and then the forces and their directions will be obtained.

Corol. 2. It is all the same thing, whether the beam is sustained by the pin B and the wall AC, or by two cords BE, AF, acting in the directions DB, DA, and with the forces HG, HD.

PROBLEM 29.

To determine the Quantities and Directions of the Forces exerted by a heavy beam AB, *at its two Extremities and its Centre of Gravity, bearing against a perp. wall at its upper end* B.

From

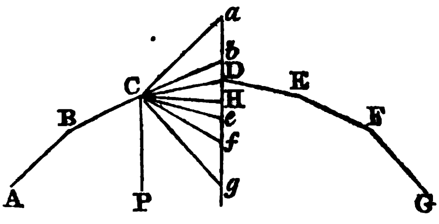


From B draw BC perp. to the face of the wall BE, which will be the direction of the force at B; also through G, the centre of gravity, draw CGD perp. to the horizontal line AE, then CD is the direction of the weight of the beam; and because these two forces meet in the point C, the third force or push A, must be in CA, directly from C; so that the three forces are in the directions CD, BC, CA, or in the directions CD, DA, CA; and, these last three forming a triangle, the three forces are not only in those directions, but are also proportional to these three lines; viz, the weight in or on the beam, as the line CD; the push against the wall at B, as the horizontal line AD; and the thrust at the bottom, as the line AC.

Some of the foregoing problems will be found useful in different cases of carpentry, especially in adapting the framing of the roofs of buildings, so as to be nearest in equilibrio in all their parts. And the last problem, in particular, will be very useful in determining the push or thrust of any arch against its piers or abutments, and thence to assign their thickness necessary to resist that push. The following problem will also be of great use in adjusting the form of a mansard roof, or of an arch, and the thickness of every part, so as to be truly balanced in a state of just equilibrium.

PROBLEM 30.

Let there be any number of lines, or bars, or beams, AB, BC, CD, DE, &c, all in the same vertical plane, connected together and freely moveable about the joints or angles A, B, C, D, E, &c, and kept in equilibrio by their own weights, or by weights only laid on the angles: It is required to assign the proportion of those weights; as also the force or push in the direction of the said lines; and the horizontal thrust at every angle.



Through any point, as D, draw a vertical line aDHg, &c: to which, from any point, as C, draw lines in the direction of, or parallel to, the given lines or beams, viz, Ca parallel to AB, and Cb parallel to BC, and Ce to DE, and Cf to EF, and Cg to FG, &c; also

also CH parallel to the horizon, or perpendicular to the vertical line aDg, in which also all these parallels terminate.

Then will all those lines be exactly proportional to the forces acting or exerted in the directions to which they are parallel, and of all the three kinds, viz, vertical, horizontal, and oblique. That is, the oblique forces or thrusts in direction of the bars . . . . AB, BC, CD, DE, EF, FG,
are proportional to their parallels ca, cb, CD, ce, cf, cg;
and the vertical weights on the angles B, C, D, E, F &c,
are as the parts of the vertical . . ab, bD, De, ef, fg,
and the weight of the whole frame ABCDEFG,
is proportional to the sum of all the verticals, or to ag; also the horizontal thrust at every angle, is every where the same constant quantity, and is expressed by the constant horizontal line CH.

Demonstration. All these proportions of the forces derive and follow immediately from the general well-known property, in Statics, that when any forces balance and keep each other in equilibrio, they are respectively in proportion as the lines drawn parallel to their directions, and terminating each other.

Thus, the point or angle B is kept in equilibrio by three forces, viz, the weight laid and acting vertically downward on that point, and by the two oblique forces or thrusts of the two beams AB, CB, and in these directions. But Ca is parallel to AB, and Cb to BC, and ab to the vertical weight; these three forces are therefore proportional to the three lines ab, Ca, Cb.

In like manner, the angle C is kept in its position by the weight laid and acting vertically on it, and by the two oblique forces or thrusts in the direction of the bars BC, CD: consequently these three forces are proportional to the three lines bD, Cb, CD, which are parallel to them.

Also, the three forces keeping the point D in its position, are proportional to their three parallel lines De, CD, Ce. And the three forces balancing the angle E, are proportional to their three parallel lines ef, Ce, Cf. And the three forces balancing the angle F, are proportional to their three parallel lines fg, Cf, Cg. And so on continually, the oblique forces or thrusts in the directions of the bars or beams, being always proportional to the parts of the lines parallel to them, intercepted by the common vertical line; while the vertical forces or weights, acting or laid on the angles, are proportional to the parts of this vertical line intercepted by the two lines parallel to the lines of the corresponding angles.

Again, with regard to the horizontal force or thrust: since the

the line DC represents, or is proportional to the force in the direction DC, arising from the weight or pressure on the angle D; and since the oblique force DC is equivalent to, and resolves into, the two DH, HC, and in those directions, by the resolution of forces, viz, the vertical force DH, and the horizontal force HC; it follows, that the horizontal force or thrust at the angle D, is proportional to the line CH; and the part of the vertical force or weight on the angle D, which produces the oblique force DC, is proportional to the part of the vertical line DH.

In like manner, the oblique force *c*b, acting at C, in the direction CB, resolves into the two *b*H, HC; therefore the horizontal force or thrust at the angle C, is expressed by the line CH, the very same as it was before for the angle D; and the vertical pressure at C, arising from the weights on both D and C, is denoted by the vertical line *b*H.

Also, the oblique force *a*C, acting at the angle B, in the direction BA, resolves into the two *a*H, HC; therefore again the horizontal thrust at the angle B, is represented by the line CH, the very same as it was at the points C and D; and the vertical pressure at B, arising from the weights on B, C, and D, is expressed by the part of the vertical line *a*H.

Thus also, the oblique force C*e*, in direction DE, resolves into the two CH, H*e*, being the same horizontal force with the vertical H*e*; and the oblique force C*f*, in direction EF, resolves into the two CH, H*f*; and the oblique force C*g*, in direction FG, resolves into the two CH, H*g*; and the oblique force C*g*, in direction FG, resolves into the two CH, H*g*; and so on continually, the horizontal force at every point being expressed by the same constant line CH; and the vertical pressures on the angles by the parts of the verticals, viz, *a*H the whole vertical pressure at B, from the weights on the angles B, C, D: and *b*H the whole pressure on C from the weights on C and D; and DH the part of the weight on D causing the oblique force DC; and H*e* the other part of the weight on D causing the oblique pressure DE; and H*f* the whole vertical pressure at E from the weights on D and E; and H*g* the whole vertical pressure on F arising from the weights laid on D, E, and F. And so on.

So that, on the whole, *a*H denotes the whole weight on the points from D to A; and H*g* the whole weight on the points from D to G; and *ag* the whole weight on all the points on both sides; while *ab*, *b*D, D*e*, *ef*, *fg* express the several particular weights, laid on the angles B, C, D, E, F.

Also, the horizontal thrust is every where the same constant quantity, and is denoted by the line CH.

Lastly,

Lastly, the several oblique forces or thrusts, in the directions AB, BC, CD, DE, EF, FG, are expressed by, or are proportional to, their corresponding parallel lines, c*a*, c*b*, cD, c*e*, c*f*, c*g*.

Corol. 1. It is obvious, and remarkable, that the lengths of the bars AB, BC, &c, do not affect or alter the proportions of any of these loads or thrusts; since all the lines c*a*, c*b*, *ab*, &c, remain the same, whatever be the lengths of AB, BC, &c. The positions of the bars, and the weights on the angles depending mutually on each other, as well as the horizontal and oblique thrusts. Thus, if there be given the position of DC, and the weights or loads laid on the angles D, C, B; set these on the vertical, DH, D*b*, *ba*, then c*b*, c*a* give the directions or positions of CB, BA, as well as the quantity or proportion CH of the constant horizontal thrust.

Corol. 2. If CH be made radius; then it is evident that H*a* is the tangent, and c*a* the secant of the elevation of c*a* or AB above the horizon; also H*b* is the tangent and c*b* the secant of the elevation of c*b* or CB; also HD and CD the tangent and secant of the elevation of CD; also H*e* and c*e* the tangent and secant of the elevation of c*e* or DE; also H*f* and c*f* the tangent and secant of the elevation of EF; and so on; also the parts of the vertical *ab*, *b*D, *ef*, *fg*, denoting the weights laid on the several angles, are the differences of the said tangents of elevations. Hence then in general,

1st. The oblique thrusts, in the directions of the bars, are to one another, directly in proportion as the secants of their angles of elevation above the horizontal directions; or, which is the same thing, reciprocally proportional to the cosines of the same elevations, or reciprocally proportional to the sines of the vertical angles, *a*, *b*, D, *e*, *f*, *g*, &c, made by the vertical line with the several directions of the bars; because the secants of any angles are always reciprocally in proportion as their cosines.

2. The weight or load laid on each angle, is directly proportional to the difference between the tangents of the elevations above the horizon, of the two lines which form the angle.

3. The horizontal thrust at every angle, is the same constant quantity, and has the same proportion to the weight on the top of the uppermost bar, as radius has to the tangent of the elevation of that bar. Or, as the whole vertical *ag*, is to the line CH, so is the weight of the whole assemblage of bars, to the horizontal thrust. Other properties also, concerning the weights and the thrusts, might be pointed out, but they are less simple and elegant than the above, and are therefore omitted;

omitted; the following only excepted, which are inserted here on account of their usefulness.

Corol. 3. It may hence be deduced also, that the weight or pressure laid on any angle, is directly proportional to the continual product of the sine of that angle and of the secants of the elevations of the bars or lines which form it. Thus, in the triangle *b*CD, in which the side *b*D is proportional to the weight laid on the angle C, because the sides of any triangle are to one another as the sines of their opposite angles, therefore as sin. D : C*b* : : sin. *b*CD : *b*D; that is, *b*D is as $\frac{\sin . b\text{CD}}{\sin . \text{D}} \times \text{C}b$; but the sine of angle D is the cosine of the elevation DCH, and the cosine of any angle is reciprocally proportional to the secant, therefore *b*D is as sin. *b*CD × sec. DCH × C*b*; and C*b* being as the secant of the angle *b*CH of the elevation of *b*C or BC above the horizon, therefore *b*D is as sin. *b*CD × sec. *b*CH × sec. DCH; and the sine of *b*CD being the same as the sine of its supplement BCD; therefore the weight on the angle C, which is as *b*D, is as the sin. BCD × sec. DCH × sec. *b*CH, that is, as the continual product of the sine of that angle, and the secants of the elevations of its two sides above the horizon.

Corol. 4. Further, it easily appears also, that the same weight on any angle C, is directly proportional to the sine of that angle BCD, and inversely proportional to the sines of the two parts BCP, DCP, into which the same angle is divided by the vertical line CP. For the secants of angles are reciprocally proportional to their cosines or sines of their complements: but BCP = C*b*H, is the complement of the elevation *b*CH, and DCP is the complement of the elevation DCH; therefore the secant of *b*CH × secant of DCH is reciprocally as the sin. BCP × sin. DCP; also the sine of *b*CD is = the sine of its supplement BCD; consequently the weight on the angle C, which is proportional to sin. *b*CD × sec. *b*CH × sec. DCH, is also proportional to $\frac{\sin . \text{BCD}}{\sin . \text{BCP} \times \sin . \text{DCP}}$, when the whole frame or series of angles is balanced, or kept in equilibrio, by the weights on the angles; the same as in the preceding proposition.

Scholium. The foregoing proposition is very fruitful in its practical consequences, and contains the whole theory of arches, which may be deduced from the premises by supposing the constituting bars to become very short, like arch stones, so as to form the curve of an arch. It appears too, that the horizontal thrust, which is constant or uniformly the same

same throughout, is a proper measuring unit, by means of which to estimate the other thrusts and pressures, as they are all determinable from it and the given positions; and the value of it, as appears above, may be easily computed from the uppermost or vertical part alone, or from the whole assemblage together, or from any part of the whole, counted from the top downwards.

The solution of the foregoing proposition depends on this consideration, viz, that an assemblage of bars or beams, being connected together by joints at their extremities, and freely moveable about them, may be placed in such a vertical position, as to be exactly balanced, or kept in equilibrio, by their mutual thrusts and pressures at the joints; and that the effect will be the same if the bars themselves be considered as without weight, and the angles be pressed down by laying on them weights which shall be equal to the vertical pressures at the same angles, produced by the bars in the case when they are considered as endued with their own natural weights. And as we have found that the bars may be of any length, without affecting the general properties and proportions of the thrusts and pressures, therefore by supposing them to become short, like arch stones, it is plain that we shall then have the same principles and properties accommodated to a real arch of equilibration, or one that supports itself in a perfect balance. It may be further observed, that the conclusions here derived, in this proposition and its corollaries, exactly agree with those derived in a very different way, in my principles of bridges, viz, in propositions 1 and 2, and their corollaries.

PROBLEM 31.

If the whole figure in the last problem be inverted, or turned round the horizontal line AG *as an axis, till it be completely reversed, or in the same vertical plane below the first position, each angle* D, *d, &c, being in the same plumb line; and if weights i, k, l, m, n, which are respectively equal to the weights laid on the angles* B, C, D, E, F, *of the first figure, be now suspended by threads from the corresponding angles b, c, d, e, f, of the lower figure; it is required to show that those weights keep this figure in exact equilibrio, the same as the former, and all the tensions or forces in the latter case, whether vertical or horizontal or oblique, will be exactly equal to the corresponding forces of weight or pressure or thrust in the like directions of the first figure.*

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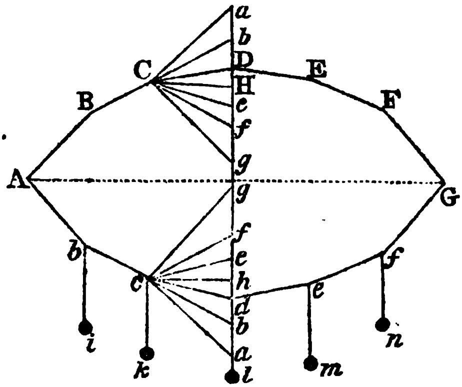
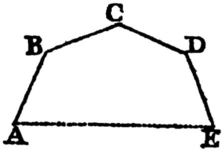


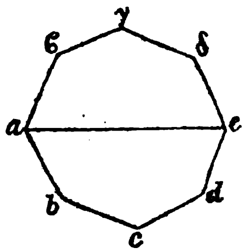
This necessarily happens, from the equality of the weights, and the similarity of the positions, and actions of the whole in both cases. Thus, from the equality of the corresponding weights, at the like angles, the ratios of the weights, *ab*, *bd*, *dh*, *he*, &c, in the lower figure, are the very same as those, *ab*, *b*D, DH, H*e*, &c, in the upper figure; and from the equality of the constant horizontal forces CH, *ch*, and the similarity of the positions, the corresponding vertical lines, denoting the weights, are equal, namely, $ab = ab$, $b\text{D} = bd$, $\text{DH} = dh$, &c. The same may be said of the oblique lines also, *ca*, *cb*, &c, which being parallel to the beams A*b*, *bc*, &c, will denote the tensions of these, in the direction of their length, the same as the oblique thrusts or pushes in the upper figures. Thus, all the corresponding weights and actions, and positions, in the two situations, being exactly equal and similar, changing only drawing and tension for pushing and thrusting, the balance and equilibrium of the upper figure is still preserved the same in the hanging festoon or lower one.

Scholium. The same figure, it is evident, will also arise, if the same weights, *i*, *k*, *l*, *m*, *n*, be suspended at like distances, A*b*, *bc*, &c, on a thread, or cord, or chain, &c, having in itself little or no weight. For the equality of the weights, and their directions and distances, will put the whole line, when they come to equilibrium, into the same festoon shape of figure. So that, whatever properties are inferred in the corollaries to the foregoing prob. will equally apply to the festoon or lower figure hanging in equilibrio.

This is a most useful principle in all cases of equilibriums, especially to the mere practical mechanist, and enables him in an experimental way to resolve problems, which the best mathematicians have found it no easy matter to effect by

mere computation. For thus, in a simple and easy way he obtains the shape of an equilibrated arch or bridge; and thus also he readily obtains the positions of the rafters in the frame of an equilibrated curb or mansard roof; a single instance of which may serve to show the extent and uses to which it may be applied. Thus, if it should be required to make a curb frame roof having a given width AE, and consisting of four rafters AB, BC, CD, DE, which shall either be equal or in any given proportion to each other. There can be no doubt but that the best form of the roof will be that which puts all its parts in equilibrio, so that there may be no unbalanced parts which may require the aid of ties or stays to keep the frame in its position Here the mechanic has nothing to do but to take four like but small pieces, that are either equal or in the same given proportions as those proposed, and connect them closely together at the joints A, B, C, D, E, by pins or strings, so as to be freely moveable about them; then suspend this from two pins *a*, *e*, fixed in a horizontal line, and the chain of the pieces will arrange itself in such a festoon or form, *abcde*, that all its parts will come to rest in equilibrio. Then, by inverting the figure, it will exhibit the form and frame of a curb roof $a\beta\gamma\delta e$, which will also be in equilibrio, the thrusts of the pieces now balancing each other, in the same manner as was done by the mutual pulls or tensions of the hanging festoon *a b c d e*. By varying the distance *ae*, of the points of suspension, moving them nearer to, or farther off, the chain will take different forms; then the frame ABCDE may be made similar to that form which has the most pleasing or convenient shape, found above as a model.





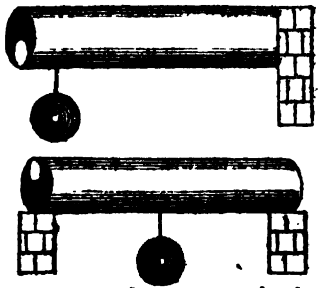
Indeed this principle is exceeding fruitful in its practical consequences. It is easy to perceive that it contains the whole theory of the construction of arches: for each stone of an arch may be considered as one of the rafters or beams in the foregoing frames, since the whole is sustained by the mere principle of equilibration, and the method, in its application, will afford some elegant and simple solutions of the most difficult cases of this important problem.

PROBLEM

PROBLEM 32.

Of all Hollow Cylinders, whose Lengths and the Diameters of the Inner and Outer Circles continue the same, it is required to show what will be the Position of the Inner Circle when the Cylinder is the Strongest Laterally.

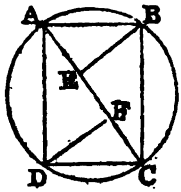
Since the magnitude of the two circles are constant, the area of the solid space, included between their two circumferences, will be the same, whatever be the position of the inner circle, that is, there is the same number of fibres to be broken, and in this respect the strength will be always the same. The strength then can only vary according to the situation of the centre of gravity of the solid part, and this again will depend on the place where the cylinder must first break, or on the manner in which it is fixed.



Now, by cor. 8 art. 251 Statics, the cylinder is strongest when the hollow, or inner circle, is nearest to that side where the fracture is to end, that is, at the bottom when it breaks first at the upper side, or when the cylinder is fixed only at one end as in the first figure. But the reverse will be the case when the cylinder is fixed at both ends; and consequently when it opens first below, or ends above, as in the 2d figure annexed.

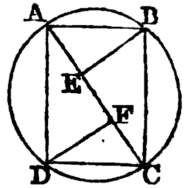
PROBLEM 33.

To determine the Dimensions of the Strongest Rectangular Beam that can be cut out of a Given Cylinder.



Let AB, the breadth of the required beam, be denoted by b, AD the depth by d, and the diameter AC of the cylinder by D. Now when AB is horizontal, the lateral strength is denoted by bd^2 (by art. 248 Statics), which is to be a maximum. But $AD^2 = AC^2 - AB^2$, or $d^2 = D^2 - b^2$; theref. $bd^2 = (D^2 - b^2)b = D^2b - b^3$ is a maximum: in fluxions $D^2\dot{b} - 3b^2\dot{b} = 0 = D^2 - 3b^2$, or $D^2 = 3b^2$; also $d^2 = D^2 - b^2 = 3b^2 - b^2 = 2b^2$. Conseq. $b^2 : d^2 : D^2 :: 1 : 2 : 3$, that is, the squares of the breadth, and of the depth, and of the cylinder's diameter, are to one another respectively as the three numbers 1, 2, 3.

Corol.



Corol. 1. Hence results this easy practical construction: divide the diameter AC into three equal parts, at the points E, F; erect the perpendiculars EB, FD; and join the points B, D to the extremities of the diameter: so shall ABCD be the rectangular end of the beam as required. For, because AE, AB, AC are in continued proportion (theor. 87 Geom.), theref. AE : AC : : $AB^2 : AC^2$; and in like manner AF : AC : : $AD^2 : AC^2$; hence AE : AF : AC : : $AB^2 : AD^2 : AC^2$: : 1 : 2 : 3.

Corol. 2. The ratios of the three b, d, D, being as the three $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$, or as 1, 1·414, 1·732, are nearly as the three 5, 7, 8·6, or more nearly as 12, 17, 20·8.

Corol. 3. A square beam cut out of the same cylinder, would have its side $= D\sqrt{\frac{1}{2}} = \frac{1}{2}D\sqrt{2}$. And its solidity would be to that of the strongest beam, as $\frac{1}{2}D^2$ to $\frac{1}{3}D^2\sqrt{2}$, or as 3 to $2\sqrt{2}$, or as 3 to 2·828; while its strength would be to that of the strongest beam, as $(D\sqrt{\frac{1}{2}})^3$ to $D\sqrt{\frac{1}{3}} \times \frac{2}{3}D^2$, or as $\frac{1}{4}\sqrt{2}$ to $\frac{2}{9}\sqrt{3}$, or as $9\sqrt{2}$ to $8\sqrt{3}$, or nearly as 101 to 110.

Corol. 4. Either of these beams will exert the greatest lateral strength, when the diagonal of its end is placed vertically, by art. 252 Statics.

Corol. 5. The strength of the whole cylinder will be to that of the square beam, when placed with its diagonal vertically, as the area of the circle to that of its inscribed square. For, the centre of the circle will be the centre of gravity of both beams, and is at the distance of the radius from the lowest point in each of them; conseq. their strengths will be as their areas, by art. 243 Statics.

PROBLEM 34.

To determine the Difference in the Strength of a Triangular Beam, according as it lies with the Edge or with the Flat Side Upwards.

In the same beam, the area is the same, and therefore the strength can only vary with the distance of the centre of gravity from the highest or lowest point; but, in a triangle, the distance of the centre of gravity from an angle, is double of its distance from the opposite side; therefore the strength of the beam will be as 2 to 1 with the different sides upwards, under different circumstances, viz, when the centre of gravity is farthest from the place where fracture ends, by art. 243 Statics, that is, with the angle upwards when the beam is supported

supported at both ends; but with the side upwards, when it is supported only at one end, (art. 252 Statics), because in the former case the beam breaks first below, but the reverse in the latter case.

PROBLEM 35.

Given the Length and Weight of a Cylinder or Prism, placed Horizontally with one end firmly fixed, and will just support a given weight at the other end without breaking; it is required to find the Length of a Similar Prism or Cylinder which, when supported in like manner at one end, shall just bear without breaking another given weight at the unsupported end.

Let l denote the length of the given cylinder or prism, d the diameter or depth of its end, w its weight, and u the weight hanging at the unsupported end; also let the like capitals L, D, W, U denote the corresponding particulars of the other prism or cylinder. Then, the weights of similar solids of the same matter being as the cubes of their lengths, as $l^3 : \text{L}^3 :: w : \frac{\text{L}^3}{l^3}w$, the weight of the prism whose length is L. Now $\frac{1}{2}wl$ will be the stress on the first beam by its own weight w acting at its centre of gravity, or at half its length; and lu the stress of the added weight u at its extremity, their sum $(\frac{1}{2}w + u)l$ will therefore be the whole stress on the given beam: in like manner the whole stress on the other beam, whose weight is W or $\frac{\text{L}^3}{l^3}w$, will be $(\frac{1}{2}\text{W}+\text{U})\text{L}$ or $(\frac{\text{L}^3}{2l^3}w+\text{U})\text{L}$.

But the lateral strength of the first beam is to that of the second, as d^3 to D^3 (art. 246 Statics), or as l^3 to L^3; and the strengths and stresses of the two beams must be in the same ratio, to answer the conditions of the problem; therefore as $(\frac{1}{2}w+u)l : (\frac{\text{L}^3}{2l^3}w+\text{U})\text{L} :: l^3 : \text{L}^3$; this analogy, turned into an equation, gives $\text{L}^3 - \frac{w+2u}{w}l\text{L}^2 + \frac{2}{w}l^3\text{U} = 0$, a cubic equation from which the numeral value of L may be easily determined, when those of the other letters are known.

Corol. 1. When U vanishes, the equation gives $\text{L}^3 = \frac{w+2u}{w}l\text{L}^2$, or $\text{L} = \frac{w+2u}{w}l$, whence $w : w+2u :: l : \text{L}$, for the length of the beam, which will but just support its own weight.

Corol. 2. If a beam just only support its own weight, when fixed at one end; then a beam of double its length, fixed at both ends, will also just sustain itself: or if the one just break, the other will do the same.

PROBLEM 36.

Given the Length and Weight of a Cylinder or Prism, fixed Horizontally as in the foregoing problem, and a weight which when hung at a given point, Breaks the Prism: it is required to determine how much longer the Prism, of equal Diameter or of equal Breadth and Depth, may be extended before it Break, either by its own weight, or by the addition of any other adventitious weight.

Let l denote the length of the given prism, w its weight, and u a weight attached to it at the distance d from the fixed end; also let L denote the required length of the other prism, and U the weight attached to it at the distance D. Now the strain occasioned by the weight of the first beam is $\frac{1}{2}wl$, and that by the weight u at the distance d, is du, their sum $\frac{1}{2}wl + du$ being the whole strain. In like manner $\frac{1}{2}WL + DU$ is the strain on the second beam; but $l : L :: w : \frac{Lw}{l} = W$ the weight of this beam, theref. $\frac{wL^2}{2l} + DU =$ its strain. But the strength of the beam, which is just sufficient to resist these strains, is the same in both cases; therefore $\frac{wL^2}{2l} + DU = \frac{1}{2}wl + du$, and hence, by reduction, the required length $L = \sqrt{(l \times \frac{wl + 2du - 2DU}{w})}$.

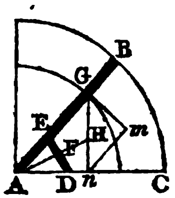
Corol. 1. When the lengthened beam just breaks by its own weight, then $U = 0$ or vanishes, and the required length becomes $L = \sqrt{(l \times \frac{wl + 2du}{w})}$.

Corol 2. Also when U vanishes, if d become $= l$, then $L = l\sqrt{\frac{w+2u}{w}}$ is the required length.

PROBLEM 37.

Let AB *be a beam moveable about the end* A, *so as to make any angle* BAC *with the plane of the horizon* AC: *it is required to determine the position of a prop or supporter* DB *of a given length, which shall sustain it with the greatest ease in any given position; also to ascertain the angle* BAC *when the least force which can sustain* AB, *is greater than the least force in any other position.*

Let



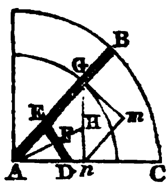
Let G be the centre of gravity of the beam; and draw Gm perp. to AB, Gn to AC, nm to Gm, and AFH to DE. Put r = AG, p = DE, w = the weight of the beam AB, and An = x. Then by the nature of the parallelogram of forces, Gn : Gm, or by sim. triangles, AG = r : An = x : : w : $\frac{wx}{r}$, the force which acting at G in the direction mG, is sufficient to sustain the beam; and by the nature of the lever, AE : AG = r : : $\frac{wx}{\text{AG}}$ the requisite force at G : $\frac{wx}{\text{AE}}$, the force capable of supporting it at E in a direction perp. to AB or parallel to mG; and again as AF : AE : : $\frac{wx}{\text{AE}}$: $\frac{wx}{\text{AF}}$, the force or pressure actually sustained by the given prop DE in a direction perp. to AF: And this latter force will manifestly be the least possible when the perp. AF upon DE is the greatest possible, whatever the angle BAC may be, which is when the triangle ADE is isosceles, or has the side AD = AE, by an obvious corol. from the latter part of prob. 6, Division of Surfaces, vol. 1.

Secondly, for a solution to the latter part of the problem, we have to find when $\frac{wx}{\text{AF}}$ is a maximum; the angles D and E being always equal to each other, while they vary in magnitude by the change in the position of AB. Let AF produced meet Gn in H: then, in the similar triangles ADF, AHn, it will be AF : An = x : : DF = $\frac{1}{2}p$: Hn, hence $\frac{x}{\text{AF}} = \frac{\text{H}n}{\frac{1}{2}p}$, and conseq. $\frac{x}{\text{F}} \times w = \frac{\text{H}n}{\frac{1}{2}p} \times w$. But, by theor. 83 Geom. and comp. AG + An = $r + x$: An = x : : Gn = $\sqrt{(r^2 - x^2)}$: Hn = $\frac{x}{r+x}\sqrt{(r^2-x^2)} = x\sqrt{\frac{r-x}{r+x}}$: consequently the force $\frac{\text{H}n}{\frac{1}{2}p} \times w$, acting on the prop, is also truly expressed by $\frac{wx}{\frac{1}{2}p}\sqrt{\frac{r-x}{r+x}}$. Then the fluxion of this made to vanish gives $x = \frac{\sqrt{5}-1}{2}r$ the cos. angle BAC = 51° 50′, the inclination required.

PROBLEM

PROBLEM 38.

Suppose the Beam AB, *instead of being moveable about the centre* A, *as in the last problem, to be supported in a given position by means of the given prop* DE : *it is required to determine the position of that prop, so that the prismatic beam* AC, *on which it stands, may be the least liable to breaking, this latter beam being only supported at its two ends* A *and* C.



Put the base AC $= b$, the prop DE $= p$, AG $= r$, the weight of AB $= w$, s and c the sine and cosine of $\angle$ A, $x =$ sin. $\angle$ E, $y =$ sin. $\angle$ D, and $z =$ AE. Then, by trigon. $z : y :: p : s$, or $\frac{y}{x} = \frac{s}{p}$, and AD $= \frac{px}{s}$; also $cw =$ the force of the beam at G in direction Gm. Let F denote the force sustaining the beam at E in the direction ED : then, because action and reaction are equal and opposite, the same force will be exerted at D in the direction DE : therefore AG . $cw =$ Fzx, and F $= \frac{rcw}{zx}$. Again, the vertical stress at D, will be as F $\times$ sine D $\times$ AD . DC $=$ Fy . AD . DC $= \frac{rcwy}{zx} \times \frac{px}{s}(b - \frac{px}{s}) =$ (substituting $\frac{s}{p}$ for its equal $\frac{y}{z}$) $\frac{rcws}{px} \times \frac{px}{s} \times \frac{bs-px}{s} = rcw \times \frac{bs-px}{s} = \frac{rcwp}{s} \times (\frac{bs}{p} - x) =$ a minimum by the problem. Conseq. $\frac{bs}{p} - x$ is a minimum, or x a maximum, that is, $x = 1$, and the angle E is a right angle. Hence the point E is easily found by this proportion, sin. A : cos. A :: ED : EA.

PROBLEM 39.

To explain the Disposition of the Parts of Machines.

When several pieces of timber, iron, or any other materials, are employed in a machine or structure of any kind, all the parts, both of the same piece, and of the different pieces in the fabric, ought to be so adjusted with respect to magnitude, that the strength in every part may be, as near as possible, in a constant proportion to the stress or strain to which they will be subjected. Thus, in the construction of any engine, the weight and pressure on every part should be investigated, and the strength apportioned accordingly. All levers, for instance, should be made strongest where they are most strained ; viz, levers of the first kind, at the fulcrum ; levers of

of the second kind, where the weight acts; and those of the third kind, where the power is applied. The axles of wheels and pulleys, the teeth of wheels, also ropes, &c, must be made stronger or weaker, as they are to be more or less acted on. The strength allotted should be more than fully competent to the stress to which the parts can ever be liable; but without allowing the surplus to be extravagant; for an over excess of strength in any part, instead of being serviceable, would be very injurious, by increasing the resistance the machine has to overcome, and thus encumbering, impeding, and even preventing the requisite motion; while, on the other hand, a defect of strength in any part will cause a failure there, and either render the whole useless, or demand very frequent repairs.

PROBLEM 40.

To ascertain the Strength of Various Substances.

The proportions that we have given on the strength and stress of materials, however true, according to the principles assumed, are of little or no use in practice, till the comparative strength of different substances is ascertained: and even then they will apply more or less accurately to different substances. Hitherto they have been applied almost exclusively to the resisting force of beams of timber; though probably no materials whatever accord less with the theory than timber of all kinds. In the theory, the resisting body is supposed to be perfectly homogeneous, or composed of parallel fibres, equally distributed round an axis, and presenting uniform resistance to rupture. But this is not the case in a beam of timber: for, by tracing the process of vegetation, it is readily seen that the ligneous coats of a tree, formed by its annual growth, are almost concentric; being like so many hollow cylinders thrust into each other, and united by a kind of medullary substance, which offers but little resistance: these hollow cylinders therefore furnish the chief strength and resistance to the force which tends to break them.

Now, when the trunk of a tree is squared, in order that it may be converted into a beam, it is plain that all the ligneous cylinders greater than the circle inscribed in the square or rectangle, which is the transverse section of the beam, are cut off at the sides; and therefore almost the whole strength or resistance arises from the cylindric trunk inscribed in the solid part of the beam; the portions of the cylindric coats, situated towards the angles, adding but little comparatively to the strength and resistance of the beam. Hence it follows that we cannot, by legitimate comparison, accurately deduce

the strength of a joist, cut from a small tree, by experiments on another which has been sawn from a much larger tree or block. As to the concentric cylinders above mentioned, they are evidently not all of equal strength: those nearest the centre, being the oldest, are also the hardest and strongest; which again is contrary to the theory, in which they are supposed uniform throughout. But yet, after all however, it is still found that, in some of the most important problems, the results of the theory and well-conducted experiments coincide, even with regard to timber: thus, for example, the experiments on rectangular beams afford results deviating but in a very slight degree from the theorem, that the strength is proportional to the product of the breadth and the square of the depth.

Experiments on the strength of different kinds of wood, are by no means so numerous as might be wished: the most useful seem to be those made by Muschenbroek, Buffon, Emerson, Parent, Banks, and Girard. But it will be at all times highly advantageous to make new experiments on the same subject; a labour especially reserved for engineers who possess skill and zeal for the advancement of their profession. It has been found by experiments, that the same kind of wood, and of the same shape and dimensions, will bear or break with very different weights: that one piece is much stronger than another, not only cut out of the same tree, but out of the same rod; and that even, if a piece of any length, planed equally thick throughout, be separated into three or four pieces of an equal length, it will often be found that these pieces require different weights to break them. Emerson observes that wood from the boughs and branches of trees is far weaker than that of the trunk or body; the wood of the large limbs stronger than that of the smaller ones; and the wood in the heart of a sound tree strongest of all; though some authors differ on this point. It is also observed that a piece of timber which has borne a great weight for a short time, has broke with a far less weight, when left upon it for a much longer time. Wood is also weaker when green, and strongest when thoroughly dried, in the course of two or three years, at least. Wood is often very much weakened by knots in it; also when cross-grained, as often happens in sawing, it will be weakened in a greater or less degree, according as the cut runs more or less across the grain. From all which it follows, that a considerable allowance ought to be made for the various strength of wood, when applied to any use where strength and durability are required.

Iron is much more uniform in its strength than wood. Yet experiments

experiments show that there is some difference arising from different kinds of ore: a difference is also found not only in iron from different furnaces, but from the same furnace, and even from the same melting; which may arise in a great measure from the different degrees of heat it has when poured into the mould.

Every beam or bar, whether of wood, iron, or stone, is more easily broken by any transverse strain, while it is also suffering any very great compression endways; so much so indeed that we have sometimes seen a rod, or a long slender beam, when used as a prop or shoar, urged home to such a degree, that it has burst asunder with a violent spring. Several experiments have been made on this kind of strain: a piece of white marble, ¼ of an inch square, and 3 inches long, bore 38lbs; but when compressed endways with 300lbs, it broke with 14½lbs. The effect is much more observable in timber, and more elastic bodies; but is considerable in all. This is a point therefore that must be attended to in all experiments; as well as the following, viz, that a beam supported at both ends, will carry almost twice as much when the ends beyond the props are kept from rising, as when the beam rests loosely on the props.

The following list of the absolute strength of several materials, is extracted from the collection made by professor Robison, from the experiments of Muschenbroek and other experimentalists. The specimens are supposed to be prisms or cylinders of one square inch transverse area, which are stretched or drawn lengthways by suspended weights, gradually increased till the bars parted or were torn asunder by the number of avoirdupois pounds, on a medium of many trials, set opposite each name.

1st. Metals.

	lbs.		lbs.
Gold, cast	22,000	Tin, cast	5,000
Silver, cast . . .	42,000	Lead, cast	860
Copper, cast . . .	34,000	Regulus of Antimony	1,000
Iron, cast	50,000	Zinc	2,600
Iron, bar	70,000	Bismuth	2,900
Steel, bar . . .	135,000		

It is very remarkable that almost all the metallic mixtures are more tenacious than the metals themselves. The change of tenacity depends much on the proportion of the ingredients; and yet the proportion which produces the most tenacious mixture, is different in the different metals. The proportion of

of ingredients here selected, is that which produces the greatest strength.

	lbs.		lbs.
2 parts gold with 1		Brass, of copper and tin	51,000
silver	28,000	3 tin, 1 lead . . .	10,200
5 pts gold, 1 copper	50,000	8 tin, 1 zinc . .	10,000
5 silver, 1 copper .	48,500	4 tin, 1 regul. antim.	12,000
4 silver, 1 tin . .	41,000	8 lead, 1 zinc . .	4,500
6 copper, 1 tin . .	60,000	4 tin, 1 lead, 1 zinc	13,000

These numbers are of considerable use in the arts. The mixtures of copper and tin are particularly interesting in the fabric of great guns. By mixing copper, whose greatest strength does not exceed 37,000, with tin which does not exceed 6000, is produced a metal whose tenacity is almost double, at the same time that it is harder and more easily wrought: it is however more fusible. We see also that a very small addition of zinc almost doubles the tenacity of tin, and increases the tenacity of lead 5 times; and a small addition of lead doubles the tenacity of tin. These are economical mixtures; and afford valuable information to plumbers for augmenting the strength of water-pipes. Also, by having recourse to these tables, the engineer can proportion the thickness of his pipes, of whatever metal, to the pressures they are to suffer.

2d. Woods, &c.

	lbs.		lbs.
Locust tree . . .	20,100	Tamarind	8,750
Jujeb	18,500	Fir	8,330
Beech, Oak	17,300	Walnut	8,130
Orange	15,500	Pitch pine . . .	7,650
Alder	13,900	Quince	6,750
Elm	13,200	Cypress	6,000
Mulberry	12,500	Poplar	5,500
Willow	12,500	Cedar	4,880
Ash	12,000	Ivory	16,270
Plum	11,800	Bone	5,250
Elder	10,000	Horn	8,750
Pomegranate . .	9,750	Whalebone	7,500
Lemon	9,250	Tooth of sea-calf . .	4,075

It is to be observed that these numbers express something more than the utmost cohesion; the weights being such as will very soon, perhaps in a minute or two, tear the rods asunder. It may be said in general, that $\frac{2}{3}$ of these weights will sensibly impair the strength after acting a considerable while, and that one-half is the utmost that can remain permanently

manently suspended at the rods with safety; and it is this last allotment that the engineer should reckon upon in his constructions. There is however considerable difference in this respect: woods of a very straight fibre, such as fir, will be less impaired by any load which is not sufficient to break them immediately. According to Mr. Emerson, the load which may be safely suspended to an inch square of various materials, is as follows:

	lbs.		lbs.
Iron	76,400	Red fir, holly, elder, plane	5,000
Brass	35,600	Cherry, hazle	4,760
Hempen rope . .	19,600	Alder, asp, birch, willow	4,290
Ivory	15,700	Freestone	914
Oak, box, yew, plum	7,850	Lead	430
Elm, ash, beech	6,070		
Walnut, plumb .	5,360		

He gives also the practical rule, that a cylinder whose diameter is d inches, loaded to $\frac{1}{4}$ of its absolute strength, will carry permanently as here annexed.

	cwts.
Iron	$135d^2$
Good rope	$22d^2$
Oak	$14d^2$
Fir	$9d^2$

Experiments on the transverse strength of bodies are easily made, and accordingly are very numerous, especially those made on timber, being the most common and the most interesting. The completest series we have seen is that given by Belidor, in his Science des Ingenieurs, and is exhibited in the following table. The first column simply indicates the number of the experiments; the column b shows the breadth of the pieces, in inches; the column d contains their depths; the column l shows the lengths; and column *lbs* shows the weights in pounds which broke them, when suspended by their middle points, being the medium of 3 trials of each piece; the accompanying words, *fixed* and *loose* denoting whether the ends were firmly fixed down, or simply lay loose on the supports.

N°.	b	d	l	*lbs.*	
1	1	1	18	406	loose.
2	1	1	18	608	fixed.
3	2	1	18	805	loose.
4	1	2	18	1580	loose.
5	1	1	36	187	loose.
6	1	1	36	283	fixed.
7	2	2	36	1585	loose.
8	$1\frac{1}{2}$	$2\frac{1}{2}$	36	1660	loose.

By

By comparing experiments 1 and 3, the strength appears proportional to the breadth.

Experiments 3 and 4 show the strength to be as the breadth multiplied by the square of the depth.

Experiments 1 and 5 show the strength nearly in the inverse ratio of the lengths, but with a sensible deficiency in the longer pieces.

Experiments 5 and 7 show the strength to be proportional to the breadth and the square of the depth.

Experiments 1 and 7 show the same thing, compounded with the inverse ratio of the length; the deficiency of which is not so remarkable here.

Experiments 1 and 2, and experiments 5 and 6, show the increase of strength, by fastening down the ends, to be in the proportion of 2 to 3; which the theory states as 2 to 4, the difference being probably owing to the manner of fixing.

Mr. Buffon made numerous experiments, both on small bars, and on large ones, which are the best. The following is a specimen of one set, made on bars of sound oak, clear of knots.

Length. feet.	Weight. lbs.	Broke with lbs.	Bent. inch.	Time. min.
7	60	5350	3·5	29′
	56	5275	4·5	22
8	68	4600	3·75	15
	63	4500	4·7	13
9	77	4100	4·85	14
	71	3950	5·5	12
10	84	3625	5·83	15
	82	3600	6·5	15
12	100	3050	7	
	98	2925	8	

Column 1 shows the length of the bar, in feet, clear between the supports.—Column 2 is the weight of the bar in lbs, the 2d day after it was felled.—Column 3 shows the number of pounds necessary for breaking the tree in a few minutes.—Col. 4 is the number of inches it bent down before breaking.—Col. 5 is the time at which it broke.—The parts next the root were always the heaviest and strongest.

The following experiments on other sizes were made in the same way; two at least of each length being taken, and the table contains the mean results. The beams were all squared, and their sides in inches are placed at the top of the columns, their

their lengths in feet being in the first column. The numbers in the other columns, are the pounds weight which broke the pieces.

	4	5	6	7	8	A
7	5312	11525	18950	32200	47649	11525
8	4550	9787	15525	26050	39750	10085
9	4025	8308	13150	22350	32800	8964
10	3612	7125	11250	19475	27750	8068
12	2987	6075	9100	16175	23450	6723
14		5300	7475	13225	19775	5763
16		4350	6362	11000	16375	5042
18		3700	5562	9245	13200	4482
20		3225	4950	8375	11487	4034
22		2975				3667
24		2162				3362
28		1775				2881

Mr. Buffon had found, by many trials, that oak timber lost much of its strength in the course of seasoning or drying; and therefore, to secure uniformity, his trees were all felled in the same season of the year, were squared the day after, and the experiments tried the third day. Trying them in this green state gave him an opportunity of observing a very curious phenomenon. When the weights were laid quickly on, nearly sufficient to break the beam, a very sensible smoke was observed to issue from the two ends with a sharp hissing sound; which continued all the time the tree was bending and cracking. This shows the great effects of the compression, and that the beam is strained through its whole length, which is shown also by its bending through the whole length.

Mr. Buffon considers the experiments with the 5-inch bars as the standard of comparison, having both extended these to greater lengths, and also tried more pieces of each length. Now, the theory determines the relative strength of bars, of the same section, to be inversely as their lengths: but most of the trials show a great deviation from this rule, probably owing, in part at least, to the weights of the pieces themselves. Thus, the 5-inch bar of 28 feet long should have half the strength of that of 14 feet or 2650, whereas it is only 1775; the bar of 14 feet should have half the strength of that of 7 feet, or 5762, but is only 5300; and so of others. The column A is added, to show the strength that each of the 5-inch bars ought to have by the theory.

Mr.

Mr. Banks, an ingenious lecturer on natural philosophy, has made many experiments on the strength of oak, deal, and iron. He found that the worst or weakest piece of dry heart of oak, 1 inch square, and 1 foot long, broke with 602lbs, and the strongest piece with 974lbs: the worst piece of deal broke with 464lbs, and the best with 690lbs. A like bar of the worst kind of cast iron 2190lbs. Bars of iron set up in positions oblique to the horizon, showed strengths nearly proportional to the sines of elevation of the pieces. Equal bars placed horizontally, on supports 3 feet distant, bore $6\frac{3}{4}$ cwt; the same at $2\frac{1}{2}$ feet distance broke only with 9 cwt.—An arched rib of $29\frac{1}{2}$ feet span, and 11 inches high in the centre, supported $99\frac{1}{2}$ cwt; it sunk in the middle $3\frac{1}{8}$ inches, and rose again $\frac{3}{8}$ on removing the load. The same rib tried without abutments, broke with 55 cwt.—Another rib, a segment of a circle, $29\frac{1}{4}$ feet span, and 3 feet high in the middle, bore $100\frac{1}{2}$ cwt, and sunk $1\frac{3}{10}$ in the middle. The same rib without abutments, broke with $64\frac{1}{2}$ cwt.

Mr. Banks made also experiments at another foundry, on like bars of 1 inch square, each yard in length weighing 9lbs, the props at 3 feet asunder.

The 1st bar broke with	963 lbs.
The 2d ditto	958
The 3d ditto	994
Bar made from the cupola, broke with . .	864
Bar equally thick in the middle, but the ends shaped into a parabola, and weighed $6\frac{3}{16}$lbs, broke with	874

From these, and many other experiments, Mr. Banks concludes, that cast iron is from $3\frac{1}{3}$ to $4\frac{1}{2}$ times stronger than oak of the same dimensions, and from 5 to $6\frac{1}{2}$ times stronger than deal.

Some Examples for Practice.

The theory, as has been before mentioned, is, That the strength of a bar, or the weight it will bear, is directly as the breadth and square of the depth divided by the length. So that, if b denote the breadth of a bar, d the depth, l the length, and w the weight it will bear; and the capitals B, D, L, W denote the like quantities in another bar; then, by the rule $\frac{bd^2}{l} : w :: \frac{BD^2}{L} : W$, which gives this general equation $bd^2LW = BD^2lw$, from which any one of the letters is easily found when the rest are given.

Now, if we take, for a standard of comparison, this experiment of Mr. Banks, that a bar of oak an inch square and a foot

foot in length, lying on a prop at each end, and its strength, or the utmost weight it can bear, on its middle, 660lbs: here $b = 1, d = 1, l = 1, w = 660$; these substituted in the above equation, it becomes $LW = 660BD^2$, from which any one of the four quantities L, W, B, D, may be found, when the other three are given, when the calculation respects oak timber. But for fir the like rule will be $LW = 440BD^2$, and for iron $LW = 2640BD^2$.

Exam. 1. Required the utmost strength of an oak beam, of 6 inches square and 8 feet long, supported at each end, or the weight to break it in the middle?

Here are given $B = 6$, $D = 6$, $L = 8$, to find $W = \frac{660BD^2}{L}$ $= \frac{660 \times 6 \times 36}{8} = 660 \times 3 \times 9 = 17820$ lbs.

Exam. 2. Required the depth of an oak beam, of the same length and strength as above, but only 3 inches breadth?

Here, as $3 : 6 :: 36 : D^2 = 72$, theref. $D = \sqrt{72} = 8{\cdot}485$ the depth.

This last beam, though as strong as the former, is but little more than $\frac{2}{3}$ of its size or quantity. And thus, by making joists thinner, a great part of the expense is saved, as in the modern style of flooring, &c.

Exam. 3. To determine the utmost strength of a deal joist of 2 inches thick and 8 inches deep, the bearing or breadth of the room being 12 feet?—Here $B = 2$, $D = 8$, $L = 12$; then the rule $LW = 440BD^2$ gives $W = \frac{440 \times B \times D^2}{L} = \frac{440 \times 2 \times 64}{12} = \frac{440 \times 32}{3} = 4693$ lbs.

Exam. 4. Required the depth of a bar of iron 2 inches broad and 8 feet long, to sustain a load of 20,000lbs?—Here $B = 2$, $L = 8$, and $W = 20{,}000$, to find D from the equation $LW = 2640BD^2$, viz, $D^2 = \frac{LW}{2640B} = \frac{8 \times 20000}{2640 \times 2} = \frac{1000}{33} = 30{\cdot}3$, and $D = \sqrt{30{\cdot}3} = 5\frac{1}{2}$ inches, the depth.

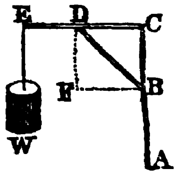
Exam. 5. To find the length of a bar of oak, an inch square, so that when supported at both ends it may just break by its own weight?—Here according to the notation and calculation in prob. 36, $l = 1$, $w = \frac{1}{2}$ of a lb, the weight of 1 foot in length, and $u = 660$lbs. Then $L = l\sqrt{\frac{w + 2a}{w}} = \sqrt{3301} = 57{\cdot}45$ feet, nearly.

Exam. 6 To find the length of an iron bar an inch square, that it may break by its own weight, when it is supported at both ends.—Here as before $l = 1$, $w = 3$lbs nearly the

weight of 1 foot in length, also $u = 2640$. Therefore $L = l\sqrt{\frac{w+2u}{w}} = 41\ 97$ feet nearly.

Note. It might perhaps have been supposed that this last result should exceed the preceding one: but it must be considered that while iron is only about 4 times stronger than oak, it is at least 8 times heavier.

Exam. 7. When a weight W is suspended from E on the arm of a crane ABCDE, it is required to find the pressure at the end D of the spur, and that at B against the upright post AC.

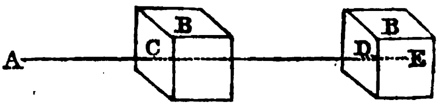


Here, by the nature of the lever, $\frac{CE}{CD}W =$ the pressure at D in the vertical direction DF: but this pressure in DF is to that in DB as DF to DB, viz, $DF : DB :: \frac{CE}{CD}W : \frac{CE \cdot DB}{DF \cdot CD}W$ the pressure in DB; and again, $DB : FB$ or $CD :: \frac{CE \cdot DB}{DF \cdot CD}W : \frac{CE}{DF}W = \frac{CE}{BC}W$ the pressure against B in direction FB.

Thus, for example, if $CE = 16$ feet, $BC = 6$, $CD = 8$, $BD = 10$, and $W = 3$ tons: then $\frac{CE \cdot BD}{BC \cdot CD}W = \frac{16 \cdot 10}{6 \cdot 8} \times 3 = 10$ tons, for the pressure on the spur DB. Also $\frac{CE}{CD}W = \frac{16}{6} \times 3 = 8$ tons, the force tending to break the bar AC at B.

PROBLEM 41.

To determine the circumstances of Space, Penetration, Velocity, and Time, arising from a Ball moving with a Given Velocity, and Striking a moveable Block of Wood, or other substance.



Let the ball move in the direction AE passing through the centre of gravity of the block B, impinging on the point C; and when the block has moved through the space CD in consequence of the blow, let the ball have penetrated to the depth DE.

Let $B =$ the mass or matter in the block,

$b =$ the same in the ball,

$s =$ CD the space moved by the block,

$x =$

$x =$ DE the penetration of the ball, and theref.
$s + x =$ CE the space described by the ball,
$a =$ the first velocity of the ball,
$v =$ the velocity of the ball at E,
$u =$ velocity of the block at the same instant,
$t =$ the time of penetration, or of the motion,
$r =$ the resisting force of the wood.

Then shall $\frac{r}{B}$ be the accelerating force of the block,

and $\frac{r}{b}$ the retarding force of the ball.

Now because the momentum $B\dot{u}$, communicated to the block in the time $\dot{t}$, is that which is lost by the ball, namely $-b\dot{v}$, therefore $B\dot{u} = -b\dot{v}$, and $Bu = -bv$. But when $v = a$, $u = 0$; therefore, by correcting, $Bu = b(a - v)$; or the momentum of the block is every where equal to the momentum lost by the ball. And when the ball has penetrated to the utmost depth, or when $u = v$, this becomes $Bu = b(a - u)$, or $ab = (B + b)u$; that is, the momentum before the stroke, is equal to the momentum after it. And the velocity communicated will be the same, whatever be the resisting force of the block, the weight being the same.

Again, (by prob. 6, Forces), it is $u^2 = \frac{4grs}{B}$, and $-v^2 = \frac{4gr}{b} \times (s + x)$, or rather, by correction, $a^2 - v^2 = \frac{4gr}{b}(s + x)$. Hence the penetration or $x = \frac{b(a^2 - v^2) - 4grs}{4gr}$. And when $v = u$, by substituting u for v, and Bu^2 for $4grs$, the greatest penetration becomes $\frac{ba^2 - (B+b)u^2}{4gr}$; and this again by writing ab for its value $(B + b)u$, gives the greatest penetration $x = \frac{Bba^2}{4gr(B+b)} = \frac{ba^2}{4gr} \times (1 - \frac{b}{B+b})$. Which is barely equal to $\frac{ba^2}{4gr}$ when the block is fixed, or infinitely great; and is always very nearly equal to the same $\frac{ba^2}{4gr}$ when B is very great in respect of b.

Hence $s + x = \frac{a^2 - u^2}{4gr} b = \frac{a^2 - \frac{a^2b^2}{(B+b)^2}}{4gr} b = \frac{B^2 + 2Bb}{(B+b)^2} \times \frac{a^2b}{4gr}$.
And theref. $B + b : B + 2b :: x : s + x$, or $B + b : b :: x : s$
and $s = \frac{bx}{B+b} = \frac{Bb^2a^2}{4gr(B+b)^2}$.

Exam. When the ball is iron, and weighs 1 pound, it penetrates

penetrates elm about 13 inches when it moves with a velocity of 1500 feet per second, in which case,

$$\frac{r}{b} = \frac{a^2}{4gr} = \frac{1500^2}{4 \times 16\frac{1}{12} \times \frac{13}{12}} = \frac{9000^2}{193 \times 13} = 32284 \text{ nearly.}$$

When B = 500lb, and $b = 1$; then $u = \frac{ab}{\text{B}+b} = \frac{1500}{501} = 3$ feet nearly per second, the velocity of the block.

Also $s = \frac{\text{B}u^2}{4gr} = \frac{500 \times 9}{4 \times 16\frac{1}{12} \times 32284} = \frac{1}{461\frac{1}{3}}$ part of a foot, or $\frac{2}{77}$ of an inch, which is the space moved by the block when the ball has completed its penetration.

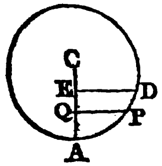
And $t = \frac{2s}{u} = \frac{2}{461\frac{1}{3} \times 3} = \frac{1}{692}$ part of a second, or

$$t = \frac{2s+2x}{v} = \frac{\frac{2}{461\frac{1}{3}} + \frac{26}{12}}{1500} = \frac{6+13 \,.\, 231}{6 \,.\, 231 \,.\, 1500} = \frac{1}{692} \text{ part of a second,}$$

the time of penetration.

PROBLEM 42.

To find the Velocity and Time of a Heavy Body descending down the Arc of a Circle, or vibrating in the Arc by a Line fixed in the Centre.



Let D be the beginning of the descent, C the centre, and A the lowest point of the circle; draw DE and PQ perpendicular to AC. Then the velocity in P being the same as in Q by falling through EQ, it will be $v = 2\surd(g \times \text{EQ}) = 8\surd(a-x)$, when $a = \text{AE}$, $x = \text{AQ}$.

But the flux. of the time $\dot{t}$ is $= \frac{-\dot{\text{AP}}}{v}$, and $\dot{\text{AP}} = \frac{r\dot{x}}{\surd(2rx-x^2)}$ where $r =$ the radius AC. Theref. $\dot{t} = \frac{r}{8} \times \frac{-\dot{x}}{\surd(2rx-x^2) \times \surd(a-x)}$

$$= \frac{d}{16} \times \frac{-\dot{x}}{\surd(ax-x^2) \times \surd(d-x)} = \frac{-\surd d}{16} \times \frac{\dot{x}}{\surd(ax-x^2) \times \surd(1-\frac{x}{d})},$$

where $d = 2r$ the diameter.

Or $\dot{t} = \frac{-\surd d}{16} \times \frac{\dot{x}}{\surd(ax-x^2)}\left(1 + \frac{x}{2d} + \frac{1 \,.\, 3x^2}{2 \,.\, 4d^2} + \frac{1 \,.\, 3 \,.\, 5x^3}{2 \,.\, 4 \,.\, 6d^3} \,\&\text{c}\right)$, by developing $\surd(1 - \frac{x}{d})$ in a series.

But the fluent of $\frac{\dot{x}}{\surd(ax-x^2)}$ is $\frac{2}{a} \times$ arc to radius $\frac{1}{2}a$ and vers. x, or it is the arc whose rad. is 1 and vers. $\frac{2x}{a}$: which call A. And let the fluents of the succeeding terms, without the coefficients, be, B, C, D, E, &c. Then will the fluxion of any one

one, as $\dot{Q}$, at n distance from A, be $\dot{Q} = x^n\dot{A} = x\dot{P}$, which suppose also = the flux. of $bP - dx^{n-1}\sqrt{(ax - x^2)} = b\dot{P} - d(n-1)\dot{x}x^{n-2}\sqrt{(ax-x^2)} - d\dot{x}x^{n-2} \times \frac{\frac{1}{2}ax - x^2}{\sqrt{(ax-x^2)}} = b\dot{P} - d\dot{x} \times \frac{(n-\frac{1}{2})ax^{n-1} - nx^n}{\sqrt{(ax-x^2)}} = b\dot{P} - d(n-\frac{1}{2})a\dot{P} + dnx\dot{P}$.

Hence, by equating the coefficients of the like terms, $d = \frac{1}{n}$; $b = \frac{2n-1}{2n}a$; and $Q = \frac{(2n-1)aP - 2x^{n-1}\sqrt{(ax-x^2)}}{2n}$

Which being substituted, the fluential terms become $\frac{\sqrt{d}}{16} \times (-A - \frac{1}{2d}.\frac{aA - 2\sqrt{(ax-x^2)}}{2} - \frac{1.3}{2.4d^2}.\frac{3aB - 2x\sqrt{(ax-x^2)}}{4} - \frac{1.3.5}{2.4.6d^3}.\frac{5aC - 2x^2\sqrt{(ax-x^2)}}{6} - \&c)$. Or the same fluents will be found by art. 80 Fluxions.

But when, $x = a$, those terms become barely $\frac{3\cdot1416\sqrt{d}}{16} \times (-1 - \frac{1^2a}{2^2d} - \frac{1^2.3^2a^2}{2^2.4^2d^2} - \frac{1^2.3^2.5^2a^3}{2^2.4^2.6^2d^3} - \&c)$; which being subtracted, and x taken $= 0$, there arises for the whole time of descending down DA, or the corrected value of $t = \frac{3\cdot1416\sqrt{d}}{16} \times (1 + \frac{1^2a}{2^2d} + \frac{1^2.3^2a^2}{2^2.4^2d^2} + \frac{1^2.3^2.5^2a^3}{2^2.4^2.6^2d^3} + \&c)$.

When the arc is small, as in the vibration of the pendulum of a clock, all the terms of the series may be omitted after the second, and then the time of a semi-vibration t is nearly $= \frac{1\cdot5708}{4}\sqrt{\frac{r}{2}} \times (1 + \frac{a}{8r})$. And theref. the times of vibration of a pendulum, in different arcs, are as $8r + a$, or 8 times the radius added to the versed sine of the arc.

If D be the degrees of the pendulum's vibration, on each side of the lowest point of the small arc, the radius being r, the diameter d, and $3\cdot1416 = p$; then is the length of that arc $A = \frac{prD}{180} = \frac{pdD}{360}$. But the versed sine in terms of the arc is $a = \frac{A^2}{2r} - \frac{A^4}{24r^3} + \&c = \frac{A^2}{d} - \frac{A^4}{3d^3} + \&c$. Therefore $\frac{a}{d} = \frac{A^2}{d^2} - \frac{A^4}{3d^4} + \&c = \frac{p^2D^2}{360^2} - \frac{p^4D^4}{3\cdot360^4} + \&c$, or only $= \frac{p^2D^2}{360^2}$ the first term, by rejecting all the rest of the terms on account of their smallness, or $\frac{a}{d} = \frac{a}{2r}$ nearly $= \frac{D^2}{13131}$. This value then being substituted for $\frac{a}{d}$ or $\frac{a}{2r}$ in the last near value of the time, it becomes $t = \frac{1\cdot5708}{4}\sqrt{\frac{r}{2}} \times (1 + \frac{D^2}{52524})$ nearly.

nearly. And therefore the times of vibration in different small arcs, are as $52524 + \text{D}^2$, or as 52524 added to the square of the number of degrees in the arc.

Hence it follows that the time lost in each second, by vibrating in a circle, instead of the cycloid, is $\frac{\text{D}^2}{52524}$; and consequently the time lost in a whole day of 24 hours, or $24 \times 60 \times 60$ seconds, is $\frac{5}{3}\text{D}^2$ nearly. In like manner, the seconds lost per day by vibrating in the arc of Δ degrees, is $\frac{5}{3}\Delta^2$. Therefore, if the pendulum keep true time in one of these arcs, the seconds lost or gained per day, by vibrating in the other, will be $\frac{5}{3}(\text{D}^2 - \Delta^2)$. So, for example, if a pendulum measure true time in an arc of 3 degrees, it will lose $11\frac{2}{3}$ seconds a day by vibrating 4 degrees; and $26\frac{2}{3}$ seconds a day by vibrating 5 degrees; and so on.

And in like manner, we might proceed for any other curve, as the ellipse, hyperbola, parabola, &c.

Scholium. By comparing this with the results of the problems 13 and 14, Prac. Ex. on Forces, it will appear that the times in the cycloid, and in the arc of a circle, and in any chord of the circle, are respectively as the three quantities.

$$1, \; 1 + \frac{a}{8r} \text{ \&c, and } \frac{1}{\cdot 7854}$$

or nearly as the three quantities 1, $1 + \frac{a}{8r}$, $1\cdot27324$; the first and last being constant, but the middle one, or the time in the circle, varying with the extent of the arc of vibration. Also the time in the cycloid is the least, but in the chord the greatest; for the greatest value of the series, in this prob. when $a = r$, on the arc AD is a quadrant, is $1\cdot18014$; and in that case the proportion of the three times is as the numbers 1, $1\cdot18014$, $1\cdot27324$. Moreover the time in the circle approaches to that in the cycloid, as the arc decreases, and they are very nearly equal when that arc is very small.

PROBLEM 43.

To find the time and Velocity of a Chain, consisting of very small links, descending from a smooth horizontal plane; the Chain being 100 *inches long, and one inch of it hanging off the Plane at the commencement of Motion.*

Put $a = 1$ inch, the length at the beginning;
$l = 100$ the whole length of the chain;
$x =$ any variable length of the plane.
Then x is the motive force to move the body,
and $\frac{x}{l} = f$ the accelerative force.

Hence

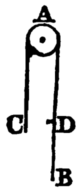
Hence $v\dot{v} = 2gf\dot{s} = 2g \times \frac{x}{l} \times \dot{x} = \frac{2gx\dot{x}}{l}$.

The fluents give $v^2 = \frac{2gx^2}{l}$. But $v = 0$ when $x = a$, theref. by correction, $v^2 = 2g \times \frac{x^2-a^2}{l}$, and $v = \sqrt{(2g \times \frac{x^2-a^2}{l})}$ the velocity for any length x. And when the chain just quits the plane, $x = l$, and then the greatest velocity is $\sqrt{(2g \times \frac{l^2-a^2}{l})} = \sqrt{(2 \times 193 \times \frac{100^2-1^2}{100})} = \sqrt{\frac{386 \times 9999}{100}} =$ 196·45902 inches, or 16·371585 feet, per second.

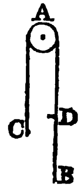
Again $\dot{t}$ or $\frac{\dot{s}}{v} = \sqrt{\frac{l}{2g}} \times \frac{\dot{x}}{\sqrt{(x^2-a^2)}}$; the correct fluent of which is $t = \sqrt{\frac{l}{2g}} \times \log. \frac{x+\sqrt{(x^2-a^2)}}{a}$, the time for any length x. And when $x = l = 100$, it is $t = \sqrt{\frac{100}{386}} \times \log. \frac{100+\sqrt{9999}}{1} =$ 2·69676 seconds, the time when the last of the chain just quits the plane.

PROBLEM 44.

To find the Time and Velocity of a Chain, of very small Links, quitting a Pulley, by passing freely over it: the whole Length being 200 Inches, and the one End hanging 2 Inches below the other at the Beginning.



Put $a = 2$, $l = 200$, and $x =$ BD any variable difference of the two parts AB, AC. Then $\frac{x}{l} = f$, and $v\dot{v}$ or $2gf\dot{s} = 2g \cdot \frac{x}{l} \cdot \frac{1}{2}\dot{x} = \frac{gx\dot{x}}{l}$. Hence the correct fluent is $v^2 = g \times \frac{x^2-a^2}{l}$, and $v = \sqrt{(g \times \frac{x^2-a^2}{l})}$, the general expression for the veloc. And when $x = l$, or when C arrives at A, it is $v = \sqrt{(g \times \frac{l^2-a^2}{l})} = \sqrt{(193 \times \frac{200^2-2^2}{200})} = \sqrt{(386 \times \frac{100^2-1^2}{100})} = \sqrt{\frac{386 \times 9999}{100}} =$ 196·45902 inches, or 16·371585 feet for the greatest velocity when the chain just quits the pulley.



Again, $\dot{t}$ or $\frac{\dot{s}}{v} = \frac{\dot{x}}{2v} = \sqrt{\frac{l}{4g}} \times \frac{\dot{x}}{\sqrt{(x^2-a^2)}}$. And the correct fluent is $t = \sqrt{\frac{l}{4g}} \times \log. \frac{x+\sqrt{(x^2-a^2)}}{a}$, the general expression for the time. And when $x = l$, it becomes $t =$ $\sqrt{}$

$\sqrt{\frac{l}{4g}} \times \log. \frac{l+\sqrt{(l^2-a^2)}}{a} = \sqrt{\frac{200}{772}} \times \log. \frac{200+\sqrt{(200^2-2^2)}}{2} =$ $\sqrt{\frac{100}{386}} \times \log. \frac{100+\sqrt{9999}}{1} = 2{\cdot}69676$ seconds, the whole time when the chain just quits the pulley.

So that the velocity and time at quitting the pulley in this prob. and the plane in the last prob. are the same; the distance descended 99 being the same in both. For though the weight l moved in this latter case, be double of what it was in the former, the moving force x is also double, because here the one end of the chain shortens as much as the other end lengthens, so that the space descended $\frac{1}{2}x$ is doubled, and becomes x; and hence the accelerative force $\frac{x}{l}$ or f is the same in both; and of course the velocity and time the same for the same distance descended.

PROBLEM 45.

To find the Number of Vibrations made by two Weights, connected by a very fine Thread, passing freely over a Tack or a Pulley, while the less Weight is drawn up to it by the Descent of the heavier Weight at the other End.

Suppose the motion to commence at equal distances below the pulley at B; and that the weights are 1 and 2 pounds.

Put a = AB, half the length of the thread;
b = $39\frac{1}{8}$ inc. or $3\frac{25}{96}$ feet, the second's pend.
x = Bw = BW, any space passed over;
z = the number of vibrations.

Then $\frac{W-w}{W+w} = f = \frac{1}{3}$ is the accelerating force.

And hence v or $\sqrt{4gfs} = \sqrt{4gfx}$, and $\dot{t}$ or $\frac{\dot{x}}{v} = \frac{\dot{x}}{\sqrt{4gfx}}$.

But, by the nature of pendulums, $\sqrt{(a \pm x)} : \sqrt{b} :: 1$ vibr. : $\sqrt{\frac{b}{a \pm x}}$ the vibrations per second made by either weight, namely, the longer or shorter, according as the upper or under sign is used, if the threads were to continue of that length for 1 second. Hence, then, as

$$1'' : \dot{t} :: \sqrt{\frac{b}{a \pm x}} : \dot{z} = \dot{t}\sqrt{\frac{b}{a \pm x}} = \sqrt{\frac{b}{4gf}} \times \frac{\dot{x}}{\sqrt{(ax \pm x^2)}},)$$

the fluxion of the number of vibrations.

Now when the upper sign + takes place, the fluent is $z = 2\sqrt{\frac{b}{4gf}} \times \mathrm{l}.\frac{\sqrt{x}+\sqrt{(a+x)}}{\sqrt{a}} = \sqrt{\frac{b}{4gf}} \times \mathrm{l}.\frac{a+2x+2\sqrt{(ax+x^2)}}{a}$.

and

And when $x = a$, the same then becomes $z = \sqrt{\frac{b}{gf}} \times$ log. $1 + \sqrt{2} = \sqrt{\frac{3b}{g}} \times$ log. $1 + \sqrt{2} = \sqrt{\frac{117\frac{3}{4}}{193}} \times$ log. $1 + \sqrt{2} =$ ·688511, the whole number of vibrations made by the descending weight.

But when the lower sign, or $-$, takes place, the fluent is $\sqrt{\frac{b}{4gf}} \times$ arc to rad. 1 and vers. $\frac{2x}{a}$. Which, when $x = a$, gives $\frac{1}{2}p \sqrt{\frac{b}{gf}} = 3{\cdot}1416 \times \sqrt{\frac{3 \times 39\frac{1}{4}}{4 \times 193}} = \frac{3{\cdot}1416}{2} \times \sqrt{\frac{117\frac{3}{4}}{193}} =$ 1·227091, the whole number of vibrations made by the lesser or ascending weight.

Schol. It is evident that the whole number of vibrations, in each case, is the same, whatever the length of the thread is. And that the greater number is to the less, as 1·5708 to the hyp. log. of $1 + \sqrt{2}$.

Farther, the number of vibrations performed in the same time t, by an invariable pendulum, constantly of the same length a, is $\sqrt{\frac{b}{gf}} =$ ·781190. For, the time of descending the space a, or the fluent of $\dot{t} = \frac{\dot{x}}{\sqrt{4gfx}}$, when $x = a$, is $t = \sqrt{\frac{a}{gf}}$. And, by the nature of pendulums, $\sqrt{a} : \sqrt{b} ::$ 1 vibr. : $\sqrt{\frac{b}{a}}$ the number of vibrations performed in 1 second; hence $1'' : t :: \sqrt{\frac{b}{a}} : t\sqrt{\frac{b}{a}} = \sqrt{\frac{b}{gf}}$, the constant number of vibrations.

So that the three numbers of vibrations, namely, of the ascending, constant, and descending pendulums, are proportional to the numbers 1·5708, 1, and hyp. log. $1 + \sqrt{2}$, or as 1·5708, 1, and ·88137; whatever be the length of the thread.

REMARK.

The solution here given by Dr. Hutton to this 45th problem, is erroneous; one of his errors in the solution consists in his not attending to the difference of tension in the pendulum as it ascends, descends, or continues of an invariable length; his method will give vibrations to the descending pendulum, even when the tension is infinitely small or nothing. A true investigation of the problem affords several curious results; but *in some cases* we are led to very tedious computations.

PROBLEM 46.

To determine the Circumstances of the Ascent and Descent of two unequal Weights, suspended at the two Ends of a Thread passing over a Pulley : the Weight of the Thread and of the Pulley being considered in the Solution.

Let $l =$ the whole length of the thread
$a =$ the weight of the same ;
$b =$ AW the dif. of lengths at first ;
$d =$ W$-w$ the dif. of the two weights ;
$c =$ a weight applied to the circumference, such as to be equal to its whole wt. and friction reduced to the circumference ;
$s =$ W$+w+a+c$ the sum of the weights moved.

Then the weight of b is $\frac{ab}{l}$, and $d-\frac{ab}{l}$ is the moving force at first. But if x denote any variable space descended by W, or ascended by w, the difference of the lengths of the thread will be altered $2x$; so that the difference will then be $b-2x$, and its weight $\frac{b-2x}{l}a$; conseq. the motive force there will be $d-\frac{b-2x}{l}a = \frac{dl-ab+2ax}{l}$ and theref. $\frac{dl-ab+2ax}{sl} = f$ the accelerating force there. Hence then $\dot{v} = 2gf\dot{x} = 2g\dot{x} \times \frac{dl-ab+2ax}{sl}$; the fluents of which give $v^2 = 4gx \times \frac{dl-ab+ax}{sl}$ or $v = 2\sqrt{\frac{ag}{sl}} \times \sqrt{(ex+x^2)}$ the general expression for the velocity, putting $e = \frac{dl-ab}{a}$. And when $x = b$, or W becomes as far below w as it was above it at the beginning, it is barely $v = 2\sqrt{\frac{bdg}{s}}$ for the velocity at that time. Also, when a, the weight of the thread, is nothing, the velocity is only $2\sqrt{\frac{dgx}{s}}$, as it ought.

Again, for the time $\dot{t}$ or $\frac{\dot{x}}{v} = \frac{1}{2}\sqrt{\frac{sl}{ag}} \times \frac{\dot{x}}{\sqrt{(ex+x^2)}}$; the fluents of which give $t = \sqrt{\frac{sl}{ag}} \times \log. \frac{\sqrt{x}+\sqrt{(e+x)}}{\sqrt{e}}$ the general expression for the time of descending any space x.

And if the radicals be expanded in a series, and the log. of it be taken, the same time will become

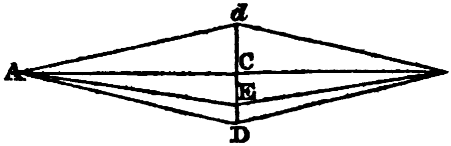
$$t = \sqrt{\frac{sx}{dg}} \times \sqrt{\frac{dl}{dl-ab}} \times \left(1-\frac{x}{6e}+\frac{3x^2}{40e^2} \&c\right).$$

Which therefore becomes barely $\sqrt{\frac{sx}{dg}}$ when a, the weight of the thread, is nothing ; as it ought.

PROBLEM

PROBLEM 47.

To find the Velocity and Time of Vibration of a small Weight, fixed to the middle of a Line, or fine Thread void of Gravity, and stretched by a given Tension; the extent of the Vibration being very small.



Let l = AC half the length of the thread;
a = CD the extent of the vibration;
x = CE any variable distance from C;
w = wt. of the small body fixed to the middle;
W = a wt. which, hung at each end of the thread, will be equal to the constant tension at each end, acting in the direction of the thread.

Now, by the nature of forces, AE : CE : : W the force in direction EA : the force in direction EC. Or, because AC is nearly = AE, the vibration being very small, taking AC instead of AE, it is AC : CE : : W : $\frac{Wx}{l}$ the force in EC arising from the tension in EA. Which will be also the same for that in EB. Therefore the sum is $\frac{2Wx}{l}$ = the whole motive force in EC arising from the tensions on both sides. Consequently $\frac{2Wx}{lw} = f$ the accelerative force there. Hence the equation of the fluxions $v\dot{v}$ or $2gf\dot{s} = \frac{-4gWx\dot{x}}{lw}$; and the flus. $v^2 = -\frac{4gWx^2}{lw}$. But when $x = a$, this is $-\frac{4gWa^2}{lw}$, and should be $= 0$; theref. the correct fluents are $v^2 = 4gW \times \frac{a^2-x^2}{lw}$, and hence $v = \sqrt{(4gW \times \frac{a^2-x^2}{lw})}$ the velocity of the little body w at any point E. And when $x = 0$, it is $v = 2a\sqrt{\frac{gW}{lw}}$ for the greatest velocity at the point C.

Now if we suppose w = 1 grain, W = 5lb troy, or 28800 grains, and $2l$ = AB = 3 feet; the velocity at C becomes $a\sqrt{\frac{8 \times 16\frac{1}{12} \times 28800}{3}} = 1111\frac{2}{3}a$. So that

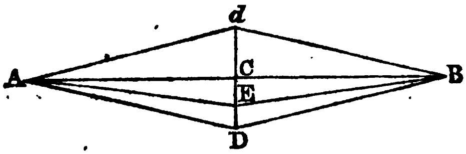
if $a = \frac{1}{10}$ inc. the greatest veloc. is $9\frac{5}{18}$ ft. per sec.
if $a = .1$ inc. the greatest veloc. is $92\frac{37}{60}$ ft. per sec.
if $a = 6$ inc. the greatest veloc. is $555\frac{7}{10}$ ft. per sec. To

To find the time t, it is $\dot{t}$ or $\frac{-\dot{x}}{v} = \frac{1}{2}\sqrt{\frac{lw}{wg}} \times \frac{-\dot{x}}{\sqrt{(a^2-x^2)}}$. Hence the correct fluent is $t = \frac{1}{2}\sqrt{\frac{wl}{wg}} \times$ arc to cosine $\frac{x}{a}$ and radius 1, for the time in DE. And when $x = 0$, the whole time in DC, or of half a vibration, is $\cdot 7854 \sqrt{\frac{wl}{wg}}$; and conseq. the time of a whole vibration through Dd is $1\cdot 5708 \sqrt{\frac{wl}{wg}}$.

Using the foregoing numbers, namely $w = 1$, $w = 28800$, and $2l = 3$ feet; this expression for the time gives $\frac{1111\frac{2}{9}}{3\cdot 1416} = 353\frac{2}{9}$, the number of vibrations per second. But if $w = 2$, there would be 250 vibrations per second; and if $w = 100$, there would be $35\frac{1}{3}\frac{3}{5}$ vibrations per second.

PROBLEM 48.

To determine the same as in the last Problem, when the Distance CD *bears some sensible Proportion to the Length* AB; *the Tension of the Thread however being still supposed a Constant Quantity.*



Using here the same notation as in the last problem, and taking the true variable length AE for AC, it is AE or EB : CE :: $2w : \frac{2wx}{AE} = \frac{2wx}{\sqrt{(l^2+x^2)}}$ the whole motive force from the two equal tensions w in AE and EB; and theref. $\frac{2w}{w} \times \frac{x}{\sqrt{(l^2+x^2)}} = f$ is the accelerative force at E. Theref. the fluxional equation is $v\dot{v}$ or $2gf\dot{s} = \frac{4wg}{w} \times \frac{-x\dot{x}}{\sqrt{(l^2+x^2)}}$; and the fluents $v^2 = \frac{8wg}{w} \times -\sqrt{(l^2+x^2)}$. But when $x = a$, these are $0 = \frac{8wg}{w} \times -\sqrt{(l^2+a^2)}$; therefore the correct fluents are $v^2 = \frac{8wg}{w} \times [\sqrt{(l^2+a^2)} - \sqrt{(l^2+x^2)}] = \frac{8wg}{w} \times$ (AD − AE). And hence $v = \sqrt{[\frac{8wg}{w} \times (\text{AD} - \text{AE})]}$ the general expression for the velocity at E. And when E arrives at C, it gives the greatest

greatest velocity there $=\sqrt{[\frac{8wg}{w} \times (\text{AD} - \text{AC})]}$. Which when $W = 28800$, $w = 1$, $2l = 3$ feet, and CD $= 6$ inches or $\frac{1}{2}$ a foot, is $\sqrt{(8 \times 28800 \times 16\frac{1}{12} \times \frac{\sqrt{10}-3}{2}} = 548\frac{1}{3}$ feet per second. Which came out $555\frac{7}{10}$ in the last problem, by using always AC for AE in the value of f. But when the extent of the vibrations is very small, as $\frac{1}{10}$ of an inch, as it commonly is, this greatest velocity here will be $\sqrt{8 \times 28800 \times 16\frac{1}{12} \times \frac{1}{43200}} = 9\frac{1}{4}$ nearly, which in the last problem was $9\frac{8}{11}$ nearly.

To find the time, it is $\dot{t}$ or $\frac{-\dot{x}}{v} = \sqrt{\frac{w}{8wg}} \times \frac{-\dot{x}}{\sqrt{[c - \sqrt{(l^2+x^2)}]}}$, making $c = \text{AD} = \sqrt{(l^2 + a^2)}$. To find the fluent the easier, multiply the numer. and denom. both by $\sqrt{[c + \sqrt{(l^2 + x^2)},]}$ so shall $\dot{t} = \sqrt{\frac{w}{8wg}} \times \frac{-\dot{x}}{\sqrt{(a^2-x^2)}} \times \sqrt{[c + \sqrt{(l^2 + x^2)}]}$. Expand now the quantity $\sqrt{[c + \sqrt{(l^2 + x^2)}]}$ in a series, and put $d = c + l$, so shall $\dot{t} = \sqrt{\frac{wd}{8wg}} \times \frac{-\dot{x}}{\sqrt{(a^2-x^2)}}(1 + \frac{x^2}{4dl} - \frac{2d+l}{32d^2l^3}x^4 + \frac{4d^2+2dl+l^2}{128d^3l^5}x^6 - \frac{40d^3+8d^2l+12dl^2+5l^3}{2048d^4l^7}x^8\&c)$. Now the fluent of the first term $\frac{\dot{x}}{\sqrt{(a^2-x^2)}}$ is $=$ the arc to sine $\frac{x}{a}$ and radius 1, which arc call A; and let P, Q be the fluents of any other two successive terms, without the coefficients, the distance of Q from the first term A being n; then it is evident that $\dot{Q} = x^2\dot{P} = x^{2n}\dot{A}$, and $\dot{P} = x^{2n-2}\dot{A}$. Assume theref. $Q = bP - ex^{2n-1}\sqrt{(a^2 - x^2)}$; then is $\dot{Q}$ or $x^2\dot{P} = b\dot{P} - (2n-1) ex^{2n-2}\dot{x}\sqrt{(a^2 - x^2)} + \frac{ex^{2n}\dot{x}}{\sqrt{(a^2-x^2)}} = b\dot{P} - \frac{(2n-1)ea^2x^{2n-2}\dot{x}}{\sqrt{(a^2-x^2)}} + \frac{(2n-1)ex^{2n}\dot{x}}{\sqrt{(a^2-x^2)}} + \frac{ex^{2n}\dot{x}}{\sqrt{(a^2-x^2)}} = b\dot{P} - (2n-1)ea^2\dot{P} + (2n-1)ex^2\dot{P} + ex^2\dot{P} = b\dot{P} - (2n-1)ea^2\dot{P} + 2nex^2\dot{P}$. Then, comparing the coefficients of the like terms, we find $1 = 2en$, and $b = (2n-1)ea^2$; from which are obtained $e = \frac{1}{2n}$, and $b = \frac{2n-1}{2n}a^2$. Consequently $Q = \frac{(2n-1)a^2P - x^{2n-1}\sqrt{(a^2-x^2)}}{2n}$, the general equation between any two successive terms, and by means of which the series may be continued as far as we please. And hence, neglecting the coefficients, putting A $=$ the first term, namely the arc whose sine is $\frac{x}{a}$, and B, C, D, &c, the following terms, the series is as follows, $A + \frac{a^2A - x\sqrt{(a^2-x^2)}}{2} +$

3a²B

$\frac{3a^2\text{B}-x^3\sqrt{(a^2-x^2)}}{4}+\frac{5a^2\text{C}-x^5\sqrt{(a^2-x^2)}}{6}$&c. Now when $x=0$, this series $=0$; and when $x=a$, the series becomes $\frac{1}{2}p+\frac{a^2\text{A}}{2}+\frac{3a^2\text{B}}{4}+\frac{5a^2\text{C}}{6}$ &c, where $p=3\cdot1416$, or the series is $\frac{1}{2}p(1+\frac{1}{2}a^2+\frac{1.3}{2.4}a^4+\frac{1.3.5}{2.4.6}a^6$ &c.)

So that, by taking in the coefficients, the general time of passing over any distance DE will be

$\sqrt{\frac{w(c+l)}{8\text{w}g}}\times\frac{1}{2}p\times(1+\frac{1}{4dl}\cdot\frac{1}{2}a^2-\frac{2d+l}{32d^2l^3}\cdot\frac{1.3}{2.4}a^4$ &c, $-$ arc sin. $\frac{x}{a}-\frac{1}{4dl}\cdot\frac{a^2\text{A}-x\sqrt{(a^2-x^2)}}{2}+\frac{2d+l}{32d^2l^3}\cdot\frac{3a^2\text{B}-x^3\sqrt{(a^2-x^2)}}{4}$ &c.

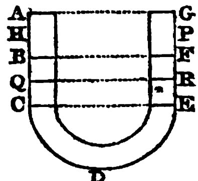
And hence, taking $x=0$, and doubling, the time of a whole vibration, or double the time of passing over CD will be equal to $\frac{1}{2}p\sqrt{\frac{w(c+l)}{2\text{w}g}}\times(1+\frac{1}{4dl}\cdot\frac{1}{2}a^2-\frac{2d+l}{32d^2l^3}\cdot\frac{1.3}{2.4}a^4+\frac{4d^2+2dl+l^2}{128d^3l^5}\cdot\frac{1.3.5}{2.4.6}a^6-\frac{40d^3+8d^2l+12dl^2+5l^3}{2048d^4l^7}\cdot\frac{1.3.5.7}{2.4.6.8}a^8$ &c.)

Which, when $a=0$, or $c=l$, becomes only $\frac{1}{2}p\sqrt{\frac{wl}{\text{w}g}}$, the same as in the last problem, as it ought.

Taking here the same numbers as in the last problem, viz, $l=\frac{3}{2}$, $a=\frac{1}{4}$, $w=2$, $\text{w}=28800$, $g=16\frac{1}{12}$; then $\frac{1}{2}p\sqrt{\frac{w(c+l)}{2\text{w}g}}=\cdot0040514$, and the series is $1+\cdot006762-\cdot000175+\cdot000003$ &c $=1\cdot006590$; therefore $\cdot0040514\times1\cdot006590=\cdot0040965=\frac{1}{245\frac{1}{8}}$ is the time of one whole vibration, and consequently $245\frac{1}{8}$ vibrations are performed in a second; which were 250 in the last problem.

PROBLEM 49.

It is proposed to determine the Velocity, and the time of Vibration, of a Fluid in the Arms of a Canal or bent Tube.



Let the tube ABCDEF have its two branches AC, GE vertical, and the lower part CDE in any position whatever, the whole being of an uniform diameter or width throughout. Let water, or quicksilver, or any other fluid, be poured in, till it stand in equilibrio, at any horizontal line BF. Then let one surface be pressed or pushed down by shaking, from B to C, and the other will ascend through the equal space FG; after which let them be permitted

mitted freely to return. The surfaces will then continually vibrate in equal times between AC and EG. The velocity and times of which oscillations are therefore required.

When the surfaces are any where out of a horizontal line, as at P and Q, the parts of the fluid in QDR, on each side, below QR, will balance each other; and the weight of the part in PR, which is equal to 2PF, gives motion to the whole. So that the weight of the part 2PF is the motive force by which the whole fluid is urged, and therefore $\frac{\text{wt of } 2PF}{\text{whole wt.}}$ is the accelerative force. Which weights being proportional to their lengths, if l be the length of the whole fluid, or axis of the tube filled, and a = FG or BC; then is $\frac{a}{l}$ the accelerative force. Putting theref. $x = GP$ any variable distance, v the velocity, and t the time; then $PF = a - x$, and $\frac{2a - 2x}{l} = f$ the accelerative force; hence $v\dot{v}$ or $2gf\dot{s} = \frac{4g}{l}(a\dot{x} - x\dot{x})$; the fluent of which give $v^2 = \frac{4g}{l}(2ax - x^2)$, and $v = \sqrt{(4g \times \frac{2ax - x^2}{l})}$ is the general expression for the velocity at any term. And when $x = a$, it becomes $v = 2a\sqrt{\frac{g}{l}}$ for the greatest velocity at B and F.

Again, for the time, we have $\dot{t}$ or $\frac{\dot{s}}{v} = \frac{1}{2}\sqrt{\frac{l}{g}} \times \frac{\dot{x}}{\sqrt{(2ax - x^2)}}$; the fluents of which give $t = \frac{1}{2}\sqrt{\frac{l}{g}} \times$ arc to versed sine $\frac{x}{a}$ and radius 1, the general expression for the time. And when $x = a$, it becomes $t = \frac{1}{4}p\sqrt{\frac{l}{g}}$ for the time of moving from G to F, p being = 3·1416; and consequently $\frac{1}{2}p\sqrt{\frac{l}{g}}$ the time of a whole vibration from G to E, or from C to A. And which therefore is the same, whatever AB is, the whole length l remaining the same.

And the time of vibration is also equal to the time of the vibration of a pendulum whose length is $\frac{1}{2}l$, or half the length of the axis of the fluid. So that, if the length l be $78\frac{1}{4}$ inches, it will oscillate in 1 second.

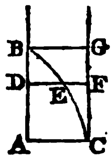
Scholium. This reciprocation of the water in the canal, is nearly similar to the motion of the waves of the sea. For the time of vibration is the same, however short the branches are, provided the whole length be the same. So that when the

the height is small, in proportion to the length of the canal; the motion is similar to that of a wave, from the top to the bottom or hollow, and from the bottom to the top of the next wave; being equal to two vibrations of the canal; the whole length of a wave, from top to top, being double the length of the canal. Hence the wave will move forward by a space nearly equal to its breadth, in the time of two vibrations of a pendulum whose length is ($\frac{1}{2}l$) half the length of the canal, or one-fourth of the breadth of a wave, or in the time of one vibration of a pendulum whose length is the whole breadth of the wave, since the times of vibration are as the square roots of their lengths. Consequently, waves whose breadth is equal to $39\frac{1}{8}$ inches, or $3\frac{25}{96}$ feet, will move over $3\frac{25}{96}$ feet in a second, or $195\frac{5}{8}$ feet in a minute, or nearly 2 miles and a quarter in an hour. And the velocity of greater or less waves will be increased or diminished in the subduplicate ratio of their breadths.

Thus, for instance, for a wave of 18 inches breadth, as $\sqrt{39\frac{1}{8}} : 39\frac{1}{8} :: \sqrt{18} : \sqrt{(39\frac{1}{8} \times 18)} = \frac{3}{2}\sqrt{313} = 26{\cdot}5377$ the velocity of the wave of 18 inches breadth.

PROBLEM 50.

To determine the Time of emptying any Ditch, or Inundation, &c, by a Cut or Notch, from the Top to the Bottom of it.



Let $x =$ AB the variable height of water at any time;
$b =$ AC the breadth of the cut;
$d =$ the whole or first depth of water;
$A =$ the area of the surface of the water in the ditch;
$g = 16\frac{1}{12}$ feet.

The velocity at any point D, is as $\sqrt{\text{BD}}$, that is, as the ordinate DE of a parabola BEC, whose base is AC, and altitude AB. Therefore the velocities at all the points in AB, are as all the ordinates of the parabola. Consequently the quantity of water running through the cut ABGC, in any time, is to the quantity which would run through an equal aperture placed all at the bottom in the same time, as the area of the parabola ABC, to the area of the parallelogram ABGC, that is, as 2 to 3.

But $\sqrt{g} : \sqrt{x} :: 2g : 2\sqrt{gx}$ the velocity at AC; therefore $\frac{2}{3} \times 2\sqrt{gx} \times bx = \frac{4}{3}bx\sqrt{gx}$ is the quantity discharged per second through ABGC; and consequently $\frac{4bx\sqrt{gx}}{3A}$ is the velocity per second of the descending surface. Hence then $\frac{4bx\sqrt{gx}}{3A} : -\dot{x} :: 1'' : \frac{-3A\dot{x}}{4bx\sqrt{gx}} = \dot{t}$ the fluxion of the time of descending.

Now

Now when A the surface of the water is constant, or the ditch is equally broad throughout, the correct fluent of this fluxion gives $t = \frac{3A}{2b\sqrt{g}} \times \frac{\sqrt{d}-\sqrt{x}}{\sqrt{dx}}$ for the general time of sinking the surface to any depth x. And when $x = 0$, this expression is infinite; which shows that the time of a complete exhaustion is infinite.

But if $d = 9$ feet, $b = 2$ feet, $A = 21 \times 1000 = 21000$, and it be required to exhaust the water down to $\frac{1}{16}$ of a foot deep; then $x = \frac{1}{16}$, and the above expression becomes $\frac{3 \times 21000}{4 \times 4\frac{1}{96}} \times \frac{3-\frac{1}{4}}{\frac{3}{4}} = 14400''$, or just 4 hours for that time. And if it be required to depress it 8 feet, or till 1 foot depth of water remain in the ditch, the time of sinking the water to that point will be $43'\ 38''$.

Again, if the ditch be the same depth and length as before, but 20 feet broad at bottom, and 22 at top; then the descending surface will be a variable quantity, and, (by prob. 16 Prac. Ex. on Forces), it will be $\frac{90+x}{90} \times 20000$; hence in this case the flux. of the time, or $\frac{-3A\dot{x}}{4bx\sqrt{gx}}$, becomes $\frac{-500}{3b\sqrt{g}} \times \frac{90+x}{x\sqrt{x}}\dot{x}$; the correct fluent of which is $t = \frac{1000}{3b\sqrt{g}} \times \left(\frac{90-x}{\sqrt{x}} - \frac{90-d}{\sqrt{d}}\right)$ for the time of sinking the water to any depth x.

Now when $x = 0$, this expression for the complete exhaustion becomes infinite.

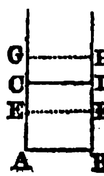
But if . . $x = 1$ foot, the time t is $42'\ 56''\frac{1}{2}$.

And when $x = \frac{1}{16}$ foot, the time is $3^h\ 50'\ 28''\frac{1}{2}$.

PROBLEM 51.

To determine the Time of filling the Ditches of a Fortification 6 Feet deep with Water, through the Sluice of a Trunk of 3 Feet Square, the Bottom of which is level with the Bottom of the Ditch, and the Height of the supplying Water is 9 Feet above the Bottom of the Ditch.

Let ACDB represent the area of the vertical sluice, being a square of 9 square feet, and AB level with the bottom of the ditch. And suppose the ditch filled to any height AE, the surface being then at EF.



Put $a = 9$ the height of the head or supply;

$b = 3 = AB = AC$;

$g = 16\frac{1}{12}$;

$A =$ the area of a horizontal section of the ditches;

$x = a - AE$, the height of the head above EF.

Then

Then $\sqrt{g} : \sqrt{x} :: 2g : 2\sqrt{gx}$ the velocity with which the water presses through the part AEFB; and theref. $2\sqrt{gx} \times$ AEFB $= 2b\sqrt{gx}(a-x)$ is the quantity per second running through AEFB. Also, the quantity running per second through ECDF is $2\sqrt{gx} \times \frac{11}{12}$ECDF $= \frac{11}{6}b\sqrt{gx}\,(b - a + x)$ nearly. For the real quantity is, by proceeding as in the last prob. the difference between two parab. segs. the alt. of the one being x, its base b, and the alt. of the other $a - b$; and the medium of that dif. between its greatest state at AB, where it is $\frac{9}{10}$AD, and its least state at CD, where it is 0, is nearly $\frac{11}{12}$ED. Consequently the sum of the two, or $\frac{1}{6}b\sqrt{gx}$ $(a + 11b - x)$ is the quantity per second running in by the whole sluice ACDB. Hence then $\frac{1}{6}b\sqrt{gx} \times \frac{a+11b-x}{A} = v$ is the rate or velocity per second with which the water rises in the ditches; and so $v : -\dot{x} :: 1'' : \dot{t} = -\frac{\dot{x}}{v} = \frac{-6A}{b\sqrt{g}} \times \frac{x^{-\frac{1}{2}}\dot{x}}{c-x}$ the fluxion of the time of filling to any height AE, putting $c = a + 11b$.

Now when the ditches are of equal width throughout, A is a constant quantity, and in that case the correct fluent of this fluxion is $t = \frac{6A}{b\sqrt{gc}} \times \log.(\frac{\sqrt{c}+\sqrt{a}}{\sqrt{c}-\sqrt{a}} \times \frac{\sqrt{c}-\sqrt{x}}{\sqrt{c}+\sqrt{x}})$ the general expression for the time of filling to any height AE, or $a - x$, not exceeding the height AC of the sluice. And when $x =$ AC $= a - b = d$ suppose, then $t = \frac{6A}{b\sqrt{gc}} \times \log.$ $(\frac{\sqrt{c}+\sqrt{a}}{\sqrt{c}-\sqrt{a}} . \frac{\sqrt{c}-\sqrt{d}}{\sqrt{c}+\sqrt{d}})$ is the time of filling to CD the top of the sluice.

Again, for filling to any height GH above the sluice, x denoting as before $a -$ AG the height of the head above GH, $2\sqrt{gx}$ will be the velocity of the water through the whole sluice AD: and therefore $2b^2\sqrt{gx}$ the quantity per second, and $\frac{2b^2\sqrt{gx}}{A} = v$ the rise per second of the water in the ditches; consequently $v : -\dot{x} :: 1'' : \dot{t} = -\frac{\dot{x}}{v} = \frac{-A}{2b^2\sqrt{g}} \times \frac{\dot{x}}{\sqrt{x}}$ the general fluxion of the time; the correct fluent of which being 0 when $x = a - b = d$, is $t = \frac{A}{b^2\sqrt{g}}\,(\sqrt{d} - \sqrt{x})$ the time of filling from CD to GH.

Then the sum of the two times, namely, that of filling from AB to CD, and that of filling from CD to GH, is $\frac{A}{b\sqrt{g}}[\frac{\sqrt{d}-\sqrt{x}}{b} + \frac{6}{\sqrt{c}}\log.(\frac{\sqrt{c}+\sqrt{a}}{\sqrt{c}-\sqrt{a}} . \frac{\sqrt{c}-\sqrt{d}}{\sqrt{c}+\sqrt{d}})]$ for the whole time

time required. And, using the numbers in the prob., this becomes $\frac{A}{3\sqrt{g}}[\frac{\sqrt{6}-\sqrt{3}}{3}+\frac{6}{\sqrt{42}}\times 1.(\frac{\sqrt{42}+\sqrt{9}}{\sqrt{42}-\sqrt{9}}\cdot\frac{\sqrt{42}-\sqrt{6}}{\sqrt{42}+\sqrt{6}})]$ $= 0{\cdot}03577277A$, the time in terms of A the area of the length and breadth, or horizontal section of the ditches. And if we suppose that area to be 200000 square feet, the time required will be $7154''$, or $1^h\ 59'\ 14''$.

And if the sides of the ditch slope a little, so as to be a little narrower at the bottom than at top, the process will be nearly the same, substituting for A its variable value, as in the preceding problem. And the time of filling will be very nearly the same as that above determined.

PROBLEM 25.

But if the Water, from which the Ditches are to be filled, be the Tide, which at Low Water is below the Bottom of the Trunk, and rises to 9 Feet above the Bottom of it by a regular Rise of One Foot in Half an Hour; it is required to ascertain the Time of Filling it to 6 Feet high, as before in the last Problem.

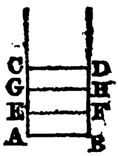
Let ACDB represent the sluice; and when the tide has risen to any height GH, below CD the top of the sluice, without the ditches, let EF be the mean height of the water within. And put $b = 3 = AB = AC$;

$g = 16\frac{1}{12}$;

$A =$ horizontal section of the ditches;

$x = AG$;

$z = AE$.



Then $\sqrt{g} : \sqrt{EG} :: 2g : 2\sqrt{g}(x-z)$ the velocity of the water through AEFB; and

$\sqrt{g} : \sqrt{EG} :: \frac{4}{3}g : \frac{4}{3}\sqrt{g(x-z)}$ the mean vel. through EGHF; theref. $2bz\sqrt{g(x-z)}$ is the quantity per sec. through AEFB; and $\frac{4}{3}b(x-z)\sqrt{g(x-z)}$ is the same through EGHF; conseq. $\frac{2}{3}b\sqrt{g}\times(2x+z)\sqrt{(x-z)}$ is the whole through AGHB per second. This quantity divided by the surface A, gives $\frac{2b\sqrt{g}}{3A}\times(2x+z)\sqrt{(x-z)} = v$ the velocity per second with which EF, or the surface of the water in the ditches, rises. Therefore

$$v : \dot{z} :: 1'' : \dot{t} = \frac{\dot{z}}{v} = \frac{3A}{2b\sqrt{g}} \times \frac{\dot{z}}{(2x+z)\sqrt{(x-z)}}.$$

But, as GH rises uniformly 1 foot in $30'$ or $1800''$, therefore $1 : AG :: 1800'' : 1800x = t$ the time of the tide rising through AG; conseq. $\dot{t} = 1800\dot{x} = \frac{3A}{2b\sqrt{g}} \times \frac{\dot{z}}{(2x+z)\sqrt{(x-z)}}$, or $m\dot{z} = (2x+z)\sqrt{(x-z)}\cdot\dot{x}$ is the fluxional equa. expressing the relation between x and z; where $m = \frac{A}{1200b\sqrt{g}} = \frac{3200}{231}$ or $13\frac{197}{231}$ when $A = 200000$ square feet. Now

Now to find the fluent of this equation, assume $z = Ax^{\frac{5}{2}} + Bx^{\frac{8}{2}} + Cx^{\frac{11}{2}} + Dx^{\frac{14}{2}}$ &c. So shall

$$\sqrt{(x-z)} = x^{\frac{1}{2}} - \frac{A}{2}x^{\frac{4}{2}} - \frac{A^2+4B}{8}x^{\frac{7}{2}} - \frac{A^3+4AB+8C}{16}x^{\frac{10}{2}} \text{ \&c},$$

$$2x + z = 2x + Ax^{\frac{5}{2}} + Bx^{\frac{8}{2}} + Cx^{\frac{11}{2}} \text{ \&c},$$

$$(2x+z)\sqrt{(x-z)}\dot{x} = 2x^{\frac{3}{2}}\dot{x} * - \frac{3A^2}{4}x^{\frac{9}{2}}\dot{x} - \frac{A^3+6AB}{4}x^{\frac{12}{2}}\dot{x} \text{ \&c},$$

and $m\dot{z} = \frac{5}{2}mAx^{\frac{3}{2}}\dot{x} + \frac{8}{2}mBx^{\frac{6}{2}}\dot{x} + \frac{11}{2}mCx^{\frac{9}{2}}\dot{x} + \frac{14}{2}mDx^{\frac{12}{2}}\dot{x}$ &c.
Then equate the coefficients of the like terms,

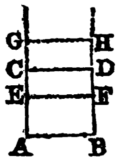
so shall	and consequently
$\frac{5}{2}mA = 2,$	$A = \frac{4}{5m},$
$\frac{8}{2}mB = 0,$	$B = 0,$
$\frac{11}{2}mC = -\frac{3}{4}A^2,$	$C = -\frac{24}{275m^3},$
$\frac{14}{2}mD = -\frac{3}{4}A^3 - \frac{3}{2}AB,$	$D = -\frac{16}{875m^4},$
&c ;	&c.

Which values of A, B, C, &c, substituted in the assumed value of z, give

$$z = \frac{4}{5m}x^{\frac{5}{2}} * - \frac{24}{275m^3}x^{\frac{11}{2}} - \frac{16}{875m^4}x^{\frac{14}{2}} \text{ \&c};$$

$$\text{or } z = \frac{4}{5m}x^{\frac{5}{2}} \text{ very nearly.}$$

And when $x = 3 = $ AC, then $z = \cdot886$ of a foot, or $10\frac{2}{3}$ inches, = AE, the height of the water in the ditches when the tide is at CD or 3 feet high without, or in the first hour and half of time.



Again, to find the time, after the above, when EF arrives at CD, or when the water in the ditches arrives as high as the top of the sluice.

The notation remaining as before,
then $2bz\sqrt{g(x-z)}$ per sec. runs through AF,
and $\frac{2}{3}b(3-z)\sqrt{g(x-z)}$ per sec. thro' ED nearly;
therefore $\frac{2}{3}b\sqrt{g}\times(12+z)\sqrt{(x-z)}$ is the whole per second through AD nearly.

conseq. $\frac{2b\sqrt{g}}{5A} \times (12+z)\sqrt{(x-z)} = v$ is the velocity per second of the point E ; and therefore,

$v : \dot{z} :: 1'' : \dot{t} = \frac{\dot{z}}{v} = \frac{5A}{2b\sqrt{g}} \times \frac{\dot{z}}{(12+z)\sqrt{(x-z)}} = 1800\ \dot{x}$, or

$m\dot{z} = (12+z)\sqrt{(x-z)}\,.\,\dot{x}$, where $m = \frac{A}{720b\sqrt{g}} = 23\frac{2}{25}$ nearly.

Assume

Assume $z = \text{A}x^{\frac{3}{2}} + \text{B}x^{\frac{4}{2}} + \text{C}x^{\frac{5}{2}} + \text{D}x^{\frac{6}{2}}$ &c. So shall

$\sqrt{(x-z)} = x^{\frac{1}{2}} - \frac{\text{A}}{2}x^{\frac{2}{2}} - \frac{\text{A}^2+4\text{B}}{8}x^{\frac{3}{2}} - \frac{\text{A}^3+4\text{AB}+8\text{C}}{16}x^{\frac{4}{2}}$ &c;

$12+z = 12 + \text{A}x^{\frac{3}{2}} + \text{B}x^{\frac{4}{2}} + \text{C}x^{\frac{5}{2}}$ &c;

$(12+z)\,.\,\sqrt{(x-z)}\,.\,\dot{x} = 12x^{\frac{1}{2}}\dot{x} - 6\text{A}x^{\frac{2}{2}}\dot{x} - (\frac{3}{8}\text{A}^2+6\text{B})x^{\frac{3}{2}}\dot{x}$ &c;

$m\dot{z} = \frac{3}{2}m\text{A}x^{\frac{1}{2}}\dot{x} + \frac{4}{2}m\text{B}x^{\frac{2}{2}}\dot{x} + \frac{5}{2}m\text{C}x^{\frac{3}{2}}\dot{x}$ &c.

Then, equating the like terms, &c, we have

$\text{A} = \frac{8}{m}$, $\text{B} = -\frac{24}{m^2}$, $\text{C} = \frac{96}{5m^3}$, $\text{D} = \frac{64}{3m^4}$ nearly, &c.

Hence $z = \frac{8}{m}x^{\frac{3}{2}} - \frac{24}{m^2}x^2 + \frac{96}{5m^3}x^{\frac{5}{2}} + \frac{64}{3m^4}x^3$ &c.

Or $z = \frac{8}{m}x^{\frac{3}{2}}$ nearly.

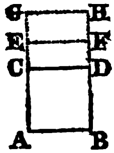
But, by the first process, when $x = 3$, $z = \cdot 886$; which substituted for them, we have $z = \cdot 886$, and the series $= \cdot 1{\cdot}63$; therefore the correct fluents are

$$z - \cdot 886 = -1{\cdot}63 + \frac{8}{m}x^{\frac{3}{2}} - \frac{24}{m^2}x^2 \text{ \&c},$$

$$\text{or } z + \cdot 774 = \frac{8}{m}x^{\frac{3}{2}} - \frac{24}{m^2}x^2 \text{ \&c}.$$

And when $z = 3 = \text{AC}$, it gives $x = 6{\cdot}369$ for the height of the tide without, when the ditches are filled to the top of the sluice, or 3 feet high; which answers to $3^{\text{h}}\ 11'\ 4''$.

Lastly, to find the time of rising the remaining 3 feet above the top of the sluice; let



$x =$ CG the height of the tide above CD,
$z =$ CE ditto in the ditches above CD;
and the other dimensions as before.
Then $\sqrt{g} : \sqrt{\text{EG}} :: 2g : 2\sqrt{g(x-z)} =$ the velocity with which the water runs through the whole sluice AD; conseq. $\text{AD} \times 2\sqrt{g(x-z)} = 18\sqrt{g(x-z)}$ is the quantity per second running through the sluice, and $\frac{18\sqrt{g}}{\text{A}}\sqrt{(x-z)} = v$ the velocity of z, or the rise of the water in the ditches, per second; hence $v : \dot{z} :: 1'' : \dot{t} = \frac{\dot{z}}{v} = \frac{\text{A}}{18\sqrt{g}} \times \frac{\dot{z}}{\sqrt{(x-z)}} = 1800\dot{x}$, and $m\dot{z} = \dot{x}\sqrt{(x-z)}$* is the fluxional equation; where $m = \frac{\text{A}}{180^2\sqrt{g}} = \frac{3200}{2079}$.

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* *Note.* The fluxional equation $m\dot{z} = \dot{x}\sqrt{(x-z)}$ may be integrated without series.—EDITOR.

To

To find the fluent,

Assume $z = Ax^{\frac{3}{2}} + Bx^{\frac{4}{2}} + Cx^{\frac{5}{2}} + Dx^{\frac{6}{2}}$ &c.

Then $x - z = x - Ax^{\frac{3}{2}} - Bx^{\frac{4}{2}} - Cx^{\frac{5}{2}}$ &c,

$$\dot{x}\sqrt{(x-z)} = x^{\frac{1}{2}}\dot{x} - \frac{A}{2}x^{\frac{2}{2}}\dot{x} - \frac{A^2+4B}{8}x^{\frac{3}{2}}\dot{x} \text{ \&c;}$$

$$m\dot{z} = \tfrac{3}{2}nAx^{\frac{1}{2}}\dot{x} + \tfrac{4}{2}nBx^{\frac{2}{2}}\dot{x} + \tfrac{5}{2}nCx^{\frac{3}{2}}\dot{x} \text{ \&c.}$$

Then equating the like terms gives

$$A = \frac{2}{3n},\ B = \frac{-1}{6n^2},\ C = \frac{1}{90n^3},\ D = \frac{-1}{810n^4}, \text{ \&c.}$$

Hence $z = \frac{2}{3n}x^{\frac{3}{2}} - \frac{1}{6n^2}x^2 + \frac{1}{90n^3}x^{\frac{5}{2}} - \frac{1}{810n^4}x^3$ &c.

But, by the second case, when $z = 0$, $x = 3{\cdot}369$, which being used in the series, it is $1{\cdot}936$; therefore the correct fluent is $z = -1{\cdot}936 + \frac{2}{3n}x^{\frac{3}{2}} - \frac{1}{6n^2}x^2$ &c. And when $z = 3$, $x = 7$; the heights above the top of the sluice; answering to 6 and 10 feet above the bottom of the ditches. That is, for the water to rise to the height of 6 feet within the ditches, it is necessary for the tide to rise to 10 feet without, which just answers to 5 hours; and so long it would take to fill the ditches 6 feet deep with water, their horizontal area being 200000 square feet.

Further, when $x = 6$, then $z = 2{\cdot}117$ the height above the top of the sluice; to which add 3, the height of the sluice, and the sum $5{\cdot}117$, is the depth of water in the ditches in 4 hours and a half, or when the tide has risen to the height of 9 feet without the ditches.

Note. In the foregoing problems, concerning the efflux of water, it is taken for granted that the velocity is the same as that which is due to the whole height of the surface of the supplying water: a supposition which agrees with the principles of the greater number of authors: though some make the velocity to be that which is due to the half height only: and others make it still less.

Also in some places, where the difference between two parabolic segments was to be taken, in estimating the mean velocity of the water through a variable orifice, I have used a near mean value of the expression; which makes the operation of finding the fluents much more easy, and is at the same time sufficiently exact for the purpose in hand.

We may further add a remark here concerning the method of finding the fluents of the three fluxional forms that occur in the solution of this problem, viz, the three forms $m\dot{z} = (2x + z)\sqrt{(x - z)}\dot{x}$, and $m\dot{z} = (12 + z)\sqrt{(x - z)}\dot{x}$, and

$m\dot{z} = \sqrt{(x - z)}\dot{x}$, the fluents of which are found by assuming the fluent mz in an infinite series ascending in terms of x with indeterminate coefficients A, B, C, &c, which coefficients are afterwards determined in the usual way, by equating the corresponding terms of two similar and equal series, the one series denoting one side of the fluxional equation, and the other series the other side. By similar series, is meant when they have equal or like exponents; though it is not necessary that the exponents of all the terms should be like or pairs, but only some of them, as those that are not in pairs will be cancelled or expelled by making their coefficients = 0 or nothing. Now the general way to make the two series similar, is to assume the fluent z equal to a series in terms of x, either ascending or descending, as here

$$z = x^r + x^{r+s} + x^{r+2s} \text{ \&c for ascending,}$$
$$\text{or } z = x^r + x^{r-s} + x^{r-2s} \text{ \&c for a descending}$$

series, having the exponents r, $r \pm s$, $r \pm 2s$, &c in arithmetical progression, the first term r, and common difference s; without the general coefficients A, B, C, &c, till the values of the exponents be determined. In terms of this assumed series for z, find the values of the two sides of the given fluxional equation, by substituting in it the said series instead of z; then put the exponent of the first term of the one side equal that of the other, which will give the value of the first exponent r; in like manner put the exponents of the two 2d terms equal, which will give the value of the common difference s; and hence the whole series of exponents r, $r \pm s$, $r \pm 2s$, &c, becomes known.

Thus, for the last of the three fluxional equations above mentioned, viz, $m\dot{z} = \sqrt{(x-z)}\dot{x}$, or only $\dot{z} = \sqrt{(x-z)}\dot{x}$; having assumed as above $z = x^r + x^{r+s}$ &c, and taking the fluxion, then $\dot{z} = x^{r-1}\dot{x} + x^{r+s-1}\dot{x} +$ &c, omitting the coefficients; and the other side of the equation $\sqrt{(x-z)}\dot{x} = \sqrt{(x - x^r - x^{r+s} \text{ \&c})} = x^{\frac{1}{2}}\dot{x} - x^{r-\frac{1}{2}}\dot{x}$ &c. Now the exponents of the first terms made equal, give $r - 1 = \frac{1}{2}$, theref. $r = 1 + \frac{1}{2} = \frac{3}{2}$; and those of the 2d terms made equal, give $r + s - 1 = r - \frac{1}{2}$, theref. $s - 1 = -\frac{1}{2}$, and $s = 1 - \frac{1}{2} = \frac{1}{2}$; conseq. the whole assumed series of exponents r, $r + s$, $r + 2s$, &c, become $\frac{3}{2}$, $\frac{4}{2}$, $\frac{5}{2}$, &c, as assumed above.

Again, for the 2d equation $m\dot{z}$ or $\dot{z} = (12 + z)\sqrt{(x-z)}\dot{x} = (a + z)\sqrt{(x-z)}\dot{x}$; assuming $z = x^r + x^{r+s}$ &c as before, then $\dot{z} = x^{r-1}\dot{x} + x^{r+s-1}\dot{x}$ &c, and $\sqrt{(x-z)}\dot{x} = x^{\frac{1}{2}}\dot{x} - x^{r-\frac{1}{2}}\dot{x}$ &c, both as above; this mult. by $a + z$ or $a + x^r + x^{r+s}$ &c, gives $ax^{\frac{1}{2}}\dot{x} - ax^{r-\frac{1}{2}}\dot{x}$ &c: then equating the first exponents gives $r - 1 = \frac{1}{2}$ or $r = \frac{3}{2}$, and $r + s - 1 = r - \frac{1}{2}$, or $s = 1 - \frac{1}{2} = \frac{1}{2}$; hence

hence the series of exponents is $\frac{3}{2}$, $\frac{4}{2}$, $\frac{5}{2}$, &c, the same as the former, and as assumed above.

Lastly, assuming the same form of series for z and $\dot{z}$ as in the above two cases, for the 1st fluxional equation also, viz, $m\dot{z} = (2x+z)\sqrt{(x-z)}\dot{x}$: then $\sqrt{(x-z)}\dot{x} = x^{\frac{1}{2}}\dot{x} - x^{r-\frac{1}{2}}\dot{x}$ &c, which mult. by $2x + z$, gives $2x^{\frac{3}{2}}\dot{x} - x^{r+\frac{1}{2}}\dot{x}$ &c: here equating the first exponents gives $r - 1 = \frac{3}{2}$ or $r = \frac{5}{2}$, and equating the 2d exponents gives $r + s - 1 = r + \frac{1}{2}$, or $s = \frac{3}{2}$; hence the series of exponents in this case is $\frac{5}{2}$, $\frac{8}{2}$, $\frac{11}{2}$, &c, as used for this case above. Then, in every case, the general coefficients A, B, C, &c, are joined to the assumed terms x^r, x^{r+s}, &c, and the whole process conducted as in the three series just referred to.

Such then is the regular and legitimate way of proceeding, to obtain the form of the series with respect to the exponents of the terms. But, in many cases we may perceive at sight, without that formal process, what the law of the exponents will be, as I indeed did in the solutions in the series above referred to; and any person with a little practice may easily do the same.

PROBLEM 53.

To determine the fall of the Water in the Arches of a Bridge.

The effects of obstacles placed in a current of water, such as the piers of a bridge, are, a sudden steep descent, and an increase of velocity in the stream of water, just under the arches, more or less in proportion to the quantity of the obstruction and velocity of the current: being very small and hardly perceptible where the arches are large and the piers few or small, but in a high and extraordinary degree at London-bridge, and some others, where the piers and the sterlings are so very large, in proportion to the arches. This is the case, not only in such streams as run always the same way, but in tide rivers also, both upward and downward, but much less in the former than in the latter. During the time of flood, when the tide is flowing upward, the rise of the water is against the under side of the piers; but the difference between the two sides gradually diminishes as the tide flows less rapidly towards the conclusion of the flood. When this has attained its full height, and there is no longer any current, but a stillness prevails in the water for a short time, the surface assumes an equal level, both above and below bridge. But, as soon as the tide begins to ebb or return again, the resistance of the piers against the stream, and the contraction of the waterway, cause a rise of the surface above and under the arches, with a full and a more rapid descent in the

the contracted stream just below. The quantity of this rise, and of the consequent velocity below, keep both gradually increasing, as the tide continues ebbing, till at quite low water, when the stream or natural current being the quickest, the fall under the arches is the greatest. And it is the quantity of this fall which it is the object of this problem to determine.

Now, the motion of free running water is the consequence of, and produced by the force of gravity, as well as that of any other falling body. Hence the height due to the velocity, that is, the height to be freely fallen by any body to acquire the observed velocity of the natural stream, in the river a little way above bridge, becomes known. From the same velocity also will be found that of the increased current in the narrowed way of the arches, by taking it in the reciprocal proportion of the breadth of the river above, to the contracted way in the arches; viz, by saying, as the latter is to the former, so is the first velocity, or slower motion, to the quicker. Next, from this last velocity, will be found the height due to it as before, that is, the height to be freely fallen through by gravity, to produce it. Then the difference of these two heights, thus freely fallen by gravity, to produce the two velocities, is the required quantity of the waterfall in the arches; allowing however, in the calculation for the contraction, in the narrowed passage, at the rate as observed by Sir I. Newton, in prop. 36 of the 2d book of the Principia, or by other authors, being nearly in the ratio of 25 to 21. Such then are the elements and principles on which the solution of the problem is easily made out as follows.

Let b = the breadth of the channel in feet;
v = mean velocity of the water in feet per second;
c = breadth of the waterway between the obstacles.

Now $25 : 21 :: c : \frac{21}{25}c$, the waterway contracted as above.

And $\frac{21}{25}c : b :: v : \frac{25b}{21c}v$, the velocity in the contracted way.

Also $32^2 : v^2 :: 16 : \frac{1}{64}v^2$, height fallen to gain the velocity v.

And $32^2 : (\frac{25b}{21c}v)^2 :: 16 : (\frac{25b}{21c})^2 \times \frac{1}{64}v^2$, ditto for the vel. $\frac{25b}{21c}v$.

Then $(\frac{25b}{21c})^2 \times \frac{v^2}{64} - \frac{v^2}{64}$ is the measure of the fall required.

Or $[(\frac{25b}{21c})^2 - 1] \times \frac{vv}{64}$ is a rule for computing the fall.

Or rather $\frac{1\cdot42b^2 - c^2}{64c^2} \times v^2$ very nearly, for the fall.

EXAM. 1. *For London-bridge.*

By the observations made by Mr. Labelye in 1746,
The breadth of the Thames at London-bridge is 926 feet;
The sum of the waterways at the time of low-water is 236 ft.;
Mean velocity of the stream just above bridge is $3\frac{1}{6}$ ft. per sec.
But under almost all the arches are driven into the bed great numbers of what are called dripshot piles, to prevent the bed from being washed away by the fall. These dripshot piles still further contract the waterways, at least $\frac{1}{6}$ of their measured breadth, or near 39 feet in the whole; so that the waterway will be reduced to 197 feet, or in round numbers suppose 200 feet.

Then $b = 926$, $c = 200$, $v = 3\frac{1}{6}$.

Hence $\dfrac{1\cdot42b^2 - c^2}{64c^2} = \dfrac{1217616 - 40000}{64 \times 40000} = \cdot46$.

And $v^2 = \dfrac{19^2}{6^2} = 10\frac{1}{36}$

Theref. $\cdot46 \times 10\frac{1}{36} = 4\cdot73$ft. $=$ 4ft. $8\frac{3}{4}$ in. the fall required. By the most exact observations made about the year 1736, the measure of the fall was 4 feet 9 inches.

EXAM. 2. *For Westminster-bridge.*

Though the breadth of the river at Westminster-bridge is 1220 feet; yet, at the time of the greatest fall, there is water through only the 13 large arches, which amount to but 820 feet; to which adding the breadth of the 12 intermediate piers, equal to 174 feet, gives 994 for the breadth of the river at that time; and the velocity of the water a little above the bridge, from many experiments, is not more than $2\frac{1}{4}$ ft. per second.

Here then $b = 994$, $c = 820$, $v = 2\frac{1}{4} = \frac{9}{4}$.

Hence $\dfrac{1\cdot42b^2 - c^2}{64c^2} = \dfrac{1403011 - 672400}{64 \times 672400} = \cdot01722$.

And $v^2 = \dfrac{81^2}{16} = 5\frac{1}{16}$.

Theref. $\cdot01722 \times 5\frac{1}{16} = \cdot0872$ ft. $= 1$ in. the fall required; which is about half an inch more than the greatest fall observed by Mr. Labelye.

And, for Blackfriar's-bridge, the fall will be much the same as that of Westminster.

Additions,

ADDITIONS,

BY THE EDITOR, R. ADRAIN.

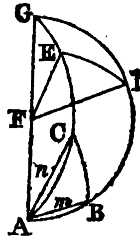
New method of determining the Angle contained by the chords of two sides of a Spherical Triangle.

See prob. v. page 77, vol. 2.

THEOREM.

If any two sides of a Spherical Triangle be produced till the continuation of each side be half the supplement of that side, the arc of a great Circle joining the extremities of the sides thus produced will be the measure of the Angle contained by the chords of those two sides.

DEMONSTRATION.



Let the two sides AB, AC of the spherical triangle ABC be produced till they meet in G, and let the supplements BG, CG, be bisected in D and E, also let the chords A*m*B, A*n*C of the arcs AB, AC be drawn; and the great circular arc DE will be the measure of the rectilineal angle contained by the chords A*m*B, A*n*C.

Let the diameter AG be the common section of the planes of ABG, ACG, and F the centre of the sphere, from which draw the straight lines FD, FE.

Since, by hypothesis, GE is the half of GC, therefore the angle at the centre GFE is equal to the angle at the circumference GA*n*C (theo. 49, Geom.), and therefore A*n*C and FE, being in the same plane, are parallel; in like manner, it is shown that FD and A*m*B are parallel, and therefore the rectilineal angles BAC and DFE are equal, and consequently, since DE is the measure of the angle DFE, it is also the measure of the angle contained by the chords A*m*B and A*n*C. Q. E. D.

New

New method of determining the oscillations of a Variable Pendulum.

The principles adopted by Dr. Hutton in the solution of his 45th problem, page 537, vol. 2, are, in my opinion, erroneous. He supposes the number of vibrations made in a given particle of time to depend on the length of the pendulum only, without considering the accelerative tension of the thread; so that by his formula we have a finite number of vibrations performed in a finite time by the descending weight, even when the ascending weight is infinitely small or nothing. Besides, the stating by which he finds the fluxion of the number of vibrations, is referred to no geometrical or mechanical principle, and appears to be nothing but a mere hypothesis. The following is a specimen of the method by which such problems may be solved according to acknowledged principles.

PROBLEM.

If two unequal weights m and m′ connected by a thread passing freely over a pulley, are suspended vertically, and exposed to the action of common gravity, it is required to investigate the number of vibrations made in a given time by the greater weight m, supposing it to descend from the point of suspension, and to make indefinitely small removals from the vertical.

SOLUTION.

Let the summit A of the vertical ABCDE be the point from which m descends, B any point in AE taken as the beginning of the plane curve BmDn described by m, which is connected with m' by the thread Am. Let mC be at right angles to AE, and put AC $= x$, Cm $= y$, Am $= r$; also let τ, t and T be the times of the descent of m through the vertical spaces AB, AC, and BC; $g = 32\frac{1}{6}$ feet, = the measure of accelerative gravity; f = the measure of the retarding force which the tension of the thread exerts on m in the direction mA, and c = the indefinitely small horizontal velocity of m at B.

As $r : x :: f : \dfrac{fx}{r}$ = the vertical action of the tension on m; and theref. $g - \dfrac{fx}{r}$ = the true accelerative force with which m is urged in a vertical direction.

Again,

Again, $r : y :: f : \frac{fy}{r}$ = the horizontal action on m produced by the tension of the thread Am. Thus the whole accelerative forces by which m is urged in directions parallel to x and y, are $g - \frac{fx}{r}$, and $\frac{fy}{r}$, the former of these forces tending to increase x, and the latter to diminish y; and therefore, by the general and well known theorem of variable motions (See *Mec. Cel.* B. 1, Chap. 2), we have the two equations

$$\frac{\ddot{x}}{\dot{t}^2} = g - \frac{fx}{r}, \text{ and } \frac{\ddot{y}}{\dot{t}^2} = -\frac{fy}{r}.$$

But by hypothesis, the angle mAC is indefinitely small, we have therefore $\frac{x}{r} = 1$, and $f = \frac{2m'g}{m+m'}$, = a given quantity; our first fluxional equation therefore becomes

$$\frac{\ddot{x}}{\dot{t}^2} = g - f,$$

of which the proper fluent is $x = \frac{1}{2}(g-f)t^2$; and by substituting for x the value just found, our second fluxional equation becomes

$$\frac{\ddot{y}}{\dot{t}^2} = -\frac{2f}{g-f} \cdot \frac{y}{t^2} \quad \text{or } \frac{t^2\ddot{y}}{\dot{t}^2} + py = 0, \text{ (putting } p = \frac{2f}{g-f} = \frac{4m'}{m-m'}\text{)}.$$

Now when p is less than $\frac{1}{4}$, let $q = \sqrt{\frac{1}{4} - p}$, and in this case the correct fluent of the equation $\frac{t^2\ddot{y}}{\dot{t}^2} + py = 0$, is easily found to be

$$\frac{y}{c} = \frac{t^{\frac{1}{2}}\tau^{\frac{1}{2}}}{2q} \cdot \left\{ \left(\frac{t}{\tau}\right)^{q} - \left(\frac{t}{\tau}\right)^{-q} \right\};$$

from which equation it is manifest that as t increases y also increases, so that m never returns to the vertical, and there are no vibrations. Again, when $p = \frac{1}{4}$, the correct fluent of the same fluxional equation is

$$\frac{y}{c} = \sqrt{t\tau} \,.\, \text{hyp. log.} \left(\frac{t}{\tau}\right).$$

So that in this case also, when t increases y increases, and the body m never returns to the vertical. Since in this case $p = \frac{4m'}{m-m'} = \frac{1}{4}$, therefore $17m' = m$, and therefore by this case and the preceding, there are no vibrations performed by the descending weight m when it is equal to or greater than 17 times the ascending weight m'.

But

But when p is greater than $\frac{1}{4}$, put $n = \sqrt{p - \frac{1}{4}}$, and in this case the correct equation of the fluents is

$$\frac{y}{c} = \frac{t^{\frac{1}{2}} \tau^{\frac{1}{2}}}{n} . \sin. (n . \text{hyp. log.} \frac{t}{\tau}).$$

This equation shows us that we shall have $y = 0$ as often as $n.$ hyp. log. $\frac{t}{\tau}$ becomes equal to any complete number of semi-circumferences: if therefore $\pi = 3{\cdot}1416$, and N = any number in the series 1, 2, 3, 4, 5, &c, we can have $y = 0$ only when $n.$ hyp. log. $\frac{t}{\tau} = N\pi$, from which we have $t = \tau . e^{\frac{N\pi}{n}}$, supposing hyp. log. $e = 1$, and therefore

$$T = \tau . \left\{ e^{\frac{N\pi}{n}} - 1 \right\},$$

which shows the relation between the number of vibrations N and the time T in which they are performed.

Hence it is manifest, that the times or durations of the several successive vibrations constitute a series in geometrical progression.

LOGARITHMS

OF THE

NUMBERS

FROM

1 to 1000.

N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0·000000	26	1·414973	51	1·707570	76	1·880814
2	0·301030	27	1·431364	52	1·716003	77	1·886491
3	0·477121	28	1·447158	53	1·724276	78	1·892095
4	0·602060	29	1·462398	54	1·732394	79	1 897627
5	0·698970	30	1·477121	55	1·740363	80	1·903090
6	0·778151	31	1·491362	56	1·748188	81	1·908485
7	0·845098	32	1·505150	57	1·755875	82	1·913814
8	0.903090	33	1·518514	58	1·763428	83	1·919078
9	0·954243	34	1·531479	59	1·770852	84	1·924279
10	1·000000	35	1·544068	60	1·778151	85	1·929419
11	1·041393	36	1·556303	61	1·785330	86	1·934498
12	1·079181	37	1·568202	62	1·792392	87	1·939519
13	1·113943	38	1·579784	63	1·799341	88	1·944483
14	1·146128	39	1·591065	64	1·806180	89	1·949390
15	1·176091	40	1·602060	65	1·812913	90	1·954243
16	1·204120	41	1·612784	66	1·819544	91	1·959041
17	1·230449	42	1·623249	67	1·826075	92	1·963788
18	1·255273	43	1·633468	68	1·832509	93	1·968483
19	1·278754	44	1·643453	69	1·838849	94	1·973128
20	1·301030	45	1·653213	70	1·845098	95	1·977724
21	1·322219	46	1·662758	71	1·851258	96	1·982271
22	1·342423	47	1·672098	72	1·857333	97	1·986772
23	1·361728	48	1·681241	73	1·863323	98	1·991226
24	1·380211	49	1·690196	74	1·869232	99	1·995635
25	1·397940	50	1·698970	75	1·875061	100	2·000000

N. B. In the following table, in the last nine columns of each page, where the first or leading figures change from 9's to 0's, points or dots are now introduced instead of the 0's through the rest of the line, to catch the eye, and to indicate that from thence the corresponding natural number in the first column stands in the next lower line, and its annexed first two figures of the Logarithm in the second column.

N.	0	1	2	3	4	5	6	7	8	9
100	000000	0434	0868	1301	1734	2166	2598	3029	3461	3891
101	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174
102	8600	9026	9451	9876	. 300	. 724	1147	1570	1993	2415
103	012837	3259	3680	4100	4521	4940	5360	5779	6197	6616
104	7033	7451	7868	8284	8700	9116	9532	9947	. 361	. 775
105	021189	1603	2016	2428	2841	3252	3664	4075	4486	4896
106	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978
107	9384	9789	195	. 600	1004	1408	1812	2216	2619	3021
108	033424	3826	4227	4628	5029	5430	5830	6230	6629	7028
109	7426	7825	8223	8620	9017	9414	9811	. 207	. 602	. 998
110	041393	1787	2182	2576	2969	3362	3755	4148	4540	4932
111	5323	5714	6105	6495	6885	7275	7664	8053	8442	6830
112	9218	9606	9993	. 380	. 766	1153	1538	1924	2309	2694
113	053078	3463	3846	4230	4613	4996	5378	5760	6142	6524
114	6905	7286	7666	8046	8426	8805	9185	9563	9942	. 320
115	060698	1075	1452	1829	2206	2582	2958	3333	3709	4083
116	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815
117	8186	8557	8928	9298	9668	. . 38	. 407	. 776	1145	1514
118	071882	2250	2617	2985	3352	3718	4085	4451	4816	5182
119	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819
120	9181	9543	9904	. 266	. 626	. 987	1347	1707	2067	2426
121	082785	3144	3503	3861	4219	4576	4934	5291	5647	6004
122	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552
123	9905	. 258	. 611	. 963	1315	1667	2018	2370	2721	3071
124	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562
125	6910	7257	7604	7951	8298	8644	8990	9335	9681	. 026
126	100371	0715	1059	1403	1747	2091	2434	2777	3119	3462
127	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871
128	7210	7549	7888	8227	8565	8903	9241	9579	9916	. 253
129	110590	0926	1263	1599	1934	2270	2605	2940	3275	3609
130	3943	4277	4611	4944	5278	5611	5943	6276	6608	6940
131	7271	7603	7934	8265	8595	8926	9256	9586	9915	245
132	120574	0903	1231	1560	1888	2216	2544	2871	3198	3525
133	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781
134	7105	7429	7753	8076	8399	8722	9045	9368	9690	. . 12
135	130334	0655	0977	1298	1619	1939	2260	2580	2900	3219
136	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403
137	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564
138	9879	. 194	. 508	. 822	1136	1450	1763	2076	2389	2702
139	143015	3327	3639	3951	4263	4574	4885	5196	5507	5818
140	6128	6438	6748	7058	7367	7676	7985	8294	8603	8911
141	9219	9527	9835	. 142	. 449	. 756	1063	1370	1676	1982
142	152288	2594	2900	3205	3510	3815	4120	4424	4728	5032
143	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061
144	8362	8664	8965	9266	9567	9868	. 168	. 469	. 769	1068
145	161368	1667	1967	2266	2564	2863	3161	3460	3758	4055
146	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022
147	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968
148	170262	0555	0848	1141	1434	1726	2019	2311	2603	2895
149	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802

N.	0	1	2	3	4	5	6	7	8	9
150	176091	6381	6670	6959	7248	7536	7825	8113	8401	8689
151	8977	9264	9552	9839	.126	.413	.699	.980	1272	1558
152	181844	2129	2415	2700	2985	3270	3555	3839	4123	4407
153	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239
154	7521	7803	8084	8366	8647	8928	9209	9490	9771	..51
155	190332	0612	0892	1171	1451	1730	2010	2289	2567	2846
156	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623
157	5899	6176	6453	6729	7005	7181	7556	7832	8107	8382
158	8657	8932	9206	9481	9755	..29	.303	.577	.850	1124
159	201397	1670	1943	2216	2488	2761	3033	3305	3577	3848
160	4120	4391	4663	4934	5204	5475	5746	6016	6286	6556
161	6826	7096	7365	7634	7804	8173	8441	8710	8979	9247
162	9515	9783	..51	.319	.586	.853	1121	1388	1654	1921
163	212188	2454	2720	2986	3252	3518	3783	4049	4314	4579
164	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221
165	7484	7747	8010	8273	8536	8796	9060	9323	9585	9846
166	220108	0370	0631	0892	1153	1414	1675	1936	2196	2456
167	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051
168	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630
169	7887	8144	8400	8657	8913	9170	9426	9682	9938	.193
170	230449	0704	0960	1215	1470	1724	1979	2234	2488	2742
171	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276
172	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795
173	8046	8297	8548	8799	9049	9299	9550	9800	..50	.300
174	240549	0799	1048	1297	1546	1795	2044	2293	2541	2790
175	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266
176	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728
177	7973	8219	8464	8709	8954	9198	9443	9687	9932	.176
178	250420	0664	0908	1151	1395	1638	1881	2125	2368	2610
179	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031
180	5273	5514	5755	5996	6237	6477	6718	6958	7198	7439
181	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833
182	260071	0310	0548	0787	1025	1263	1501	1739	1976	2214
183	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582
184	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937
185	7172	7406	7641	7875	8110	8344	8580	8812	9046	9279
186	9513	9746	9980	.213	.446	.679	.912	1144	1377	1609
187	271842	2074	2306	2538	2770	3001	3233	3464	3696	3927
188	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232
189	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525
190	8754	8982	9211	9439	9667	9895	.123	.351	.578	.806
191	281033	1261	1488	1715	1942	2169	2396	2622	2849	3075
192	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332
193	5557	5782	6007	5232	6456	6681	6905	7130	7354	7578
194	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812
195	290035	0257	0480	0702	0925	1147	1369	1591	1813	2034
196	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246
197	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446
198	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635
199	8853	9071	9289	9507	9725	9943	.161	.378	.595	.813

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202	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282
203	7496	7710	7924	8137	8351	8564	8776	8991	9204	9417
204	9630	9843	. .56	. 268	. 481	. 693	. 906	1118	1330	1542
205	31175	1966	2177	2389	2600	2812	3023	3234	3445	3656
206	3867	4078	4289	4499	4710	4920	5130	5340	5551	5760
207	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854
208	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938
209	320146	0354	0562	0769	0977	1184	1391	1598	1805	2012
210	2219	2426	2633	2839	3046	3252	3458	3665	3871	4077
211	4282	4488	469	4899	5105	5310	5516	5721	5926	6131
212	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176
213	8380	8583	8787	8991	9194	9398	9601	9805	. . . 8	. 211
214	330414	0617	0819	1022	1225	1427	1630	1832	2034	2236
215	2438	2640	2842	3044	3246	3447	3649	3850	4051	4253
216	445-	1655	4856	5057	5257	5458	5658	5859	6059	6260
217	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257
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219	340444	0642	0841	1039	1237	1435	1632	1830	2028	2225
220	2423	2620	2817	3014	3212	3409	3606	3802	3999	4196
221	4392	4589	4785	4981	5178	5374	5570	5766	5962	6157
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223	8305	8500	8694	8889	9083	9278	9472	9666	9860	. . 54
224	350248	0442	0636	0829	1023	1216	1410	1603	1796	1989
225	2183	2375	2568	2761	2954	3147	3339	3532	3724	3916
226	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834
227	6026	6217	6408	6599	6790	6981	7172	7366	7554	7744
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229	9835	. . 25	. 215	. 404	. 593	. 783	. 972	1161	1350	1539
230	361728	1917	2105	2294	2482	2671	2859	3048	3236	3424
231	3612	3800	3988	4176	4663	4551	4739	4926	5113	5301
232	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169
233	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030
234	9216	9401	9587	9772	9958	. 143	. 328	. 513	. 698	. 883
235	371068	1253	1437	1622	1806	1991	2175	2360	2544	2728
236	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565
237	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394
238	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216
239	8398	8580	8761	8943	9124	9306	9487	9668	9849	. . 30
240	380211	0392	0573	0754	0934	1115	1296	1476	1656	1837
241	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636
242	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428
243	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212
244	7390	7568	7746	7923	8101	8279	8456	8634	8811	8989
245	9166	9343	9520	9698	9875	. . 51	. 228	. 405	. 582	. 759
246	390935	1112	1288	1464	1641	1817	1993	2169	2345	2521
247	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277
248	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025
249	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766

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252	401401	1573	1745	1917	2089	2261	2433	2605	2777	2949
253	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663
254	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370
255	6540	6710	6881	7051	7221	7391	7561	7731	7901	8070
256	8240	8410	8579	8749	8918	9087	9257	9426	9595	9764
257	9933	.102	.271	.440	.609	.777	.946	1114	1283	1451
258	411620	1788	1956	2124	2293	2461	2629	2796	2964	3132
259	3300	3467	3635	3803	3970	4137	4305	4472	4639	4806
260	4973	5140	5307	5474	5641	5808	5974	6141	6308	6474
261	6641	6807	6973	7139	7306	7472	7638	7804	7970	8135
262	8301	8467	8633	8798	8964	9129	9295	9460	9625	9791
263	9956	.121	.286	.451	.616	781	.975	1110	1275	1439
264	421604	1768	1933	2097	2261	2426	2590	2754	2918	3082
265	3246	3410	3574	3737	3901	4065	4228	4392	4555	4718
266	4882	5045	5208	5371	5534	5697	5860	6023	6186	6349
267	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973
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269	9752	9914	..75	.236	.398	.559	.720	.881	1042	1203
270	431364	1525	1685	1846	2007	2167	2328	2488	2649	2809
271	2909	3130	3290	3450	3610	3770	3930	4090	4249	4409
272	4569	4729	4888	5048	5207	5367	5526	5685	5844	6004
273	6163	6322	6481	6640	6800	6957	7116	7275	7433	7592
274	7751	7909	8067	8226	8384	8542	8701	8859	9017	9175
275	9333	9491	9648	9806	9964	.122	.279	.437	.594	.752
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277	2480	2637	2793	2950	3106	3263	3419	3576	3732	3889
278	4045	4201	4357	4513	4669	4825	4981	5137	5293	5449
279	5604	5760	5915	6071	6226	6382	6537	6692	6848	7003
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281	8706	8861	9015	9170	9324	9478	9633	9787	9941	..95
282	450249	0403	0557	0711	0865	1018	1172	1326	1479	1633
283	1786	1940	2093	2247	2400	2553	2706	2859	3012	3165
284	3318	3471	3624	3777	3930	4082	4235	4387	4540	4692
285	4845	4997	5150	5302	5454	5606	5758	5910	6062	6214
286	6366	6518	6670	6821	6973	7125	7276	7428	7579	7731
287	7882	8033	8184	8336	8487	8638	8789	8940	9091	9242
288	9392	9543	9694	9845	9995	.146	.296	.447	.597	.748
289	460898	1048	1198	1348	1499	1649	1799	1948	2098	2248
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291	3893	4042	4191	4340	4490	4639	4788	4936	5085	5234
292	5383	5532	5680	5829	5977	6126	6274	6423	6571	6719
293	6868	7016	7164	7312	7460	7608	7756	7904	8052	8200
294	8347	8495	8643	8790	8938	9085	9233	9380	9527	9675
295	9822	9969	.116	.263	.410	.557	.704	.851	.998	1145
296	471292	1488	1585	1732	1878	2025	2171	2318	2464	2610
297	2756	2903	3049	3195	3341	3487	3633	3779	3925	4071
298	4216	4362	4508	4653	4799	4944	5090	5235	5381	5526
299	5671	5816	5962	6107	6252	6397	6542	6687	6832	6976

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302	480007	0151	0294	0438	0582	0725	0869	1012	1156	1299
303	1443	1586	1729	1872	2016	2159	2302	2445	2588	2731
304	2874	3016	3159	3302	3445	3587	3730	3872	4015	4157
305	4300	4442	4585	4727	4869	5011	5153	5295	5437	5579
306	5721	5863	6005	6147	6289	6430	6572	6714	6855	6997
307	7138	7280	7421	7563	7704	7845	7986	8127	8269	8410
308	8551	8692	8833	8974	9114	9255	9396	9537	9677	9818
309	9958	..99	.239	.380	.520	.661	.801	.941	1081	1222
310	491362	1502	1642	1782	1922	2062	2201	2341	2481	2621
311	2760	2900	3040	3179	3319	3458	3597	3737	3876	4015
312	4155	4294	4433	4572	4711	4850	4989	5128	5267	5406
313	5544	5683	5822	5960	6099	6238	6376	6515	6653	6791
314	6930	7068	7206	7344	7483	7621	7759	7897	8035	8173
315	8311	8448	8586	8724	8862	8999	9137	9275	9412	9550
316	9687	9824	9962	..99	.236	.374	.511	.648	.785	.922
317	501059	1196	1333	1470	1607	1744	1880	2017	2154	2291
318	2427	2564	2700	2837	2973	3109	3246	3382	3518	3655
319	3791	3927	4063	4199	4335	4471	4607	4743	1878	5014
320	5150	5286	5421	5557	5693	5828	5964	6099	6234	6370
321	6505	6640	6776	6911	7046	7181	7316	7451	7586	7721
322	7856	7991	8126	8260	8395	8530	8664	8799	8934	9068
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326	3218	3351	3484	3617	3750	3883	4016	4149	4282	3414
327	4548	4681	4813	4946	5079	5211	5344	5476	5609	5741
328	5874	6006	6139	6271	6403	6535	6668	6800	6932	7064
329	7196	7328	7460	7592	7724	7855	7987	8119	8251	8382
330	8514	8646	8777	8909	9040	9171	9303	9434	9566	9697
331	9828	9959	..90	.221	.353	.484	.615	.745	.876	1007
332	521138	1269	1400	1530	1661	1792	1922	2053	2183	2314
333	2444	2575	2705	2835	2966	3096	3226	3356	3486	3616
334	3746	3876	4006	4136	4266	4396	4526	4656	4785	4915
335	5045	5174	5304	5434	5563	5693	582[illegible]	5951	6081	6210
336	6339	6469	6598	6727	6856	6985	7114	7243	7372	7501
337	7630	7759	7888	8016	8145	8274	8402	8531	8660	8788
338	8917	9045	9174	9302	9430	9559	9687	9815	9943	..72
339	530200	0328	0456	0584	0712	0840	0968	1096	1223	1351
340	1479	1607	1734	1862	1990	2117	2245	2372	2500	2627
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342	4026	4153	4280	4407	4534	4661	4787	4914	5041	5167
343	5294	5421	5547	5674	5800	5927	6053	6180	6306	6432
344	655[illegible]	6685	6811	6937	7063	7189	7315	7441	7567	7693
345	781[illegible]	7945	8071	8197	8322	8448	8574	8699	8825	8951
346	9076	9202	9327	9452	9578	9703	9829	9954	..79	.204
347	540329	0455	0580	0705	0830	0955	1080	1205	1330	1454
348	157[illegible]	1704	1829	1953	2078	2203	2327	2452	2576	2701
349	2825	2950	3074	3199	3323	3447	3571	3696	3820	3944

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351	5307	5431	5555	5678	5802	5925	6049	6172	6296	641
352	6543	6666	6789	6913	7036	7159	7282	7405	7529	765
353	7775	7898	8021	8144	8267	8389	8512	8635	8758	888
354	9003	9126	9249	9371	9494	9616	9739	9861	9984	.19
355	550228	0351	0473	0595	0717	0840	0962	1084	1206	132
356	1450	1572	1694	1816	1938	2060	2181	2303	2425	254
357	2668	2790	2911	3033	3155	3276	3398	3519	3640	376
358	3883	4004	4126	4247	4368	4489	4610	4731	4852	497
359	5094	5215	5336	5457	5578	5699	5820	5940	6061	618
360	6303	6423	6544	6664	6785	6905	7026	7146	7267	738
361	7507	7627	7748	7868	7988	8108	8228	8349	8469	858
362	8709	8829	8948	9068	9188	9308	9428	9548	9667	978
363	9907	..26	.146	.265	.385	.504	.624	.743	.863	.98
364	561101	1221	1340	1459	1578	1698	1817	1936	2055	217
365	2293	2412	2531	2650	2769	2887	3006	3125	3244	336
366	3481	3600	3718	3837	3955	4074	4192	4311	4429	454
367	4666	4784	4903	5021	5139	5257	5376	5494	5612	573
368	5848	5966	6084	6202	6320	6437	6555	6673	6791	690
369	7026	7144	7262	7379	7497	7614	7732	7849	7967	808
370	8202	8319	8436	8554	8671	8788	8905	9023	9140	925
371	9374	9491	9608	9725	9882	9959	..76	.193	.309	.42
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373	1709	1825	1942	2058	2174	2291	2407	2523	2639	275
374	2872	2988	3104	3220	3336	3452	3568	3684	3800	391
375	4031	4147	4263	4379	4494	4610	4726	4841	4957	507
376	5188	5303	5419	5534	5650	5765	5880	5996	6111	622
377	6341	6457	6572	6687	6802	6917	7032	7147	7262	737
378	7492	7607	7722	7836	7951	8066	8181	8295	8410	852
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380	9784	9898	..12	.126	.241	.355	.469	.583	.697	.81
381	580925	1039	1153	1267	1381	1495	1608	1722	1836	195
382	2063	2177	2291	2404	2518	2631	2745	2858	2972	308
383	3199	3312	3426	3539	3652	3765	3879	3992	4105	421
384	4331	4444	4557	4670	4783	4896	5009	5122	5235	534
385	5461	5574	5686	5799	5912	6024	6137	6250	6362	647
386	6587	6700	6812	6925	7037	7149	7262	7374	7486	759
387	7711	7823	7935	8047	8160	8272	8384	8496	8608	872
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389	9950	..61	.173	.284	.396	.507	.619	.730	.842	.95
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392	3286	3397	3508	3618	3729	3840	3950	4061	4171	428
393	4393	4503	4614	4724	4834	4945	5055	5165	5276	538
394	5496	5606	5717	5827	5937	6047	6157	6267	6377	648
395	6597	6707	6817	6927	7037	7146	7256	7366	7476	758
396	7695	7805	7914	8024	8134	8243	8353	8462	8572	868
397	8791	8900	9009	9119	9228	9337	9446	9556	9666	977
398	9883	9992	.101	.210	.319	.428	.537	.646	.755	.86
399	600973	1082	1191	1299	1408	1517	1625	1734	1843	195

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401	3144	3253	3361	3469	3577	3686	3794	3902	4010	4118
402	4226	4334	4442	4550	4658	4766	4874	4982	5089	5197
403	5305	5413	5521	5628	5736	5844	5951	6059	6166	6274
404	6381	6489	6596	6704	6811	6919	7026	7133	7241	7348
405	7455	7562	7669	7777	7884	7991	8098	8205	8312	8419
406	8526	8633	8740	8847	8954	9061	9167	9274	9381	9488
407	9594	9701	9808	9914	..21	.128	.234	.341	.447	.554
408	610660	0767	0873	0979	1086	1192	1298	1405	1511	1617
409	1723	1829	1936	2042	2148	2254	2360	2466	2572	2678
410	2784	2890	2996	3102	3207	3313	3419	3525	3630	3736
411	3842	3947	4053	4159	4264	4370	4475	4581	4686	4792
412	4897	5003	5108	5213	5319	5424	5529	5634	5740	5845
413	5950	6055	6160	6265	6370	6476	6581	6686	6790	6895
414	7000	7105	7210	7315	7420	7525	7629	7734	7839	7943
415	8048	8153	8257	8362	8466	8571	8676	8780	8884	8989
416	9093	9198	9302	9406	9511	9615	9719	9824	9928	..32
417	620136	0240	0344	0448	0552	0656	0760	0864	0968	1072
418	1176	1280	1384	1488	1592	1695	1799	1903	2007	2110
419	2214	2318	2421	2525	2628	2732	2835	2939	3042	3146
420	3249	3353	3456	3559	3663	3766	3869	3973	4076	4179
421	4282	4385	4488	4591	4695	4798	4901	5004	5107	5210
422	5312	5415	5518	5621	5724	5827	5929	6032	6135	6238
423	6340	6443	6546	6648	6751	6853	6956	7058	7161	7263
424	7366	7468	7571	7673	7775	7878	7980	8082	8185	8287
425	8389	8491	8593	8695	8797	8900	9002	9104	9206	9308
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429	2457	2559	2660	2761	2862	2963	3064	3165	3266	3367
430	3468	3569	3670	3771	3872	3973	4074	4175	4276	4376
431	4477	4578	4679	4779	4880	4981	5081	5182	5283	5383
432	5484	5584	5685	5785	5886	5986	6087	6187	6287	6388
433	6488	6588	6688	6789	6889	6989	7089	7189	7290	7390
434	7490	7590	7690	7790	7890	7990	8090	8190	8290	8389
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440	3453	3551	3650	3749	3847	3946	4044	4143	4242	4340
441	4439	4537	4636	4734	4832	4931	5029	5127	5226	5324
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453	6098	6194	6290	6386	6482	6577	6673	6769	6864	6960
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455	8011	8107	8202	8298	8393	8488	8584	8679	8774	8870
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655	6241	6308	6374	6440	6506	6573	6639	6705	6771	6838
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657	7565	7631	7698	7764	7830	7896	7962	8028	8094	8160
658	8226	8292	8358	8424	8490	8556	8622	8688	8754	8820
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670	6075	6140	6204	6269	6334	6399	6464	6528	6593	6658
671	6723	6787	6852	6917	6981	7046	7111	7175	7240	7305
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673	8015	8080	8144	8209	8273	8338	8402	8467	8531	8595
674	8660	8724	8789	8853	8918	8982	9046	9111	9175	9239
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676	9947	..11	..75	.139	.204	.268	.332	.396	.460	.525
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705	8189	8251	8312	8374	8435	8497	8559	8620	8682	8743
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735	6287	6346	6405	5465	6524	6583	6642	6701	6760	6819
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777	890421	0477	0533	0589	0645	0700	0756	0812	0868	0924
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779	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039
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783	3762	3817	3873	3928	3984	4039	4094	4150	4205	4261
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08	8959	9010	9061	9112	9163	9215	9266	9317	9368

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852	930440	0491	0542	0592	0643	0694	0745	0796	0847	0898
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854	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915
855	1966	2017	2068	2118	2169	2220	2271	2322	2372	2423
856	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930
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859	3993	4044	4094	4145	4195	4246	4269	4347	4397	4448
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864	6514	6564	6614	6665	6715	6765	6815	6865	6916	6966
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870	9519	9569	9619	9669	9719	9769	9819	9869	9918	9968
871	940018	0068	0118	0168	0218	0267	0317	0367	0417	0467
872	0516	0566	0616	0666	0716	0765	0815	0865	0915	0964
873	1014	1064	1114	1163	1213	1263	1313	1362	1412	1462
874	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958
875	2008	2058	2107	2157	2207	2256	2306	2355	2405	2455
876	2504	2554	2603	2653	2702	2752	2801	2851	2901	2950
877	3000	3049	3099	3148	3198	3247	3297	3346	3396	3445
878	3495	3544	3593	3643	3692	3742	3791	3841	3890	3939
879	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433
880	4483	4532	4581	4631	4680	4729	4779	4828	4877	4927
881	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419
882	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912
883	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403
884	6452	6501	[illegible]	6600	6649	6698	6747	6796	6845	6894
885	6943	6992	7041	7090	7140	7189	7238	7287	7336	7385
886	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875
887	7924	7973	8022	8070	8119	8168	8217	8266	8315	8365
888	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853
889	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341
890	9390	9439	9488	9536	9585	9634	9683	9731	9780	9829
891	9878	9926	9975	..24	..73	.121	.170	.219	.267	.316
892	950365	0414	0462	0511	0560	0608	0657	0706	0754	0803
893	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289
894	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775
895	1823	1872	1920	1969	2017	2066	2114	2163	2211	2260
896	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744
897	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228
898	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711
899	3760	3808	3856	3905	3953	4001	4049	4098	4146	4194

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4725	477[illegible]	4821	48[illegible]	4918	4966	501[illegible]	5062	5110	5158
5207	525[illegible]	5303	5351	5399	5447	549[illegible]	5543	5592	5640
5688	5736	5784	5832	5880	5928	5976	6024	6072	6120
6168	6216	6265	631[illegible]	6361	6409	6457	6505	6553	66[illegible]1
6649	6697	6745	6793	6840	6888	6936	6984	7032	7080
7128	7176	7224	7272	7320	7368	7416	7464	7512	7559
7607	7655	7703	7751	7799	7847	7894	7942	7990	8038
8086	8134	8181	8229	8277	8325	8373	8421	8468	8516
8564	8612	8659	8707	8755	8803	8850	8898	8946	8994
9041	9089	9137	9185	9232	9280	9328	9375	9423	9471
9518	9566	9614	9661	9709	757	9804	9852	9900	9947
9995	..42	..90	.138	.185	.233	.280	.328	.376	.423
960471	0518	0566	0613	0661	0709	0756	0804	0851	0899
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1421	1469	1516	1563	1611	1658	1706	1753	1801	1848
1895	1943	990	2038	2085	2132	2180	2227	2275	2322
2369	2417	2464	2511	2559	2606	2653	2701	2748	2795
2843	2890	2937	2985	3032	3079	3126	3174	3221	3268
3316	3363	3410	3457	3504	3552	3599	3646	3693	3741
3788	3835	3882	3929	3977	4024	4071	4118	4165	4212
4260	4307	4354	4401	4448	4495	4542	4590	4637	4684
4731	4778	4825	4872	4919	4966	5013	5061	5108	5155
5202	5249	5296	5343	5390	5437	5484	5531	5578	5625
5672	5719	5766	5813	5860	5907	5954	6001	6048	6095
6142	6189	6236	6283	6329	6376	6423	6470	6517	6564
6611	6658	5705	6752	6799	6845	6892	6939	6986	7033
7080	7127	7173	7220	7267	7314	7361	7408	7454	7501
7548	7595	7642	7688	7735	7782	7829	7875	7922	7969
8016	8062	8109	8156	8203	8249	8296	8343	8390	8436
8483	8530	8576	8623	8670	8716	8763	8810	8856	8903
8950	8996	9043	6090	9136	9183	9229	9276	9323	9369
9416	9463	9509	9556	9602	9649	9695	9742	9789	9835
9882	9928	9975	..21	..68	.114	.161	.207	.254	.300
970347	0393	0440	0486	0533	0579	0626	0672	0719	0765
0812	0858	0904	0951	0997	1044	1090	1137	1183	1229
1276	1322	1369	1415	1461	1508	1554	1601	1647	1693
1740	1786	1832	1879	1925	1971	2018	2064	2110	2157
2203	2249	2295	2342	2388	2434	2481	2527	2573	2619
2666	2712	2758	2804	2851	2897	2943	2989	3035	3082
3128	3174	3220	3266	3313	3359	3405	3451	3497	3543
3590	3636	3682	3728	3774	3820	3866	3913	3959	4005
4051	4097	4143	4189	4235	4281	4327	4374	4420	4466
4512	4558	4604	4650	4696	4742	4788	4834	4880	4926
4973	5018	5064	5110	5156	5202	5248	5294	5340	5386
5432	5478	5524	5570	5616	5662	5707	5753	5799	5843
5891	5937	5983	6029	6075	6121	6167	6212	6258	6304
6350	6396	6442	6488	6533	6579	6625	6671	6717	6763
6808	6854	6900	6946	6992	7037	7083	7129	7175	7220
7266	7312	7358	7403	7449	7495	7541	7586	7632	7678

N.	0	1	2	3	4	5	6	7	8
950	977724	7769	7815	7861	7906	7952	7998	8043	808
951	8181	8226	8272	8317	8363	8409	8454	8500	854
952	8637	8683	8728	8774	8819	8865	8911	8956	900
953	9093	9138	9184	9230	9275	9321	9366	9412	945
954	9548	9594	9639	9685	9730	9776	9821	9867	991
955	980003	0049	0094	0140	0185	0231	0276	0322	036
956	0458	0503	0549	0594	0640	0685	0730	0776	082
957	0912	0957	1003	1048	1093	1139	1184	1229	127
958	1366	1411	1456	1501	1547	1592	1637	1683	172
959	1819	1864	1909	1954	2000	2045	2090	2135	2181
960	2271	2316	2362	2407	2452	2497	2543	2588	263
961	2723	2769	2814	2859	2904	2949	2994	3040	308
962	3175	3220	3265	3310	3356	3401	3446	3491	353
963	3626	3671	3716	3762	3807	3852	3897	3942	3987
964	4077	4122	4167	4212	4257	4302	4347	4392	4437
965	4527	4572	4617	4662	4707	4752	4797	4842	4887
966	4977	5022	5067	5112	5157	5202	5247	5292	5337
967	5426	5471	5516	5561	5606	5651	5699	5741	578
968	5875	5920	5965	6010	6055	6100	6144	6189	6234
969	6324	6369	6413	6458	6503	6548	6593	6637	6682
970	6772	6817	6861	6906	6951	6996	7040	7085	713
971	7219	7264	7309	7353	7398	7443	7488	7532	7577
972	7666	7711	7756	7800	7845	7890	7934	7979	8024
973	8113	8157	8202	8247	8291	8336	8381	8425	847
974	8559	8604	8648	8693	8737	8782	8826	8871	891
975	9005	9049	9049	9138	9183	9227	9272	9316	9361
976	9450	9494	9539	9583	9628	9672	9717	9761	980
977	9895	9939	9983	..28	. .72	.117	.161	.206	.25
978	990339	0383	0428	0472	0516	0561	0605	0650	0694
979	0783	0827	0871	0916	0960	1004	1049	1093	1137
980	1226	1270	1315	1359	1403	1448	1492	1536	1580
981	1669	1713	1758	1802	1846	1890	1935	1979	2023
982	2111	2156	2200	2244	2288	2333	2377	2421	2465
983	2554	2598	2642	2686	2730	2774	2819	2863	2907
984	2995	3039	3083	3127	3172	3216	3260	3304	3348
985	3436	3480	3524	3568	3613	3657	3701	3745	3789
986	3877	3921	3965	4009	4053	4097	4141	4185	4229
987	4317	4361	4405	4449	4493	4537	4581	4625	4669
988	4757	4801	4845	4889	4933	4977	5021	5065	5108
989	5196	5240	5284	5328	5372	5416	5460	5504	5547
990	5635	5679	5723	5767	5811	5854	5898	5942	5986
991	6074	6117	6161	6205	6249	6293	6337	6380	6424
992	6512	6555	6599	6643	6687	6731	6774	6818	6862
993	6949	6993	7037	7080	7124	7168	7212	7255	7299
994	7386	7430	7474	7517	7561	7605	7648	7692	7736
995	7823	7867	7910	7954	7998	8041	8085	8129	8172
996	8259	8303	8347	8390	8434	8477	8521	8564	8608
997	8695	8739	8782	8826	8869	8913	8956	9000	9043
998	9131	9174	9218	9261	9305	9348	9392	9435	9479
999	9565	9609	9652	9696	9739	9783	9826	9870	991

0 Deg.				1 Deg.				
Sine.	Cosine.	Tang.	Cotang.	Sine	Cosine.	Tang.	Cotang.	
	10·000000			8·241855	9·999934	8·241921	11·758079	60
6·463726	10·000000	6·463726	13·536274	8·249033	9·999932	8·249102	11·750898	59
6 764756	10·000000	6·764756	13·235244	8·256094	9·999929	8·256165	11·743835	58
6·940847	10·000000	6·940847	13·059153	8·263042	9·999927	8·263115	11·736885	57
7·065786	10·000000	7·065786	12·934214	8·269881	9·999925	8·269956	11·730044	56
7·162696	10·000000	7·162696	12 837304	8·276614	9·999922	8·276691	11·723309	55
7·241877	9·999999	7·241878	12·758122	8·283243	9·999920	8·283323	11·716677	54
7·308824	9·999999	7·308825	12·691175	8·289773	9·999918	8·289856	11·710144	53
7·366816	9·999999	7·366817	12·633183	8·296207	9·999915	8 296292	11·703708	52
7·417968	9·999999	7·417970	12·582030	8·302546	9·999913	8·302634	11·697366	51
7·463726	9·999998	7·463727	12·536273	8·308794	9·999910	8·308884	11·691116	50
7·505118	9·999998	7·505120	12·494880	8·314954	9·999907	8·315046	11·684954	49
7·542906	9·999997	7·542909	12·457091	8·321027	9·999905	8·321122	11·678878	48
7·577668	9·999997	7·577672	12·422328	8·327016	9·999902	8 327114	11·672886	47
7·609853	9·999996	7·609857	12·390143	8·332924	9·999899	8·333025	11·666975	46
7·639816	9·999996	7·639820	12·360180	8·338753	9·999897	8·338856	11·661144	45
7·667845	9·999995	7·667849	12·332151	8·344504	9·999894	8·344610	11·655390	44
7·694173	9·999995	7·694179	12·305821	8·350181	9·999891	8·350289	11·649711	43
7·718997	9·999994	7·719003	12·280997	8·355783	9·999888	8·355895	11·644105	42
7·742478	9·999993	7·742484	12·257516	8·361315	9·999885	8·361430	11·638570	41
7·764754	9·999993	7·764761	12·235239	8·366777	9·999882	8·366895	11·633105	40
7·785943	9·999992	7·785951	12·214049	8·372171	9·999879	8·372292	11·627708	39
7·806146	9·999991	7·806155	12·193845	8·377499	9·999876	8·377622	11·622378	38
7·825451	9·999990	7·825460	12·174540	8·382762	9·999873	8·382889	11·617111	37
7·843934	9·999989	7·843944	12·156056	8·387962	9·999870	8·388092	11·611908	36
7·861662	9·999989	7·861674	12·138326	8·393101	9·999867	8·393234	11·606766	35
7·878695	9·999988	7·870708	12·121292	8·398179	9 999864	8·398315	11·601685	34
7·895085	9·999987	7·895099	12·104901	8·403199	9·999861	8·403338	11·596662	33
7·910879	9·999986	7·910894	12·089106	8·408161	9·999858	8·408304	11·591696	32
7·926119	9·999985	7·926134	12·073866	8·413068	9·999854	8·413213	11·586787	31
7·940842	9·999983	7·940858	12·059142	8·417919	9·999851	8·418068	11·581932	30
7·955082	9·999982	7·955100	12·044900	8·422717	9·999848	8·422869	11·577131	29
7·968870	9·999981	7·968889	12·031111	8·427462	9·999845	8·427618	11·572382	28
7·982233	9·999980	7·982253	12·017747	8·432156	9·999841	8·432315	11·567685	27
7·995198	9·999979	7·995219	12·004881	8·436800	9·999838	8·436962	11·563038	26
8·007787	9·999977	8·007809	11·992191	8·441394	9·999834	8·441560	11·558440	25
8·020021	9·999976	8·020044	11·979956	8·445941	9·999831	8·446110	11·553890	24
8·031919	9·999975	8·031945	11·968055	8·450440	9·999827	8·450613	11·549387	23
8·043501	9·999973	8·043527	11·956473	8·454893	9·999824	8·455070	11·544930	22
8·054781	9·999972	8·054809	11·945191	8·459301	9·999820	8·459481	11·540519	21
8·065776	9·999971	8·065806	11·934194	8·463665	9·999816	8·463849	11·536151	20
8·076500	9·999969	8·076531	11·923469	8·467985	9·999813	8·468172	11·531828	19
8·086965	9·999968	8·086997	11·913003	8·472263	9 999809	8·472454	11·527546	18
8·097183	9·999966	8·097217	11·902783	8·476498	9·999805	8·476693	11·523307	17
8·107167	9·999964	8·107203	11·892797	8·480693	9·999801	8·480892	11·519108	16
8·116926	9·999963	8·116963	11·883037	8·484848	9·999797	8·485050	11·514950	15
8·126471	9·999961	8·126510	11·873490	8·488963	9·999794	8·489170	11·510830	14
8 135810	9·999959	8·135851	11·864149	8·493040	9·999790	8·493250	11·506750	13
8·144953	9·999958	8·144996	11·855004	8·497078	9·999786	8·497293	11·502707	12
8·153907	9·999956	8·153952	11·846048	8·501080	9·999782	8·501298	11·498702	11
8·162681	9·999954	8·162727	11·837273	8·505045	9·999778	8·505267	11·494733	10
8·171280	9·999952	8·171328	11·828672	8·508974	9·999774	8·509200	11·490800	9
8·179713	9·999950	8·179763	11·820237	8·512867	9·999769	8·513098	11·486902	8
8·187985	9·999948	8·188036	11·811964	8·516726	9·999765	8·516961	11·483039	7
8·196102	9·999946	8·196156	11·803844	8·520551	9·999761	8·520790	11·479210	6
8·204070	9·999944	8·204126	11·765874	8·524343	9·999757	8·524586	11·475414	5
8·211895	9·999942	8·211953	11·788047	8·528102	9·999753	8·528349	11·471651	4
8·219581	9·999940	8·219641	11·780359	8·531828	9·999748	8·532080	11·467920	3
8·227134	9·999938	8·227195	11·772805	8·535523	9·999744	8·535779	11·464221	2
8·234557	9·999936	8·234621	11·765379	8·539186	9·999740	8·539447	11·460553	1
8·241855	9·999934	8·241921	11·758079	8·542819	9·999735	8·543084	11·456916	0
Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine	Cotan.	Tang.	

	2 Deg.				3 Deg.				
′	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
0	8·542819	9·999735	8·543084	11·456916	8·718800	9·999404	8·719396	11·280604	60
1	8·546422	9·999731	8·546691	11·453309	8·721204	9·999398	8·721806	11·278194	59
2	8·549995	9·999726	8·550268	11·449732	8·723595	9·999391	8·724204	11·275796	58
3	8·553539	9·999722	8·553817	11·446183	8·725972	9·999384	8·726588	11·273412	57
4	8·557054	9·999717	8·557336	11·442664	8·728337	9·999378	8·728959	11·271041	56
5	8·560540	9·999713	8·560828	11·439179	8·730688	9·999371	8·731317	11·268683	55
6	8·563999	9·999708	8·564291	11·435709	8·733027	9·999364	8·733663	11·266337	54
7	8·567431	9·999704	8·567727	11·432273	8·735354	9·999357	8·735996	11·264004	53
8	8·570836	9·999699	8·571137	11·428863	8·73[illegible]67	9·999350	8·738317	11·261683	52
9	8·574214	9·999694	8·574520	11·425480	8·739969	9·999343	8·740626	11·259374	51
10	8·577566	9·999689	8·577877	11·422123	8·742259	9·999336	8·742922	11·257078	50
11	8·580892	9·999685	8·581208	11·418792	8·744536	9·999329	8·745207	11·254793	49
12	8·584193	9·999680	8·584514	11·415486	8·746802	9·999322	8·747479	11·252521	48
13	8·587469	9·999675	8·587795	11·412205	8·749055	9·999315	8·749740	11·250260	47
14	8·590721	9·999670	8·591051	11·408949	8·751297	9·999308	8·751989	11·248011	46
15	8·593948	9·999665	8·594283	11·405717	8·753528	9·999301	8·754227	11·245773	45
16	8·597152	9·999660	8·597492	11·402508	8·755747	9·999294	8·756453	11·243547	44
17	8·600332	9·999655	8·600677	11·399323	8·757955	9·999287	8·758668	11·241332	43
18	8·603489	9·999650	8·603839	11·396161	8·760151	9·999279	8·760872	11·239128	42
19	8·606623	9·999645	8·606978	11·393022	8·762337	9·999272	8·763065	11·236935	41
20	8·609734	9·999640	8·610094	11·389909	8·764511	9·999265	8·765246	11·234754	40
21	8·612823	9·999635	8·613189	11·386811	8·766675	9·999257	8·767417	11·232583	39
22	8·615891	9·999629	8·616262	11·383738	8·768828	9·999250	8·769578	11·230422	38
23	8·618937	9·999624	8·619313	11·380687	8·770970	9·999242	8·771727	11·228273	37
24	8·621962	9·999619	8·622343	11·377657	8·773101	9·999235	8·773866	11·226134	36
25	8·624965	9·999614	8·625352	11·374648	[illegible]775223	9·999227	8·775995	11·224005	35
26	8·627948	9·999608	8·628340	11·371660	8·777333	9·999220	8·778114	11·221886	34
27	8·630911	9·999603	8·631308	11·368692	8·779434	9·999212	8·780222	11·219778	33
28	8·633854	9·999597	8·634256	11·365744	8·781524	9·999205	8·782320	11·217680	32
29	8·636776	9·999592	8·637184	11·362816	8·783605	9·999197	8·784408	11·215592	31
30	8·639680	9·999586	8·640093	11·359907	8·785675	9·999189	8·786486	11·213514	30
31	8·642563	9·999581	8·642982	11·357018	8·787736	9·999181	8·788554	11·211446	29
32	8·645428	9·999575	8·645853	11·354147	8·789787	9·999174	8·790613	11·209337	28
33	8·648274	9·999570	8·648704	11·351296	8·791828	9·999166	8·792662	11·207338	27
34	8·651102	9·999564	8·651537	11·348463	8·793859	9·999158	8·794701	11·205299	26
35	8·653911	9·999558	8·654352	11·345648	8·795881	9·999150	8·796731	11·203269	25
36	8·656702	9·999553	8·657149	11·342851	8·797894	9·999142	8·798752	11·201248	24
37	8·659475	9·999547	8·659928	11·340072	8·799897	9·999134	8·800763	11·199237	23
38	8·662230	9·999541	8·662689	11·337311	8·801892	9·999126	8·802765	11·197235	22
39	8·664968	9·999535	8·665433	11·334567	8·803876	9·999118	8·804758	11·195242	21
40	8·667689	9·999529	8·668160	11·331840	8·805852	9·999110	8·806742	11·193258	20
41	8·670393	9·999524	8·670870	11·329130	8·807819	9·999102	8·808717	11·191283	19
42	8·673080	9·999518	8·673563	11·326437	8·809777	9·999094	8·810683	11·189317	18
43	8·675751	9·999512	8·676239	11·323761	8·811726	9·999086	8·812641	11·187359	17
44	8·678405	9·999506	8·678900	11·321100	8·813667	9·999077	8·814589	11·185411	16
45	8·681043	9·999500	8·681544	11·318456	8·815599	9·999069	8·816529	11·183471	15
46	8·683665	9·999493	8·684172	11·315828	8·817522	9·999061	8·818461	11·181539	14
47	8·686272	9·999487	8·686784	11·313216	8·819436	9·999053	8·820384	11·179616	13
48	8·688863	9·999481	8·689381	11·310619	8·821343	9·999044	8·822298	11·177702	12
49	8·691438	9·999475	8·691963	11·308037	8·823240	9·999036	8·824205	11·175795	11
50	8·693998	9·999469	8·694529	11·305471	8·825130	9·999027	8·826103	11·173897	10
51	8·696543	9·999463	8·697081	11·302919	8·827011	9·999019	8·827992	11·172008	9
52	8·699073	9·999456	8·699617	11·300383	8·828884	9·999010	8·829874	11·170126	8
53	8·701589	9·999450	8·702139	11·297861	8·830749	9·999002	8·831748	11·168252	7
54	8·704090	9·999443	8·704646	11·295354	8·832607	9·998993	8·833613	11·166387	6
55	8·706577	9·999437	8·707140	11·292860	8·834456	9·998984	8·835471	11·164529	5
56	8·709049	9·999431	8·709618	11·290382	8·836297	9·998976	8·837321	11·162679	4
57	8·711507	9·999424	8·712083	11·287917	8·838130	9·998967	8·839163	11·160837	3
58	8·713952	9·999418	8·714534	11·285466	8·839956	9·998958	8·840998	11·159002	2
59	8·716383	9·999411	8·716972	11·283028	8·841774	9·998950	8·842825	11·157175	1
60	8·718800	9·999404	8·719396	11·280604	8·843585	9·998941	8·844644	11·1553[illegible]	
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Ta	

'	4 Deg. Sine.	Cosine.	Tang.	Cotang.	5 Deg. Sine.	Cosine.	Tang.	Cotang.	'
0	8·843585	9·998941	8·844644	11·155356	8·940296	9·998344	8·941952	11·058048	60
1	8·845387	9·998932	8·846455	11·153545	8·941738	9·998333	8·943404	11·056596	59
2	8·847183	9·998923	8·848260	11·151740	8·943174	9·998322	8·944852	11·055148	58
3	8·848971	9·998914	8·850057	11·149943	8·944606	9·998311	8·946295	11·053705	57
4	8·850751	9·998905	8·851846	11·148154	8·946034	9·998300	8·947734	11·052266	56
5	8·852525	9·998896	8·853628	11·146372	8·947456	9·998289	8·949168	11·050832	55
6	8·854291	9·998887	8·855403	11·144597	8·948874	9·998277	8·950597	11·049403	54
7	8·856049	9·998878	8·857171	11·142829	8·950287	9·998266	8·952021	11·047979	53
8	8·857801	9·998869	8·858932	11·141068	8·951696	9·998255	8·953441	11·046559	52
9	8·859546	9·998860	8·860686	11·139314	8·953100	9·998243	8·954856	11·045144	51
10	8·861283	9·998851	8·862433	11·137567	8·954499	9·998232	8·956267	11·043733	50
11	8·863014	9·998841	8·864173	11·135827	8·955894	9·998220	8·957674	11·042326	49
12	8·864738	9·998832	8·865906	11·134094	8·957284	9·998209	8·959075	11·040925	48
13	8·866455	9·998823	8·867632	11·132368	8·958670	9·998197	8·960473	11·039527	47
14	8·868165	9·998813	8·869351	11·130649	8·960052	9·998186	8·961866	11·038134	46
15	8·869868	9·998804	8·871064	11·128936	8·961429	9·998174	8·963255	11·036745	45
16	8·871565	9·998795	8·872770	11·127230	8·962801	9·998163	8·964639	11·035361	44
17	8·873255	9·998785	8·874469	11·125531	8·964170	9·998151	8·966019	11·033981	43
18	8·874938	9·998776	8·876162	11·123838	8·965534	9·993139	8·967394	11·032606	42
19	8·876615	9·998766	8·877849	11·122151	8·966893	9·998128	8·968766	11·031234	41
20	8·878285	9·998757	8·879529	11·120471	8·968249	9·998116	8·970133	11·029867	40
21	8·879949	9·998747	8·881202	11·118798	8·969600	9·998104	8·971496	11·028504	39
22	8·881607	9·998738	8·882869	11·117131	8·970947	9·998092	8·972855	11·027145	38
23	8·883258	9·998728	8·884530	11·115470	8·972289	9·998080	8·974209	11·025791	37
24	8·884903	9·998718	8·886185	11·113815	8·973628	9·998068	8·975560	11·024440	36
25	8·886542	9·998708	8·887833	11·112[illegible]67	8·974962	9·998056	8·976906	11·023094	35
26	8·888174	9·998699	8·889476	11·110524	8·976293	9·998044	8·978248	11·021752	34
27	8·889801	9·998689	8·891112	11·108888	8·977619	9·998032	8·979586	11·020414	33
28	8·891421	9·998679	8·892742	11·107258	8·978941	9·998020	8·980921	11·019079	32
29	8·893035	9·998669	8·894366	11·105634	8·980259	9·998008	8·982251	11·017749	31
30	8·894643	9·998659	8·895984	11·104016	8·981573	9·997996	8·983577	11·016423	30
31	8·896246	9·998649	8·897596	11·102404	8·982883	9·997984	8·984899	11·015101	29
32	8·897842	9·998639	8·899203	11·100797	8·984189	9·997972	8·986217	11·013783	28
33	8·899432	9·998629	8·900803	11·099197	8·985491	9·997959	8·987532	11·012468	27
34	8·901017	9·998619	8·902398	11·097602	8·986789	9·997947	8·988842	11·011158	26
35	8·902596	9·998609	8·903987	11·096013	8·988083	9·997935	8·990149	11·009851	25
36	8·904169	9·998599	8·905570	11·094430	8·989374	9·997922	8·991451	11·008549	24
37	8·905736	9·998589	8·907147	11·092853	8·990660	9·997910	8·992750	11·007250	23
38	8·907297	9·998578	8·908719	11·091281	8·991943	9·997897	8·994045	11·005955	22
39	8·908853	9·998568	8·910285	11·089715	8·993222	9·997885	8·495337	11·004663	21
40	8·910404	9·998558	8·911846	11·088154	8·994497	9·997872	8·996624	11·003376	20
41	8·911949	9·998548	8·913401	11·086599	8·995768	9·997860	8·997908	11·002092	19
42	8·913488	9·998537	8·914951	11·085049	8·997036	9·997847	8·999188	11·000812	18
43	8·915022	9·998527	8·916495	11·083505	8·998299	9·997835	9·000465	10·999535	17
44	8·916550	9·998516	8·918034	11·081966	8·999560	9·997822	9·001738	10·998262	16
45	8·918073	9·998506	8·919568	11·080432	9·000816	9·997809	9·003007	10·996993	15
46	8·919591	9·998495	8·921096	11·078904	9·002069	9·997797	9·004272	10·995728	14
47	8·921103	9·998485	8·922619	11·077381	9·003318	9·997784	9·005534	10·994466	13
48	8·922610	9·998474	8·924136	11·075864	9·004563	9·997771	9·006792	10·993208	12
49	8·924112	9·998464	8·925649	11·074351	9·005805	9·997758	9·008047	10·991953	11
50	8·925609	9·998453	8·927156	11·072844	9·007044	9·997745	9·009298	10·990702	10
51	8·927100	9·998442	8·928658	11·071342	9·008278	9·997732	9·010546	10·989454	9
52	8·928587	9·998431	8·930155	11·069845	9·009510	9·997719	9·011790	10·988210	8
53	8·930068	9·998421	8·931647	11·068353	9·010737	9·997706	9·013031	10·986969	7
54	8·931544	9·998410	8·933134	11·066866	9·011962	9·997693	9·014268	10·985732	6
55	8·933015	9·998399	8·934616	11·065384	9·013182	9·997680	9·015502	10·984498	5
56	8·934481	9·998388	8·936093	11·063907	9·014400	9·997667	9·016732	10·983268	4
57	8·935942	9·998377	8·937565	11·062435	9·015613	9·997654	9·017959	10·982041	3
58	8·937398	9·998366	8·939032	11·060968	9·016824	9·997641	9·019183	10·980817	2
59	8·938850	9·998355	8·940494	11·059506	9·018031	9·997628	9·020403	10·979597	1
60	8·940296	9·998344	8·941952	11·058048	9·019235	9·997614	9·021620	10·978380	0
'	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.	'

	6 Deg.				7 Deg.				
′	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
0	9.019235	9.997614	9.021620	10.978380	9.085894	9.996751	9.089144	10.910856	60
1	9.020435	9.997601	9.022834	10.977166	9.086922	9.996735	9.090187	10.909813	59
2	9.021632	9.997588	9.024044	10.975956	9.087947	9.996720	9.091228	10.908772	58
3	9.022825	9.997574	9.025251	10.974749	9.088970	9.996704	9.092266	10.907734	57
4	9.024016	9.997561	9.026455	10.973545	9.089990	9.996688	9.093302	10.906698	56
5	9.025203	9.997547	9.027655	10.972345	9.091006	9.996673	9.094336	10.905664	55
6	9.026386	9.997534	9.028852	10.971148	9.092024	9.996657	9.095367	10.904633	54
7	9.027567	9.997520	9.030046	10.969954	9.093037	9.996641	9.096395	10.903605	53
8	9.028744	9.997507	9.031237	10.968763	9.094047	9.996625	9.097422	10.902578	52
9	9.029918	9.997493	9.032425	10.967575	9.095056	9.996610	9.098446	10.901554	51
10	9.031089	9.997480	9.033609	10.966391	9.096062	9.996594	9.099468	10.900532	50
11	9.032257	9.997466	9.034791	10.965209	9.097065	9.996578	9.100487	10.899513	49
12	9.033421	9.997452	9.035969	10.964031	9.098066	9.996562	9.101504	10.898496	48
13	9.034582	9.997439	9.037144	10.962856	9.099065	9.996546	9.102519	10.897481	47
14	9.035741	9.997425	9.038316	10.961684	9.100062	9.996530	9.103532	10.896468	46
15	9.036896	9.997411	9.039485	10.960515	9.101056	9.996514	9.104542	10.895458	45
16	9.038048	9.997397	9.040651	10.959349	9.102048	9.996498	9.105550	10.894450	44
17	9.039197	9.997383	9.041813	10.958187	9.103037	9.996482	9.106556	10.893444	43
18	9.040342	9.997369	9.042973	10.957027	9.104025	9.996465	9.107559	10.892441	42
19	9.041485	9.997355	9.044130	10.955870	9.105010	9.996449	9.108560	10.891440	41
20	9.042625	9.997341	9.045284	10.954716	9.105992	9.996433	9.109559	10.890441	40
21	9.043762	9.997327	9.046434	10.953566	9.106973	9.996417	9.110556	10.889444	39
22	9.044895	9.997313	9.047582	10.952418	9.107951	9.996400	9.111551	10.888449	38
23	9.046026	9.997299	9.048727	10.951273	9.108927	9.996384	9.112543	10.887457	37
24	9.047154	9.997285	9.049869	10.950131	9.109901	9.996368	9.113533	10.886467	36
25	9.048277	9.997271	9.051008	10.948992	9.110873	9.996351	9.114521	10.885479	35
26	9.049400	9.997257	9.052144	10.947856	9.111842	9.996335	9.115507	10.884493	34
27	9.050519	9.997242	9.053277	10.946723	9.112809	9.996318	9.116491	10.883509	33
28	9.051635	9.997228	9.054407	10.945593	9.113774	9.996302	9.117472	10.882528	32
29	9.052749	9.997214	9.055535	10.944465	9.114737	9.996285	9.118452	10.881548	31
30	9.053859	9.997199	9.056659	10.943341	9.115698	9.996269	9.119429	10.880571	30
31	9.054966	9.997185	9.057781	10.942219	9.116656	9.996252	9.120404	10.879596	29
32	9.056071	9.997170	9.058900	10.941100	9.117613	9.996235	9.121377	10.878623	28
33	9.057172	9.997156	9.060016	10.939984	9.118567	9.996219	9.122348	10.877652	27
34	9.058271	9.997141	9.061130	10.938870	9.119519	9.996202	9.123317	10.876683	26
35	9.059367	9.997127	9.062240	10.937760	9.120469	9.996185	9.124284	10.875716	25
36	9.060460	9.997112	9.063348	10.936652	9.121417	9.996168	9.125249	10.874751	24
37	9.061551	9.997098	9.064453	10.935547	9.122362	9.996151	9.126211	10.873789	23
38	9.062639	9.997083	9.065556	10.934444	9.123306	9.996134	9.127172	10.872828	22
39	9.063724	9.997068	9.066655	10.933345	9.124248	9.996117	9.128130	10.871870	21
40	9.064806	9.997053	9.067752	10.932248	9.125187	9.996100	9.129087	10.870913	20
41	9.065885	9.997039	9.068846	10.931154	9.126125	9.996083	9.130041	10.869959	19
42	9.066962	9.997024	9.069938	10.930062	9.127060	9.996066	9.130994	10.869006	18
43	9.068036	9.997009	9.071027	10.928973	9.127993	9.996049	9.131944	10.868056	17
44	9.069107	9.996994	9.072113	10.927887	9.128925	9.996032	9.132893	10.867107	16
45	9.070176	9.996979	9.073197	10.926803	9.129854	9.996015	9.133839	10.866161	15
46	9.071242	9.996964	9.074278	10.925722	9.130781	9.995998	9.134784	10.865216	14
47	9.072306	9.996949	9.075356	10.924644	9.131706	9.995980	9.135726	10.864274	13
48	9.073366	9.996934	9.076432	10.923568	9.132630	9.995963	9.136667	10.863333	12
49	9.074424	9.996919	9.077505	10.922495	9.133551	9.995946	9.137605	10.862395	11
50	9.075480	9.996904	9.078576	10.921424	9.134470	9.995928	9.138542	10.861458	10
51	9.076533	9.996889	9.079644	10.920356	9.135387	9.995911	9.139476	10.860524	9
52	9.077583	9.996874	9.080710	10.919290	9.136303	9.995894	9.140409	10.859591	8
53	9.078631	9.996858	9.081773	10.918227	9.137216	9.995876	9.141340	10.858660	7
54	9.079676	9.996843	9.082833	10.917167	9.138128	9.995859	9.142269	10.857731	6
55	9.080719	9.996828	9.083891	10.916109	9.139037	9.995841	9.143196	10.856804	5
56	9.081759	9.996812	9.084947	10.915053	9.139944	9.995823	9.144121	10.855879	4
57	9.082797	9.996797	9.086000	10.914000	9.140850	9.995806	9.145044	10.854956	3
58	9.083832	9.996782	9.087050	10.912950	9.141754	9.995788	9.145966	10.854034	2
59	9.084864	9.996766	9.088098	10.911902	9.142655	9.995771	9.146885	10.853115	1
60	9.085894	9.996751	9.089144	10.910856	9.143555	9.995753	9.147803	10.852197	0
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.	′

	8 Deg.				9 Deg.				
′	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	′
0	9.143555	9.995753	9.147803	10.852197	9.194332	9.994620	9.199713	10.800287	60
1	9.144453	9.995735	9.148718	10.851282	9.195129	9.994600	9.200529	10.799471	59
2	9.145349	9.995717	9.149632	10.850368	9.195925	9.994580	9.201345	10.798655	58
3	9.146243	9.995699	9.150544	10.849456	9.196719	9.994560	9.202159	10.797841	57
4	9.147136	9.995681	9.151454	10.848546	9.197511	9.994540	9.202951	10.797029	56
5	9.148026	9.995664	9.152363	10.847637	9.198302	9.994519	9.203782	10.796218	55
6	9.148915	9.995646	9.153269	10.846731	9.199091	9.994499	9.204592	10.795408	54
7	9.149802	9.995628	9.154174	10.845826	9.199879	9.994479	9.205400	10.794400	53
8	9.150686	9.995610	9.155077	10.844923	9.200666	9.994459	9.206207	10.793793	52
9	9.151569	9.995591	9.155978	10.844022	9.201451	9.994438	9.207013	10.792987	51
10	9.152451	9.995573	9.156877	10.843123	9.202234	9.994418	9.207817	10.792183	50
11	9.153330	9.995555	9.157775	10.842225	9.203017	9.994398	9.208619	10.791381	49
12	9.154208	9.995537	9.158671	10.841329	9.203797	9.994377	9.209420	10.790580	48
13	9.155083	9.995519	9.159565	10.840435	9.204577	9.994357	9.810220	10.789780	47
14	9.155957	9.995501	9.160457	10.839543	9.205354	9.994336	9.211018	10.788982	46
15	9.156830	9.995482	9.161347	10.838653	9.206131	9.994316	9.211815	10.788185	45
16	9.157700	9.995464	9.162236	10.837764	9.206906	9.994295	9.212611	10.787389	44
17	9.158569	9.995446	9.163123	10.836877	9.207679	9.994274	9.213405	10.786595	43
18	9.159435	9.995427	9.164008	10.835992	9.208452	9.994254	9.214198	10.785802	42
19	9.160301	9.995409	9.164892	10.835108	9.209222	9.994233	9.214989	10.785011	41
20	9.161164	9.995390	9.165774	10.834226	9.209992	9.994212	9.215780	10.784220	40
21	9.162025	9.995372	9.166654	10.833346	9.210760	9.994191	9.216568	10.783432	39
22	9.162885	9.995353	9.167532	10.832468	9.211526	9.994171	9.217356	10.782644	38
23	9.163743	9.995334	9.168409	10.831591	9.212291	9.994150	9.218142	10.781858	37
24	9.164600	9.995316	9.169284	10.830716	9.213055	9.994129	9.218926	10.781074	36
25	9.165454	9.995297	9.170157	10.829843	9.213818	9.994108	9.219710	10.780290	35
26	9.166307	9.995278	9.171029	10.828971	9.214579	9.994087	9.220492	10.779508	34
27	9.167159	9.995260	9.171899	10.828101	9.215338	9.994066	9.221272	10.778728	33
28	9.168008	9.995241	9.172767	10.827233	9.216097	9.994045	9.222052	10.777948	32
29	9.168856	9.995222	9.173634	10.826366	9.216854	9.994024	9.222830	10.777170	31
30	9.169702	9.995203	9.174499	10.825501	9.217609	9.994003	9.223607	10.776393	30
31	9.170547	9.995184	9.175362	10.824638	9.218363	9.993982	9.224382	10.775618	29
32	9.171389	9.995165	9.176224	10.823776	9.219116	9.993960	9.225156	10.774844	28
33	9.172230	9.995146	9.177084	10.822916	9.219868	9.993939	9.225929	10.774071	27
34	9.173070	9.995127	9.177942	10.822058	9.220618	9.993918	9.225700	10.773300	26
35	9.173908	9.995108	9.178799	10.821201	9.221367	9.993897	9.227471	10.772529	25
36	9.174744	9.995089	9.179655	10.820345	9.222115	9.993875	9.228239	10.771761	24
37	9.175578	9.995070	9.180508	10.819492	9.222861	9.993854	9.229007	10.770993	23
38	9.176411	9.995051	9.181360	10.818640	9.223606	9.993832	9.229773	10.770227	22
39	9.177242	9.995032	9.182211	10.817789	9.224349	9.993811	9.230539	10.769461	21
40	9.178072	9.995013	9.183059	10.816941	9.225092	9.993789	9.231302	10.768698	20
41	9.178900	9.994993	9.183907	10.816093	9.225833	9.993768	9.232065	10.767935	19
42	9.179726	9.994974	9.184752	10.815248	9.226573	9.993746	9.232826	10.767174	18
43	9.180551	9.994955	9.185597	10.814403	9.227311	9.993725	9.233586	10.766414	17
44	9.181374	9.994935	9.186439	10.813561	9.228048	9.993703	9.234345	10.765655	16
45	9.182196	9.994916	9.187280	10.812720	9.228784	9.993681	9.235103	10.764897	15
46	9.183016	9.994896	9.188120	10.811880	9.229518	9.993660	9.235859	10.764141	14
47	9.183834	9.994877	9.188958	10.811042	9.230252	9.993638	9.236614	10.763386	13
48	9.184651	9.994857	9.189794	10.810206	9.230984	9.993616	9.237368	10.762632	12
49	9.185466	9.994838	9.190629	10.809371	9.231715	9.993594	9.238120	10.761880	11
50	9.186280	9.994818	9.191462	10.808538	9.232444	9.993572	9.238872	10.761128	10
51	9.187092	9.994798	9.192294	10.807706	9.233172	9.993550	9.239622	10.760378	9
52	9.187903	9.994779	9.193124	10.806876	9.233899	9.993528	9.240371	10.759629	8
53	9.188712	9.994759	9.193953	10.806047	9.234625	9.993506	9.241118	10.758882	7
54	9.189519	9.994739	9.194780	10.805220	9.235349	9.993484	9.241865	10.758135	6
55	9.190325	9.994720	9.195606	10.804394	9.236073	9.993462	9.242610	10.757390	5
56	9.191130	9.994700	9.196430	10.803570	9.236795	9.993440	9.243354	10.756646	4
57	9.191933	9.994680	9.197253	10.802747	9.237515	9.993418	9.244097	10.755903	3
58	9.192734	9.994660	9.198074	10.801926	9.238235	9.993396	9.244839	10.755161	2
59	9.193534	9.994640	9.198894	10.801106	9.238953	9.993374	9.245579	10.754421	1
60	9.194332	9.994620	9.199713	10.800287	9.239670	9.993351	9.246319	10.753681	0
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.	′

'	10 Deg. Sine.	Cosine.	Tang.	Cotang.	11 Deg. Sine.	Cosine.	Tang.	Cotang.	
0	9.239670	9.993351	9.246319	10.753681	9.280599	9.991947	9.288652	10.711348	60
1	9.240386	9.993329	9.247057	10.752943	9.281248	9.991922	9.289326	10.710674	59
2	9.241101	9.993307	9.247794	10.752206	9.281897	9.991897	9.289999	10.710001	58
3	9.241814	9.993284	9.248530	10.751470	9.282544	9.991873	9.290671	10.709329	57
4	9.242526	9.993262	9.249264	10.750736	9.283190	9.991848	9.291342	10.708658	56
5	9.243237	9.993240	9.249998	10.750002	9.283836	9.991823	9.292013	10.707987	55
6	9.243947	9.993217	9.250730	10.749270	9.284480	9.991799	9.292682	10.707318	54
7	9.244656	9.993195	9.251461	10.748539	9.285124	9.991774	9.293350	10.706650	53
8	9.245363	9.993172	9.252191	10.747809	9.285766	9.991749	9.294017	10.705983	52
9	9.246069	9.993149	9.252920	10.747080	9.286408	9.991724	9.294684	10.705316	51
10	9.246775	9.993127	9.253648	10.746352	9.287048	9.991699	9.295349	10.704651	50
11	9.247478	9.993104	9.254374	10.745626	9.287688	9.991674	9.296013	10.703987	49
12	9.248181	9.993081	9.255100	10.744900	9.288326	9.991649	9.296677	10.703323	48
13	9.248883	9.993059	9.255824	10.744176	9.288964	9.991624	9.297339	10.702661	47
14	9.249583	9.993036	9.256547	10.743453	9.289600	9.991599	9.298001	10.701999	46
15	9.250282	9.993013	9.257269	10.742731	9.290236	9.991574	9.298662	10.701338	45
16	9.250980	9.992990	9.257990	10.742010	9.290870	9.991549	9.299322	10.700678	44
17	9.251677	9.992967	9.258710	10.741290	9.291504	9.991524	9.299980	10.700020	43
18	9.252373	9.992944	9.259429	10.740571	9.292137	9.991498	9.300638	10.699362	42
19	9.253067	9.992921	9.260146	10.739854	9.292768	9.991473	9.301295	10.698705	41
20	9.253761	9.992898	9.260863	10.739137	9.293399	9.991448	9.301951	10.698049	40
21	9.254453	9.992875	9.261578	10.738422	9.294029	9.991422	9.302607	10.697393	39
22	9.255144	9.992852	9.262292	10.737708	9.294658	9.991397	9.303261	10.696739	38
23	9.255834	9.992829	9.263005	10.736995	9.295286	9.991372	9.303914	10.696086	37
24	9.256523	9.992806	9.263717	10.736283	9.295913	9.991346	9.304567	10.695433	36
25	9.257211	9.992783	9.264428	10.735572	9.296539	9.991321	9.305218	10.694782	35
26	9.257898	9.992759	9.265138	10.734862	9.297164	9.991295	9.305869	10.694131	34
27	9.258583	9.992736	9.265847	10.734153	9.297788	9.991270	9.306519	10.693481	33
28	9.259268	9.992713	9.266555	10.733445	9.298412	9.991244	9.307168	10.692832	32
29	9.259951	9.992690	9.267261	10.732739	9.299034	9.991218	9.307816	10.692184	31
30	9.260633	9.992666	9.267967	10.732033	9.299655	9.991193	9.308463	10.691537	30
31	9.261314	9.992643	9.268671	10.731329	9.300276	9.991167	9.309109	10.690891	29
32	9.261994	9.992619	9.269375	10.730625	9.300895	9.991141	9.309754	10.690246	28
33	9.262673	9.992596	9.270077	10.729923	9.301514	9.991115	9.300399	10.689601	27
34	9.263351	9.992572	9.270779	10.729221	9.302132	9.991090	9.311042	10.688958	26
35	9.264027	9.992549	9.271479	10.728521	9.302748	9.991064	9.311685	10.688315	25
36	9.264703	9.992525	9.272178	10.727822	9.303364	9.991038	9.312327	10.687673	24
37	9.265377	9.992501	9.272876	10.727124	9.303979	9.991012	9.312968	10.687032	23
38	9.266051	9.992478	9.273573	10.726427	9.304593	9.990986	9.313608	10.686392	22
39	9.266723	9.992454	9.274269	10.725731	9.305207	9.990960	9.314247	10.685753	21
40	9.267395	9.992430	9.274964	10.725036	9.305819	9.990934	9.314885	10.685115	20
41	9.268065	9.992406	9.275658	10.724342	9.306430	9.990908	9.315523	10.684477	19
42	9.268734	9.992382	9.276351	10.723649	9.307041	9.990882	9.316159	10.683841	18
43	9.269402	9.992359	9.277043	10.722957	9.307650	9.990855	9.316795	10.683205	17
44	9.270069	9.992335	9.277734	10.722266	9.308259	9.990829	9.317430	10.682570	16
45	9.270735	9.992311	9.278424	10.721576	9.308867	9.990803	9.318064	10.681936	15
46	9.271400	9.992287	9.279113	10.720887	9.309474	9.990777	9.318697	10.681303	14
47	9.272064	9.992263	9.279801	10.720199	9.310080	9.990750	9.319330	10.680670	13
48	9.272726	9.992239	9.280488	10.719512	9.310685	9.990724	9.319961	10.680039	12
49	9.273388	9.992214	9.281174	10.718826	9.311289	9.990697	9.320592	10.679408	11
50	9.274049	9.992190	9.281858	10.718142	9.311893	9.990671	9.321222	10.678778	10
51	9.274708	9.992166	9.282542	10.717458	9.312495	9.990645	9.321851	10.678149	9
52	9.275367	9.992142	9.283225	10.716775	9.313097	9.990618	9.322479	10.677521	8
53	9.276025	9.992118	9.283907	10.716093	9.313698	9.990591	9.323106	10.676894	7
54	9.276681	9.992093	9.284588	10.715412	9.314297	9.990565	9.323733	10.676267	6
55	9.277337	9.992069	9.285268	10.714732	9.314897	9.990538	9.324358	10.675642	5
56	9.277991	9.992044	9.285947	10.714053	9.315495	9.990511	9.324983	10.675017	4
57	9.278645	9.992020	9.286624	10.713376	9.316092	9.990485	9.325607	10.674393	3
58	9.279297	9.991996	9.287301	10.712699	9.316689	9.990458	9.326231	10.673769	2
59	9.279948	9.991971	9.287977	10.712023	9.317284	9.990431	9.326853	10.673147	1
60	9.280599	9.991947	9.288652	10.711348	9.317879	9.990404	9.327475	10.672525	0
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.	'

	12 Deg.				13 Deg.				
	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
0	9.317879	9.990404	9.327475	10.672525	9.352088	9.988724	9.363364	10.636636	60
1	9.318473	9.990378	9.328095	10.671905	9.352635	9.988695	9.363940	10.636060	59
2	9.319066	9.990351	9.328715	10.671285	9.353181	9.988666	9.364515	10.635485	58
3	9.319658	9.990324	9.329334	10.670666	9.353726	9.988636	9.365090	10.634910	57
4	9.320249	9.990297	9.329953	10.670047	9.354271	9.988607	9.365664	10.634336	56
5	9.320840	9.990270	9.330570	10.669430	9.354815	9.988578	9.366237	10.633763	55
6	9.321430	9.990243	9.331187	10.668813	9.355358	9.988548	9.366810	10.633190	54
7	9.322019	9.990215	9.331803	10.668197	9.355901	9.988519	9.367382	10.632618	53
8	9.322607	9.990188	9.332418	10.667582	9.356443	9.988489	9.367953	10.632047	52
9	9.323194	9.990161	9.333033	10.666967	9.356984	9.988460	9.368524	10.631476	51
10	9.323780	9.990134	9.333646	10.666354	9.357524	9.988430	9.369094	10.630906	50
11	9.324366	9.990107	9.334259	10.665741	9.358064	9.988401	9.369663	10.630337	49
12	9.324950	9.990079	9.334871	10.665129	9.358603	9.988371	9.370232	10.629768	48
13	9.325534	9.990052	9.335482	10.664518	9.359141	9.988342	9.370799	10.629201	47
14	9.326117	9.990025	9.336093	10.663907	9.359678	9.988312	9.371367	10.628633	46
15	9.326700	9.989997	9.336702	10.663298	9.360215	9.988282	9.371933	10.628067	45
16	9.327281	9.989970	9.337311	10.662689	9.360752	9.988252	9.372499	10.627501	44
17	9.327862	9.989942	9.337919	10.662081	9.361287	9.988223	9.373064	10.626936	43
18	9.328442	9.989915	9.338527	10.661473	9.361822	9.988193	9.373629	10.626371	42
19	9.329021	9.989887	9.339133	10.660867	9.362356	9.988163	9.374193	10.625807	41
20	9.329599	9.989860	9.339739	10.660261	9.362889	9.988133	9.374756	10.625244	40
21	9.330176	9.989832	9.340344	10.659656	9.363422	9.988103	9.375319	10.624681	39
22	9.330753	9.989804	9.340948	10.659052	9.363954	9.988073	9.375881	10.624119	38
23	9.331329	9.989777	9.341552	10.658448	9.364485	9.988043	9.376442	10.623558	37
24	9.331903	9.989749	9.342155	10.657845	9.365016	9.988013	9.377003	10.622997	36
25	9.332478	9.989721	9.342757	10.657243	9.365546	9.987983	9.377563	10.622437	35
26	9.333051	9.989693	9.343358	10.656642	9.366075	9.987953	9.378122	10.621878	34
27	9.333624	9.989665	9.343958	10.656042	9.366604	9.987922	9.378681	10.621319	33
28	9.334195	9.989637	9.344558	10.655442	9.367131	9.987892	9.379239	10.620761	32
29	9.334767	9.989610	9.345157	10.654843	9.367659	9.987862	9.379797	10.620203	31
30	9.335337	9.989582	9.345755	10.654245	9.368185	9.987832	9.380354	10.619646	30
31	9.335906	9.989553	9.346353	10.653647	9.368711	9.987801	9.380910	10.619090	29
32	9.336475	9.989525	9.346949	10.653051	9.369236	9.987771	9.381466	10.618534	28
33	9.337043	9.989497	9.347545	10.652455	9.369761	9.987740	9.382020	10.617980	27
34	9.337610	9.989469	9.348141	10.651859	9.370285	9.987710	9.382575	10.617425	26
35	9.338176	9.989441	9.348735	10.651265	9.370808	9.987679	9.383129	10.616871	25
36	9.338742	9.989413	9.349329	10.650671	9.371330	9.987649	9.383682	10.616318	24
37	9.339307	9.989385	9.349922	10.650078	9.371852	9.987618	9.384234	10.615766	23
38	9.339871	9.989356	9.350514	10.649486	9.372373	9.987588	9.384786	10.615214	22
39	9.340434	9.989328	9.351106	10.648894	9.372894	9.987557	9.385337	10.614663	21
40	9.340996	9.989300	9.351697	10.648303	9.373414	9.987526	9.385888	10.614112	20
41	9.341558	9.989271	9.352287	10.647713	9.373933	9.987496	9.386438	10.613562	19
42	9.342119	9.989243	9.352876	10.647124	9.374452	9.987465	9.386987	10.613013	18
43	9.342679	9.989214	9.353465	10.646535	9.374970	9.987434	9.387536	10.612464	17
44	9.343239	9.989186	9.354053	10.645947	9.375487	9.987403	9.388084	10.611916	16
45	9.343797	9.989157	9.354640	10.645360	9.376003	9.987372	9.388631	10.611369	15
46	9.344355	9.989128	9.355227	10.644773	9.376519	9.987341	9.389178	10.610822	14
47	9.344912	9.989100	9.355813	10.644187	9.377035	9.987310	9.389724	10.610276	13
48	9.345469	9.989071	9.356398	10.643602	9.377549	9.987279	9.390270	10.609730	12
49	9.346024	9.989042	9.356982	10.643018	9.378063	9.987248	9.390815	10.609185	11
50	9.346579	9.989014	9.357566	10.642434	9.378577	9.987217	9.391360	10.608640	10
51	9.347134	9.988985	9.358149	10.641851	9.379089	9.987186	9.391903	10.608097	9
52	9.347687	9.988956	9.358731	10.641269	9.379601	9.987155	9.392447	10.607553	8
53	9.348240	9.988927	9.359313	10.640687	9.380113	9.987124	9.392989	10.607011	7
54	9.348792	9.988898	9.359893	10.640107	9.380624	9.987092	9.393531	10.606469	6
55	9.349343	9.988869	9.360474	10.639526	9.381134	9.987061	9.394073	10.605927	5
56	9.349893	9.988840	9.361053	10.638947	9.381643	9.987030	9.394614	10.605386	4
57	9.350443	9.988811	9.361632	10.638368	9.382152	9.986998	9.395154	10.604846	3
58	9.350992	9.988782	9.362210	10.637790	9.382661	9.986967	9.395694	10.604306	2
59	9.351540	9.988753	9.362787	10.637213	9.383168	9.986936	9.396233	10.603767	1
60	9.352088	9.988724	9.363364	10.636636	9.383675	9.986904	9.396771	10.603229	0
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.	

′	14 Deg.				15 Deg.			
	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.
0	9.383675	9.986904	9.396771	10.603229	9.412996	9.984944	9.428052	10.571948
1	9.384182	9.986873	9.397309	10.602691	9.413467	9.984910	9.428558	10.571442
2	9.384687	9.986841	9.397846	10.602154	9.413938	9.984876	9.429062	10.570938
3	9.385192	9.986809	9.398383	10.601617	9.414408	9.984842	9.429566	10.570434
4	9.385697	9.986778	9.398919	10.601081	9.414878	9.984808	9.430070	10.569930
5	9.386201	9.986746	9.399455	10.600545	9.415347	9.984774	9.430573	10.569427
6	9.386704	9.986714	9.399990	10.600010	9.415815	9.984740	9.431075	10.568925
7	9.387207	9.986683	9.400524	10.599476	9.416283	9.984706	9.431577	10.568423
8	9.387709	9.986651	9.401058	10.598942	9.416751	9.984672	9.432079	10.567921
9	9.388210	9.986619	9.401591	10.598409	9.417217	9.984638	9.432580	10.567420
10	9.388711	9.986587	9.402124	10.597876	9.417684	9.984603	9.433080	10.566920
11	9.389211	9.986555	9.402656	10.597344	9.418150	9.984569	9.433580	10.566420
12	9.389711	9.986523	9.403187	10.596813	9.418615	9.984535	9.434080	10.565920
13	9.390210	9.986491	9.403718	10.596282	9.419079	9.984500	9.434579	10.565421
14	9.390708	9.986459	9.404249	10.595751	9.419544	9.984466	9.435078	10.564922
15	9.391206	9.986427	9.404778	10.595222	9.420007	9.984432	9.435576	10.564424
16	9.391703	9.986395	9.405308	10.594692	9.420470	9.984397	9.436073	10.563927
17	9.392199	9.986363	9.405836	10.594164	9.420933	9.984363	9.436570	10.563430
18	9.392695	9.986331	9.406364	10.593636	9.421395	9.984328	9.437067	10.562933
19	9.393191	9.986299	9.406892	10.593108	9.421857	9.984294	9.437563	10.562437
20	9.393685	9.986266	9.407419	10.592581	9.422318	9.984259	9.438059	10.561941
21	9.394179	9.986234	9.407945	10.592055	9.422778	9.984224	9.438554	10.561446
22	9.394673	9.986202	9.408471	10.591529	9.423238	9.984190	9.439048	10.560952
23	9.395166	9.986169	9.408996	10.591004	9.423697	9.984155	9.439563	10.560457
24	9.395658	9.986137	9.409521	10.590479	9.424156	9.984120	9.440036	10.559964
25	9.396150	9.986104	9.410045	10.589955	9.424615	9.984085	9.440529	10.559471
26	9.396641	9.986072	9.410569	10.589431	9.425073	9.984050	9.441022	10.558978
27	9.397132	9.986039	9.411092	10.588908	9.425530	9.984015	9.441514	10.558486
28	9.397621	9.986007	9.411615	10.588385	9.425987	9.983981	9.442006	10.557994
29	9.398111	9.985974	9.412137	10.587863	9.426443	9.983946	9.442497	10.557503
30	9.398600	9.985942	9.412658	10.587342	9.426899	9.983911	9.442988	10.557012
31	9.399088	9.985909	9.413179	10.586821	9.427354	9.983875	9.443479	10.556521
32	9.399575	9.985876	9.413699	10.586301	9.427809	9.983840	9.443968	10.556032
33	9.400062	9.985843	9.414219	10.585781	9.428263	9.983805	9.444458	10.555542
34	9.400549	9.985811	9.414738	10.585262	9.428717	9.983770	9.444947	10.555053
35	9.401035	9.985778	9.415257	10.584743	9.429170	9.983735	9.445435	10.554565
36	9.401520	9.985745	9.415775	10.584225	9.429623	9.983700	9.445923	10.554077
37	9.402005	9.985712	9.416293	10.583707	9.430075	9.983664	9.446411	10.553589
38	9.402489	9.985679	9.416810	10.583190	9.430527	9.983629	9.446898	10.553102
39	9.402972	9.985646	9.417326	10.582674	9.430978	9.983594	9.447384	10.552616
40	9.403455	9.985613	9.417842	10.582158	9.431429	9.983558	9.447870	10.552130
41	9.403938	9.985580	9.418358	10.581642	9.431879	9.983523	9.448356	10.551644
42	9.404420	9.985547	9.418873	10.581127	9.432329	9.983487	9.448841	10.551159
43	9.404901	9.985514	9.419387	10.580613	9.432778	9.983452	9.449326	10.550674
44	9.405382	9.985480	9.419901	10.580099	9.433226	9.983416	9.449810	10.550190
45	9.405862	9.985447	9.420415	10.579585	9.433675	9.983381	9.450294	10.549706
46	9.406341	9.985414	9.420927	10.579073	9.434122	9.983345	9.450777	10.549223
47	9.406820	9.985381	9.421440	10.578560	9.434569	9.983309	9.451260	10.548740
48	9.407299	9.985347	9.421952	10.578048	9.435016	9.983273	9.451743	10.548257
49	9.407777	9.985314	9.422463	10.577537	9.435462	9.983238	9.452225	10.547775
50	9.408254	9.985280	9.422974	10.577026	9.435908	9.983202	9.452706	10.547294
51	9.408731	9.985247	9.423484	10.576516	9.436353	9.983166	9.453187	10.546813
52	9.409207	9.985213	9.423993	10.576007	9.436798	9.983130	9.453668	10.546332
53	9.409682	9.985180	9.424503	10.575497	9.437242	9.983094	9.454148	10.545852
54	9.410157	9.985146	9.425011	10.574989	9.437686	9.983058	9.454628	10.545372
55	9.410632	9.985113	9.425519	10.574481	9.438129	9.983022	9.455107	10.544893
56	9.411106	9.98507[illegible]	9.426027	10.573973	9.438572	9.982986	9.455586	10.544414
57	9.411579	9.985045	9.426534	10.573469	9.439014	9.982950	9.456064	10.543936
58	9.412052	9.985011	9.427041	10.572959	9.439456	9.982914	9.456542	10.543458
59	9.412524	9.984978	9.427547	10.572453	9.439897	9.982878	9.457019	10.542981
60	9.412996	9.984944	9.428052	10.571948	9.440338	9.982842	9.457496	10.542504
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.

	16 Deg.				17 Deg.				
	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
0	9.440338	9.982842	9.457496	10.542504	9.465935	9.980596	9.485339	10.514661	60
1	9.440778	9.982805	9.457973	10.542027	9.466348	9.980558	9.485791	10.514209	59
2	9.441218	9.982769	9.458449	10.541551	9.466761	9.980519	9.486242	10.513758	58
3	9.441658	9.982733	9.458925	10.541075	9.467173	9.980480	9.486693	10.513307	57
4	9.442096	9.982696	9.459400	10.540600	9.467585	9.980442	9.487143	10.512857	56
5	9.442535	9.982660	9.459875	10.540125	9.467996	9.980403	9.487593	10.512407	55
6	9.442973	9.982624	9.460349	10.539651	9.468407	9.980364	9.488043	10.511957	54
7	9.443410	9.982587	9.460823	10.539177	9.468817	9.980325	9.488492	10.511508	53
8	9.443847	9.982551	9.461297	10.538703	9.469227	9.980286	9.488941	10.511059	52
9	9.444284	9.982514	9.461770	10.538230	9.469637	9.980247	9.489390	10.510610	51
10	9.444720	9.982477	9.462242	10.537758	9.470046	9.980208	9.489838	10.510162	50
11	9.445155	9.982441	9.462715	10.537285	9.470455	9.980169	9.490286	10.509714	49
12	9.445590	9.982404	9.463186	10.536814	9.470863	9.980130	9.490733	10.509267	48
13	9.446025	9.982367	9.463658	10.536342	9.471271	9.980091	9.491180	10.508820	47
14	9.446459	9.982331	9.464128	10.535872	9.471679	9.980052	9.491627	10.508373	46
15	9.446893	9.982294	9.464599	10.535401	9.472086	9.980012	9.492073	10.507927	45
16	9.447326	9.982257	9.465069	10.534931	9.472492	9.979973	9.492519	10.507481	44
17	9.447759	9.982220	9.465539	10.534461	9.472898	9.979934	9.492965	10.507035	43
18	9.448191	9.982183	9.466008	10.533992	9.473304	9.979895	9.493410	10.506590	42
19	9.448623	9.982146	9.466477	10.533523	9.473710	9.979855	9.493854	10.506146	41
20	9.449054	9.982109	9.466945	10.533055	9.474115	9.979816	9.494299	10.505701	40
21	9.449485	9.982072	9.467413	10.532587	9.474519	9.979776	9.494743	10.505257	39
22	9.449915	9.982035	9.467880	10.532120	9.474923	9.979737	9.495186	10.504814	38
23	9.450345	9.981998	9.468347	10.531653	9.475327	9.979697	9.495630	10.504370	37
24	9.450775	9.981961	9.468814	10.531186	9.475730	9.979658	9.496073	10.503927	36
25	9.451204	9.981924	9.469280	10.530720	9.476133	9.979618	9.496515	10.503485	35
26	9.451632	9.981886	9.469746	10.530254	9.476536	9.979579	9.496957	10.503043	34
27	9.452060	9.981849	9.470211	10.529789	9.476938	9.979539	9.497399	10.502601	33
28	9.452488	9.981812	9.470676	10.529324	9.477340	9.979499	9.497841	10.502159	32
29	9.452915	9.981774	9.471141	10.528859	9.477741	9.979459	9.498282	10.501718	31
30	9.453342	9.981737	9.471605	10.528395	9.478142	9.979420	9.498722	10.501278	30
31	9.453768	9.981700	9.472069	10.527931	9.478542	9.979380	9.499163	10.500837	29
32	9.454194	9.981662	9.472532	10.527468	9.478942	9.979340	9.499603	10.500397	28
33	9.454619	9.981626	9.472995	10.527005	9.479342	9.979300	9.500042	10.499958	27
34	9.455044	9.981587	9.473457	10.526543	9.479741	9.979260	9.500481	10.499519	26
35	9.455469	9.981549	9.473919	10.526081	9.480140	9.979220	9.500920	10.499080	25
36	9.455893	9.981512	9.474381	10.525619	9.480539	9.979180	9.501359	10.498641	24
37	9.456316	9.981474	9.474842	10.525158	9.480937	9.979140	9.501797	10.498203	23
38	9.456739	9.981436	9.475303	10.524697	9.481334	9.979100	9.502235	10.497765	22
39	9.457162	9.981399	9.475763	10.524237	9.481731	9.979059	9.502672	10.497328	21
40	9.457584	9.981361	9.476223	10.523777	9.482128	9.979019	9.503109	10.496891	20
41	9.458006	9.981323	9.476683	10.523317	9.482525	9.978979	9.503546	10.496454	19
42	9.458427	9.981285	9.477142	10.522858	9.482921	9.978939	9.503982	10.496018	18
43	9.458848	9.981247	9.477601	10.522399	9.483316	9.978898	9.504418	10.495582	17
44	9.459268	9.981209	9.478059	10.521941	9.483712	9.978858	9.504854	10.495146	16
45	9.459688	9.981171	9.478517	10.521483	9.484107	9.978817	9.505289	10.494711	15
46	9.460108	9.981133	9.478975	10.521025	9.484501	9.978777	9.505724	10.494276	14
47	9.460527	9.981095	9.479432	10.520568	9.484895	9.978737	9.506159	10.493841	13
48	9.460946	9.981057	9.479889	10.520111	9.485289	9.978696	9.506593	10.493407	12
49	9.461364	9.981019	9.480345	10.519655	9.485682	9.978655	9.507027	10.492973	11
50	9.461782	9.980981	9.480801	10.519199	9.486075	9.978615	9.507460	10.492540	10
51	9.462199	9.980942	9.481257	10.518743	9.486467	9.978574	9.507893	10.492107	9
52	9.462616	9.980904	9.481712	10.518288	9.486860	9.978533	9.508326	10.491674	8
53	9.463032	9.980866	9.482167	10.517833	9.487251	9.978493	9.508759	10.491241	7
54	9.463448	9.980827	9.482621	10.517379	9.487643	9.978452	9.509191	10.490809	6
55	9.463864	9.980789	9.483075	10.516925	9.488034	9.978411	9.509622	10.490378	5
56	9.464279	9.980750	9.483529	10.516471	9.488424	9.978370	9.510054	10.489946	4
57	9.464694	9.980712	9.483982	10.516018	9.488814	9.978329	9.510485	10.489515	3
58	9.465108	9.980673	9.484435	10.515565	9.489204	9.978288	9.510916	10.489084	2
59	9.465522	9.980635	9.484887	10.515113	9.489593	9.978247	9.511346	10.488654	1
60	9.465935	9.980596	9.485339	10.514661	9.489982	9.978206	9.511776	10.488224	0
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.	

	18 Deg.				19 Deg.				
′	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
0	9.489982	9.978206	9.511776	10.488224	9.512642	9.975670	9.536972	10.463028	60
1	9.490371	9.978165	9.512206	10.487794	9.513009	9.975627	9.537382	10.462618	59
2	9.490759	9.978124	9.512635	10.487365	9.513375	9.975583	9.537792	10.462208	58
3	9.491147	9.978083	9.513064	10.486936	9.513741	9.975539	9.538202	10.461798	57
4	9.491535	9.978042	9.513493	10.486507	9.514107	9.975496	9.538611	10.461389	56
5	9.491922	9.978001	9.513921	10.486079	9.514472	9.975452	9.539020	10.460980	55
6	9.492308	9.977959	9.514349	10.485651	9.514837	9.975408	9.539429	10.460571	54
7	9.492695	9.977918	9.514777	10.485223	9.515202	9.975365	9.539837	10.460163	53
8	9.493081	9.977877	9.515204	10.484796	9.515566	9.975321	9.540245	10.459755	52
9	9.493466	9.977835	9.515631	10.484369	9.515930	9.975277	9.540653	10.459347	51
10	9.493851	9.977794	9.516057	10.483943	9.516294	9.975233	9.541061	10.458939	50
11	9.494236	9.977752	9.516484	10.483516	9.516657	9.975189	9.541468	10.458532	49
12	9.494621	9.977711	9.516910	10.483090	9.517020	9.975140	9.541875	10.458125	48
13	9.495005	9.977669	9.517335	10.482665	9.517382	9.975101	9.542281	10.457719	47
14	9.495388	9.977628	9.517761	10.482239	9.517745	9.975057	9.542688	10.457312	46
15	9.4957[illegible]2	9.977586	9.518186	10.481814	9.518107	9.975013	9.543094	10.456906	45
16	9.496154	9.977544	9.518610	10.481390	9.518468	9.974969	9.543499	10.456501	44
17	9.496537	9.977503	9.519034	10.480966	9.518829	9.974925	9.543905	10.456095	43
18	9.496919	9.977461	9.519458	10.480542	9.519190	9.974880	9.544310	10.455690	42
19	9.497301	9.977419	9.519882	10.480118	9.519551	9.974836	9.544715	10.455285	41
20	9.497682	9.977377	9.520305	10.479695	9.519911	9.974792	9.545119	10.454881	40
21	9.498064	9.977335	9.520728	10.479272	9.520271	9.974748	9.545524	10.454476	39
22	9.498444	9.977293	9.521151	10.478849	9.520631	9.974703	9.545928	10.454072	38
23	9.498825	9.977251	9.521573	10.478427	9.520990	9.974659	9.546331	10.453669	37
24	9.499204	9.977209	9.521995	10.478005	9.521349	9.974614	9.546735	10.453265	36
25	9.499584	9.977167	9.522417	10.477583	9.521707	9.974570	9.547138	10.452862	35
26	9.499963	9.977125	9.522838	10.477162	9.522066	9.974525	9.547540	10.452460	34
27	9.500342	9.977083	9.523259	10.476741	9.522424	9.974481	9.547943	10.452057	33
28	9.500721	9.977041	9.523680	10.476320	9.522781	9.974436	9.548345	10.451655	32
29	9.501099	9.976999	9.524100	10.475900	9.523138	9.974391	9.548747	10.451253	31
30	9.501476	9.976957	9.524520	10.475480	9.523495	9.974347	9.549149	10.450851	30
31	9.501854	9.976914	9.524940	10.475060	9.523852	9.974302	9.549550	10.450450	29
32	9.502231	9.976872	9.525360	10.474641	9.524208	9.974257	9.549951	10.450049	28
33	9.502607	9.976830	9.525778	10.474222	9.524564	9.974212	9.550352	10.449648	27
34	9.502984	9.976787	9.526197	10.473803	9.524920	9.974167	9.550752	10.449248	26
35	9.503360	9.976745	9.526615	10.473385	9.525275	9.974122	9.551153	10.448847	25
36	9.503735	9.976702	9.527033	10.472967	9.525630	9.974077	9.551552	10.448448	24
37	9.504110	9.976660	9.527451	10.472549	9.525984	9.974032	9.551952	10.448048	23
38	9.504485	9.976617	9.527868	10.472132	9.526339	9.973987	9.552351	10.447649	22
39	9.504860	9.976574	9.528285	10.471715	9.526693	9.973942	9.552750	10.447250	21
40	9.505234	9.976532	9.528702	10.471298	9.527046	9.973897	9.553149	16.446851	20
41	9.505608	9.976489	9.529119	10.470881	9.527400	9.973852	9.553548	10.446452	19
42	9.505981	9.976446	9.529535	10.470465	9.527753	9.973807	9.553946	10.446054	18
43	9.506354	9.976404	9.529951	10.470049	9.528105	9.973761	9.554344	10.445656	17
44	9.506727	9.976361	9.530366	10.469634	9.528458	9.973716	9.554741	10.445259	16
45	9.507099	9.976318	9.530781	10.469219	9.528810	9.973671	9.555139	10.444861	15
46	9.507471	9.976275	9.531196	10.468804	9.529161	9.973625	9.555536	10.444464	14
47	9.507843	9.976232	9.531611	10.468389	9.529513	9.973580	9.555933	10.444067	13
48	9.508214	9.976189	9.532025	10.467975	9.529864	9.973535	9.556329	10.443671	12
49	9.508585	9.976146	9.532439	10.467561	9.530215	9.973489	9.556725	10.443275	11
50	9.508956	9.976103	9.532853	10.467147	9.530565	9.973444	9.557121	10.442879	10
51	9.509326	9.976060	9.533266	10.466734	9.530915	9.973398	9.557517	10.442483	9
52	9.509696	9.976017	9.533679	10.466321	9.531265	9.973352	9.557913	10.442087	8
53	9.510065	9.975974	9.534092	10.465908	9.531614	9.973307	9.558308	10.441692	7
54	9.510434	9.975930	9.534504	10.465496	9.531963	9.973261	9.558703	10.441297	6
55	9.510803	9.975887	9.534916	10.465084	9.532312	9.973215	9.559097	10.440903	5
56	9.511172	9.975844	9.535328	10.464672	9.532661	9.973169	9.559491	10.440509	4
57	9.511540	9.975800	9.535739	10.464261	9.533009	9.973124	9.559885	10.440115	3
58	9.511907	9.975757	9.536150	10.463850	9.533357	9.973078	9.560279	10.439721	2
59	9.512275	9.975714	9.536561	10.463439	9.533704	9.973032	9.560673	10.439327	1
60	9.512642	9.975670	9.536972	10.463028	9.534052	9.972986	9.561066	10.438934	0
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.	

	20 Deg.				21 Deg.				
′	Sine.	Cosine.	Tang.	Cotang.	Sine	Cosine.	Tang.	Cotang.	
0	9.534052	9.972986	9.561066	10.438934	9.554329	9.970152	9.584177	10.415823	60
1	9.534399	9.972940	9.561459	10.438541	9.554658	9.970103	9.584555	10.415445	59
2	9.534745	9.972894	9.561851	10.438149	9.554987	9.970055	9.584932	10.415068	58
3	9.535092	9.972848	9.562244	10.437756	9.555315	9.970006	9.585309	10.414691	57
4	9.535438	9.972802	9.562636	10.437364	9.555643	9.969957	9.585686	10.414314	56
5	9.535783	9.972755	9.563028	10.436972	9.555971	9.969909	9.586062	10.413938	55
6	9.536129	9.972709	9.563419	10.436581	9.556299	9.969860	9.586439	10.413561	54
7	9.536474	9.972663	9.563811	10.436189	9.556626	9.969811	9.586815	10.413185	53
8	9.536818	9.972617	9.564202	10.435798	9.556953	9.969762	9.587190	10.412810	52
9	9.537163	9.972570	9.564593	10.435407	9.557280	9.969714	9.587566	10.412434	51
10	9.537507	9.972524	9.564983	10.435017	9.557606	9.969665	9.587941	10.412059	50
11	9.537851	9.972478	9.565373	10.434627	9.557932	9.969616	9.588316	10.411684	49
12	9.538194	9.972431	9.565763	10.434237	9.558258	9.969567	9.588691	10.411309	48
13	9.538538	9.972385	9.566153	10.433847	9.558583	9.969518	9.589066	10.410934	47
14	9.538880	9.972338	9.566542	10.433458	9.558909	9.969469	9.589440	10.410560	46
15	9.539223	9.972291	9.566932	10.433068	9.559234	9.969420	9.589814	10.410186	45
16	9.539565	9.972245	9.567320	10.432680	9.559558	9.969370	9.590188	10.409812	44
17	9.539907	9.972198	9.567709	10.432291	9.559883	9.969321	9.590562	10.409438	43
18	9.540249	9.972151	9.568098	10.431902	9.560207	9.969272	9.590935	10.409065	42
19	9.540590	9.972105	9.568486	10.431514	9.560531	9.969223	9.591308	10.408692	41
20	9.540931	9.972058	9.568873	10.431127	9.560855	9.969173	9.591681	10.408319	40
21	9.541272	9.972011	9.569261	10.430739	9.561178	9.969124	9.592054	10.407946	39
22	9.541613	9.971964	9.569648	10.430352	9.561501	9.969075	9.592426	10.407574	38
23	9.541953	9.971917	9.570035	10.429965	9.561824	9.969025	9.592799	10.407201	37
24	9.542293	8.971870	9.570422	10.429578	9.562146	9.968976	9.593171	10.406829	36
25	9.542632	9.971823	9.570809	10.429191	9.562468	9.968926	9.593542	10.406458	35
26	9.542971	9.971776	9.571195	10.428805	9.562790	9.968877	9.593914	10.406086	34
27	9.543310	9.971729	9.571581	10.428419	9.563112	9.968827	9.594285	10.405715	33
28	9.543649	9.971682	9.571967	10.428033	9.563433	9.968777	9.594656	10.405344	32
29	9.543987	9.971635	9.572352	10.427648	9.563755	9.968728	9.595027	10.404973	31
30	9.544325	9.971588	9.572738	10.427262	9.564075	9.968678	9.595398	10.404602	30
31	9.544663	9.971540	9.573123	10.426877	9.564396	9.968628	9.595768	10.404232	29
32	9.545000	9.971493	9.573507	10.426493	9.564716	9.968578	9.596138	10.403862	28
33	9.545338	9.971446	9.573892	10.426108	9.565036	9.968528	9.596508	10.403492	27
34	9.545674	9.971398	9.574276	10.425724	9.565356	9.968479	9.596878	10.403122	26
35	9.546011	9.971351	9.574660	10.425340	9.565676	9.968429	9.597247	10.402753	25
36	9.546347	9.971303	9.575044	10.424956	9.565995	9.968379	9.597616	10.402384	24
37	9.546683	9.971256	9.575427	10.424573	9.566314	9.968329	9.597985	10.402015	23
38	9.547019	9.971208	9.575810	10.424190	9.566632	9.968278	9.598354	10.401646	22
39	9.547354	9.971161	9.576193	10.423807	9.566951	9.968228	9.598722	10.401278	21
40	9.547689	9.971113	9.576576	10.423424	9.567269	9.968178	9.599091	10.400909	20
41	9.548024	9.971066	9.576959	10.423041	9.567587	9.968128	9.599459	10.400541	19
42	9.548359	9.971018	9.577341	10.422659	9.567904	9.968078	9.599827	10.400173	18
43	9.548693	9.970970	9.577723	10.422277	9.568222	9.968027	9.600194	10.399806	17
44	9.549027	9.970922	9.578104	10.421896	9.568539	9.967977	9.600562	10.399438	16
45	9.549360	9.970874	9.578486	10.421514	9.568856	9.967927	9.600929	10.399071	15
46	9.549693	9.970827	9.578867	10.421133	9.569172	9.967876	9.601296	10.398704	14
47	9.550026	9.970779	9.579248	10.420752	9.569488	9.967826	9.601663	10.398337	13
48	9.550359	9.970731	9.579629	10.420371	9.569804	9.967775	9.602029	10.397971	12
49	9.550692	9.970683	9.580009	10.419991	9.570120	9.967725	9.602395	10.397605	11
50	9.551024	9.970635	9.580389	10.419611	9.570435	9.967674	9.602761	10.397239	10
51	9.551356	9.970586	9.580769	10.419231	9.570751	9.967624	9.603127	10.396873	9
52	9.551687	9.970538	9.581149	10.418851	9.571066	9.967573	9.603493	10.396507	8
53	9.552018	9.970490	9.581528	10.418472	9.571380	9.967522	9.603858	10.396142	7
54	9.552349	9.970442	9.581907	10.418093	9.571695	9.967471	9.604223	10.395777	6
55	9.552680	9.970394	9.582286	10.417714	9.572009	9.967421	9.604588	10.395412	5
56	9.553010	9.970345	9.582665	10.417335	9.572323	9.967370	9.604953	10.395047	4
57	9.553341	9.970297	9.583044	10.416956	9.572636	9.967319	9.605317	10.394683	3
58	9.553670	9.970249	9.583422	10.416578	9.572950	9.967268	9.605682	10.394318	2
59	9.554000	9.970200	9.583800	10.416200	9.573263	9.967217	9.606046	10.393954	1
60	9.554329	9.970152	9.584177	10.415823	9.573575	9.967166	9.606410	10.393590	0
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.	′

'	22 Deg.				23 Deg.				
'	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
0	9.573575	9.967166	9.606410	10.393590	9.591878	9.964026	9.627852	10.372148	60
1	9.573888	9.967115	9.606773	10.393227	9.592176	9.963972	9.628203	10.371797	59
2	9.574200	9.967064	9.607137	10.392863	9.592473	9.963919	9.628554	10.371446	58
3	9.574512	9.967013	9.607500	10.392500	9.592770	9.963865	9.628905	10.371095	57
4	9.574824	9.966961	9.607863	10.392137	9.593067	9.963811	9.629255	10.370745	56
5	9.575136	9.966910	9.608225	10.391775	9.593363	9.963757	9.629606	10.370394	55
6	9.575447	9.966859	9.608588	10.391412	9.593659	9.963704	9.629956	10.370044	54
7	9.575758	9.966808	9.608950	10.391050	9.593955	9.963650	9.630306	10.369694	53
8	9.576069	9.966756	9.609312	10.390688	9.594251	9.963596	9.630656	10.369344	52
9	9.576379	9.966705	9.609674	10.390326	9.594547	9.963542	9.631005	10.368995	51
10	9.576689	9.966653	9.610036	10.389964	9.594842	9.963488	9.631355	10.368645	50
11	9.576999	9.966602	9.610397	10.389603	9.595137	9.963434	9.631704	10.368296	49
12	9.577309	9.966550	9.610759	10.389241	9.595432	9.963379	9.632053	10.367947	48
13	9.577618	9.966499	9.611120	10.388880	9.595727	9.963325	9.632402	10.367598	47
14	9.577927	9.966447	9.611480	10.388520	9.596021	9.963271	9.632750	10.367250	46
15	9.578236	9.966395	9.611841	10.388159	9.596315	9.963217	9.633090	10.366901	45
16	9.578545	9.966344	9.612201	10.387799	9.596609	9.963163	9.633447	10.366553	44
17	9.578853	9.966292	9.612561	10.387439	9.596903	9.963108	9.633795	10.366205	43
18	9.579162	9.966240	9.612921	10.387079	9.597196	9.963054	9.634143	10.365857	42
19	9.579470	9.966188	9.613281	10.386719	9.597490	9.962999	9.634490	10.365510	41
20	9.579777	9.966136	9.613641	10.386359	9.597783	9.962945	9.634838	10.365162	40
21	9.580085	9.966085	9.614000	10.386000	9.598075	9.962890	9.635185	10.364815	39
22	9.580392	9.966033	9.614359	10.385641	9.598368	9.962836	9.635532	10.364468	38
23	9.580699	9.965981	9.614718	10.385282	9.598660	9.962781	9.635879	10.364121	37
24	9.581005	9.965929	9.615077	10.384923	9.598952	9.992727	9.636226	10.363774	36
25	9.581312	9.965876	9.615435	10.384565	9.599244	9.962672	9.636572	10.363428	35
26	9.581618	9.965824	9.615793	10.384207	9.599536	9.962617	9.636919	10.363081	34
27	9.581924	9.965772	9.616151	10.383849	9.599827	9.962562	9.637265	10.362735	33
28	9.582229	9.965720	9.616509	10.383491	9.600118	9.962508	9.637611	10.362389	32
29	9.582535	9.965668	9.616867	10.383133	9.600409	9.962453	9.637956	10.362044	31
30	9.582840	9.965615	9.617224	10.382776	9.600700	9.962398	9.638302	10.361698	30
31	9.583145	9.965563	9.617582	10.382418	9.600990	9.962343	9.638647	10.361355	29
32	9.583449	9.965511	9.617939	10.382061	9.601280	9.962288	9.638992	10.361008	28
33	9.583754	9.965458	9.618295	10.381705	9.601570	9.962233	9.639337	10.360663	27
34	9.584058	9.965406	9.618652	10.381348	9.601860	9.962178	9.639682	10.360318	26
35	9.584361	9.965353	9.619008	10.380992	9.602150	9.962123	9.640027	10.359973	25
36	9.584665	9.965301	9.619364	10.380636	9.602439	9.962067	9.640371	10.359629	24
37	9.584968	9.965248	9.619720	10.380280	9.602728	9.962012	9.640716	10.359284	23
38	9.585272	9.965195	9.620076	10.379924	9.603017	9.961957	9.641060	10.358940	22
39	9.585574	9.965143	9.620432	10.379568	9.603305	9.961902	9.641404	10.358596	21
40	9.585877	9.965090	9.620787	10.379213	9.603594	9.961846	9.641747	10.358253	20
41	9.586179	9.965037	9.621142	10.378858	9.603882	9.961791	9.642091	10.357909	19
42	9.586482	9.964984	9.621497	10.378503	9.604170	9.961735	9.642434	10.357566	18
43	9.586783	9.964931	9.621852	10.378148	9.604457	9.961680	9.642777	10.357223	17
44	9.587085	9.964879	9.622207	10.377793	9.604745	9.961624	9.643120	10.356880	16
45	9.587386	9.964826	9.622561	10.377439	9.605032	9.961569	9.643463	10.356537	15
46	9.587688	9.964773	9.622915	10.377085	9.605319	9.961513	9.643806	10.356194	14
47	9.587989	9.964720	9.623269	10.376731	9.605606	9.961458	9.644148	10.355852	13
48	9.588289	9.964666	9.623623	10.376377	9.605892	9.961402	9.644490	10.355510	12
49	9.588590	9.964613	9.623976	10.376024	9.606179	9.961346	9.644832	10.355168	11
50	9.588890	9.964560	9.624330	10.375670	9.606465	9.961290	9.645174	10.354826	10
51	9.589190	9.964507	9.624683	10.375317	9.606751	9.961235	9.645516	10.354484	9
52	9.589489	9.964454	9.625036	10.374964	9.607036	9.961179	9.645857	10.354143	8
53	9.589789	9.964400	9.625388	10.374612	9.607322	9.961123	9.646199	10.353801	7
54	9.590088	9.964347	9.625741	10.374259	9.607607	9.961067	9.646540	10.353460	6
55	9.590387	9.964294	9.626093	10.373907	9.607892	9.951011	9.646881	10.353119	5
56	9.590686	9.964240	9.626445	10.373555	9.608177	9.960955	9.647222	10.352778	4
57	9.590984	9.964187	9.626797	10.373203	9.608461	9.960899	9.647562	10.352438	3
58	9.591282	9.964133	9.627149	10.372851	9.608745	9.960843	9.647903	10.352097	2
59	9.591580	9.964080	9.627501	10.372499	9.609029	9.960786	9.648243	10.351757	1
60	9.591878	9.964026	9.627852	10.372148	9.609313	9.960730	9.648583	10.351417	0
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.	'

′	24 Deg. Sine.	Cosine.	Tang.	Cotang.	25 Deg. Sine.	Cosine.	Tang.	Cotang.	′
0	9·609313	9·960730	9·648583	10·351417	9·625948	9·957276	9·668673	10·331327	60
1	9·609597	9·960674	9·648923	10·351077	9·626219	9·957217	9·669002	10·330998	59
2	9·609880	9·960618	9·649990	10·350737	9·626490	9·957158	9·669332	10·330668	58
3	9·610164	9·960561	9·649602	10·350398	9·626760	9·957099	9·669661	10·330339	57
4	9·610447	9·960505	9·649942	10·350058	9·627030	9·957040	9·669991	10·330009	56
5	9·610729	9·960448	9·650281	10·349719	9·627300	9·956981	9·670320	10·329680	55
6	9·611012	9·960392	9·650620	10·349380	9·627570	9·956921	9·670649	10·329351	54
7	9·611294	9·960335	9·650959	10·349041	9·627840	9·956862	9·670977	10·329023	53
8	9·611576	9·960279	9·651297	10·348703	9·628109	9·956803	9·671306	10·328694	52
9	9·611858	9·960222	9·651636	10·348364	9·628378	9·956744	9·671635	10·328365	51
10	9·612140	9·960165	9·651974	10·348026	9·628647	9·956684	9·671963	10·328037	50
11	9·612421	9·960109	9·652312	10·347688	9·628916	9·956625	9·672291	10·327709	49
12	9·612702	9·960052	9·652650	10·347350	9·629185	9·956566	9·672619	10·327381	48
13	9·612983	9·959995	9·652988	10·347012	9·629453	9·956506	9·672947	10·327053	47
14	9·613264	9·959938	9·653326	10·346674	9·629721	9·956447	9·673274	10·326726	46
15	9·613545	9·959882	9·653663	10·346337	9·629989	9·956387	9·673602	10·326398	45
16	9·613825	9·959825	9·654000	10·346000	9·630257	9·956327	9·673929	10·326071	44
17	9·614105	9·959768	9·654337	10·345663	9·630524	9·956268	9·674257	10·325743	43
18	9·614385	9·959711	9·654674	10·345326	9·630792	9·956208	9·674584	10·325416	42
19	9·614665	9·959654	9·655011	10·344989	9·631059	9·956148	9·674911	10·325089	41
20	9·614914	9·959596	9·655348	10·344652	9·631326	9·956089	9·675237	10·324763	40
21	9·615223	9·959539	9·655684	10·344316	9·631593	9·956029	9·675564	10·324436	39
22	9·615502	9·959489	9·656020	10·343980	9·631859	9·955969	9·675890	10·324110	38
23	9·615781	9·959425	9·656356	10·343644	9·632125	9·955909	9·676217	10·323783	37
24	9·616060	9·959368	9·656692	10·343308	9·632392	9·955849	9·676543	10·323457	36
25	9·616338	9·959310	9·657028	10·342972	9·532658	9·955789	9·676869	10·323131	35
26	9·616616	9·959253	9·657364	10·342636	9·632923	9·955729	9·677194	10·322806	34
27	9·616894	9·959195	9·657699	10·342301	9·633189	9·955669	9·677520	10·322480	33
28	9·617172	9·959138	9·658034	10·341966	9·633454	9·955609	9·677846	10·322154	32
29	9·617450	9·959080	9·658369	10·341681	9·633719	9·955548	9·678171	10·321829	31
30	9·617727	9·959023	9·658704	10·341296	9·633984	9·955488	9·678496	10·321504	30
31	9·618004	9·958965	9·659039	10·340961	9·634249	9·955428	9·678821	10·321179	29
32	9·618281	9·958908	9·659373	10·340327	9·634514	9·955368	9·679146	10·320854	28
33	9·618558	9·958850	9·659708	10·340292	9·634778	9·955307	9·679471	10·320529	27
34	9·618834	9·958792	9·660042	10·339958	9·635042	9·955247	9·679795	10·320205	26
35	9·619110	9·958734	9·660376	10·339624	9·635306	9·955186	9·680120	10·319880	25
36	9·619386	9·958677	9·660710	10·339290	9·635570	9·955126	9·680444	10·319556	24
37	9·619662	9·958619	9·661043	10·338957	9·635834	9·955065	9·680768	10·319232	23
38	9·619938	9·958561	9·661377	10·338623	9·636097	9·955005	9·681092	10·318908	22
39	9·620213	9·958503	9·661710	10·338290	9·636360	9·954944	9·681416	10·318584	21
40	9·620488	9·958445	9·662043	10·337957	9·636623	9·954883	9·681740	10·318260	20
41	9·620763	9·958387	9·662376	10·337624	9·636886	9·954823	9·682063	10·317937	19
42	9·621038	9·958329	9·662709	10·337291	9·637148	9·954762	9·682387	10·317613	18
43	9·621313	9·958271	9·663042	10·336958	9·637411	9·954701	9·682710	10·317290	17
44	9·621587	9·958213	9·663375	10·336625	9·637673	9·954640	9·683033	10·316967	16
45	9·621861	9·958154	9·663707	10·336293	9·637935	9·954579	9·683356	10·316644	15
46	9·622135	9·958096	9·664039	10·335961	9·638197	9·954518	9·683679	10·316321	14
47	9·622409	9·958038	9·664371	10·335629	9·638458	9·954457	9·684001	10·315999	13
48	9·622682	9·957979	9·664703	10·335297	9·638720	9·954396	9·684324	10·315676	12
49	9·622956	9·957921	9·665035	10·334965	9·638981	9·954335	9·684646	10·315354	11
50	9·623229	9·957863	9·665366	10·334634	9·639242	9·954274	9·684968	10·315032	10
51	9·623502	9·957804	9·665698	10·334302	9·639503	9·954213	9·685290	10·314710	9
52	9·623774	9·957746	9·666029	10·333971	9·639764	9·954152	9·685612	10·314388	8
53	9·624047	9·957687	9·666360	70·333640	9·640024	9·954090	9·685934	10·314066	7
54	9·624319	9·957628	9·666691	10·333309	9·640284	9·954029	9·686255	10·313745	6
55	9·624591	9·957570	9·667021	10·332979	9·640544	9·953968	9·686577	10·313423	5
56	9·624863	9·957511	9·667352	10·332648	9·640804	9·953906	9·686898	10·313102	4
57	9·625135	9·957452	9·667682	10·332318	9·641064	9·953845	9·687219	10·312781	3
58	9·625406	9·957393	9·668013	10·331987	9·641324	9·953783	9·687540	10·312460	2
59	9·625677	9·957335	9·668343	10·331657	9·641583	9·953722	9·687861	10·312139	1
60	9·625948	9·957276	9·668673	10·331327	9·641842	9·953660	9·688182	10·311818	0

	26 Deg.				27 Deg.				
'	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
0	9.641842	9.953660	9.688182	10.311818	9.657047	9.949881	9.707166	10.292834	60
1	9.642101	9.953599	9.688502	10.311498	9.657295	9.949816	9.707478	10.292522	59
2	9.642360	9.953537	9.688823	10.311177	9.657542	9.949752	9.707790	10.292210	58
3	9.642618	9.953475	9.689143	10.310857	9.657790	9.949688	9.708102	10.291898	57
4	9.642877	9.953413	9.689463	10.310537	9.658037	9.949623	9.708414	10.291586	56
5	9.643135	9.953352	9.689783	10.310217	9.658284	9.949558	9.708726	10.291274	55
6	9.643393	9.953290	9.690103	10.309897	9.658531	9.949494	9.709037	10.290963	54
7	9.643650	9.953228	9.690423	10.309577	9.658778	9.949429	9.709349	10.290651	53
8	9.643908	9.953166	9.690742	10.309258	9.659025	9.949364	9.709660	10.290340	52
9	9.644165	9.953104	9.691062	10.308938	9.659271	9.949300	9.709971	10.290029	51
10	9.644423	9.953042	9.691381	10.308619	9.659517	9.949235	9.710282	10.289718	50
11	9.644680	9.952980	9.691700	10.308300	9.659763	9.949170	9.710593	10.289407	49
12	9.644936	9.952918	9.692019	10.307981	9.660009	9.949105	9.710904	10.289096	48
13	9.645193	9.952855	9.692338	10.307662	9.660255	9.949040	9.711215	10.288785	47
14	9.645450	9.952793	9.692656	10.307344	9.660501	9.948975	9.711525	10.288475	46
15	9.645706	9.952731	9.692975	10.307025	9.660746	9.948910	9.711836	10.288164	45
16	9.645962	9.952669	9.693293	10.306707	9.660991	9.948845	9.712146	10.287854	44
17	9.646218	9.952606	9.693612	10.306388	9.661236	9.948780	9.712456	10.287544	43
18	9.646474	9.952544	9.693930	10.306070	9.661481	9.948715	9.712766	10.287234	42
19	9.646729	9.952481	9.694248	10.305752	9.661726	9.948650	9.713076	10.286924	41
20	9.646984	9.952419	9.694566	10.305434	9.661970	9.948584	9.713386	10.286614	40
21	9.647240	9.952356	9.694883	10.305117	9.662214	9.948519	9.713696	10.286304	39
22	9.647494	9.952294	9.695201	10.304799	9.662459	9.948454	9.714005	10.285995	38
23	9.647749	9.952231	9.695518	10.304482	9.662703	9.948388	9.714314	10.285686	37
24	9.648004	9.952168	9.695836	10.304164	9.662946	9.948323	9.714624	10.285376	36
25	9.648258	9.952106	9.696153	10.303847	9.663190	9.948257	9.714933	10.285067	35
26	9.648512	9.952043	9.696470	10.303530	9.663433	9.948192	9.715242	10.284758	34
27	9.648766	9.951980	9.696787	10.303213	9.663677	9.948126	9.715551	10.284449	33
28	9.649020	9.951917	9.697103	10.302897	9.663920	9.948060	9.715860	10.284140	32
29	9.649274	9.951854	9.697420	10.302580	9.664163	9.947995	9.716168	10.283832	31
30	9.649527	9.951791	9.697736	10.302264	9.664406	9.947929	9.716477	10.283523	30
31	9.649781	9.951728	9.698053	10.301947	9.664648	9.947863	9.716785	10.283215	29
32	9.650034	9.951665	9.698369	10.301631	9.664891	9.947797	9.717093	10.282907	28
33	9.650287	9.951602	9.698685	10.301315	9.665133	9.947731	9.717401	10.282599	27
34	9.650539	9.951539	9.699001	10.300999	9.665375	9.947665	9.717709	10.282291	26
35	9.650792	9.951476	9.699316	10.300684	9.665617	9.947600	9.718017	10.281983	25
36	9.651044	9.951412	9.699632	10.300368	9.665859	9.947533	9.718325	10.281675	24
37	9.651297	9.951349	9.699947	10.300053	9.666100	9.947467	9.718633	10.281367	23
38	9.651549	9.951286	9.700263	10.299737	9.666342	9.947401	9.718940	10.281060	22
39	9.651800	9.951222	9.700578	10.299422	9.666583	9.947335	9.719248	10.280752	21
40	9.652052	9.951159	9.700893	10.299107	9.666824	9.947269	9.719555	10.280445	20
41	9.652304	9.951096	9.701208	10.298792	9.667065	9.947203	9.719862	10.280138	19
42	9.652555	9.951032	9.701523	10.298477	9.667305	9.947136	9.720169	10.279831	18
43	9.652806	9.950968	9.701837	10.298163	9.667546	9.947070	9.720476	10.279524	17
44	9.653057	9.950905	9.702152	10.297848	9.667786	9.947004	9.720783	10.279217	16
45	9.653308	9.950841	9.702466	10.297534	9.668027	9.946937	9.721089	10.278911	15
46	9.653558	9.950778	9.702781	10.297219	9.668267	9.946871	9.721396	10.278604	14
47	9.653808	9.950714	9.703095	10.296905	9.668506	9.946804	9.721702	10.278298	13
48	9.654059	9.950650	9.703409	10.296591	9.668746	9.946738	9.722009	10.277991	12
49	9.654309	9.950586	9.703722	10.296278	9.668986	9.946671	9.722315	10.277685	11
50	9.654558	9.950522	9.704036	10.295964	9.669225	9.946604	9.722621	10.277379	10
51	9.654808	9.950458	9.704350	10.295650	9.669464	9.946538	9.722927	10.277073	9
52	9.655058	9.950394	9.704663	10.295337	9.669703	9.946471	9.723232	10.276768	8
53	9.655307	9.950330	9.704976	10.295024	9.669942	9.946404	9.723538	10.276462	7
54	9.655556	9.950266	9.705290	10.294710	9.670181	9.946337	9.723844	10.276156	6
55	9.655805	9.950202	9.705603	10.294397	9.670419	9.946270	9.724149	10.275851	5
56	9.656054	9.950138	9.705916	10.294084	9.670658	9.946203	9.724454	10.275546	4
57	9.656302	9.950074	9.706228	10.293772	9.670896	9.946136	9.724760	10.275240	3
58	9.656551	9.950010	9.706541	10.293459	9.671134	9.946069	9.725065	10.274935	2
59	9.656799	9.949945	9.706854	10.293146	9.671372	9.946002	9.725370	10.274630	1
60	9.657047	9.949881	9.707166	10.292834	9.671609	9.945935	9.725674	10.274326	0
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.	'

	28 Deg.				29 Deg.				
'	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
0	9.671609	9.945935	9.725674	10.274326	9.685571	9.941819	9.743752	10.256248	60
1	9.671847	9.945868	9.725979	10.274021	9.685799	9.941749	9.744050	10.255950	59
2	9.672084	9.945800	9.726284	10.273716	9.686027	9.941679	9.744348	10.255652	58
3	9.672321	9.945733	9.726588	10.273412	9.686254	9.941609	9.744645	10.255355	57
4	9.672558	9.945666	9.726892	10.273108	9.686482	9.941539	9.744943	10.255057	59
5	9.672795	9.945598	9.727197	10.272803	9.686709	9.941469	9.745240	10.254760	55
6	9.673032	9.945531	9.727501	10.272499	9.686936	9.941398	9.745538	10.254462	54
7	9.673268	9.945464	9.727805	10.272195	9.687163	9.941328	9.745835	10.254165	53
8	9.673505	9.945396	9.728109	10.271891	9.687389	9.941258	9.746132	10.253868	52
9	9.673741	9.945328	9.728412	10.271588	9.687616	9.941187	9.746429	10.253571	51
10	9.673977	9.945261	9.728716	10.271284	9.687843	9.941117	9.746726	10.253274	50
11	9.674213	9.945193	9.729020	10.270980	9.688069	9.941046	9.747023	10.252977	49
12	9.674448	9.945125	9.729323	10.270677	9.688295	9.940975	9.747319	10.252681	48
13	9.674684	9.945058	9.729626	10.270374	9.688521	9.940905	9.747616	10.252384	47
14	9.674919	9.944990	9.729929	10.270071	9.688747	9.940834	9.747913	10.252087	46
15	9.675155	9.944922	9.730233	10.269767	9.688972	9.940763	9.748209	10.251791	45
16	9.675390	9.944854	9.730535	10.269465	9.689198	9.940693	9.748505	10.251495	44
17	9.675624	9.944786	9.730838	10.269162	9.689423	9.940622	9.748801	10.251199	43
18	9.675859	9.944718	9.731141	10.268859	9.689648	9.940551	9.749097	10.250903	42
19	9.676094	9.944650	9.731444	10.268556	9.689873	9.940480	9.749393	10.250607	41
20	9.676328	9.944582	9.731746	10.268254	9.690098	9.940409	9.749689	10.250311	40
21	9.676562	9.944514	9.732048	10.267952	9.690323	9.940338	9.749985	10.250015	39
22	9.676796	9.944446	9.732351	10.267649	9.690548	9.940267	9.750281	10.249719	38
23	9.677030	9.944377	9.732653	10.267347	9.690772	9.940196	9.750576	10.249424	37
24	9.677264	9.944309	9.732955	10.267045	9.690996	9.940125	9.750872	10.249128	36
25	9.677498	9.944241	9.733257	10.266743	9.691220	9.940054	9.751167	10.248833	35
26	9.677731	9.944172	9.733558	10.266442	9.691444	9.939982	9.751462	10.248538	34
27	9.677964	9.944104	9.733860	10.266140	9.691668	9.939911	9.751757	10.248243	33
28	9.678197	9.944036	9.734162	10.265838	9.691892	9.939840	9.752052	10.247948	32
29	9.678430	9.943967	9.734463	10.265537	9.692115	9.939768	9.752347	10.247653	31
30	9.678663	9.943899	9.734764	10.265236	9.692339	9.939697	9.752642	10.247358	30
31	9.678895	9.943830	9.735066	10.264934	9.692562	9.939625	9.752937	10.247063	29
32	9.679128	9.943761	9.735367	10.264633	9.692785	9.939554	9.753231	10.246769	28
33	9.679360	9.943693	9.735668	10.264332	9.693008	9.939482	9.753526	10.246474	27
34	9.679592	9.943624	9.735969	10.264031	9.693231	9.939410	9.753820	10.246180	26
35	9.679824	9.943555	9.736269	10.263731	9.693453	9.939339	9.754115	10.245885	25
36	9.680056	9.943486	9.736570	10.263430	9.693676	9.939267	9.754409	10.245591	24
37	9.680288	9.943417	9.736870	10.263130	9.693898	9.939195	9.754703	10.245297	23
38	9.680519	9.943348	9.737171	10.262829	9.694120	9.939123	9.754997	10.245003	22
39	9.680750	9.943279	9.737471	10.262529	9.694342	9.939052	9.755291	10.244709	21
40	9.680982	9.943210	9.737771	10.262229	9.694564	9.938980	9.755585	10.244415	20
41	9.681213	9.943141	9.738071	10.261929	9.694786	9.938908	9.755878	10.244122	19
42	9.681443	9.943072	9.738371	10.261629	9.695007	9.938836	9.756172	10.243828	18
43	9.681674	9.943003	9.738671	10.261329	9.695229	9.938763	9.756465	10.243535	17
44	9.681905	9.942934	9.738971	10.261029	9.695450	9.938691	9.756759	10.243241	16
45	9.682135	9.942864	9.739271	10.260729	9.695671	9.938619	9.757052	10.242948	15
46	9.682365	9.942795	9.739570	10.260430	9.695892	9.938547	9.757345	10.242655	14
47	9.682595	9.942726	9.739870	10.260130	9.696113	9.938475	9.757638	10.242362	13
48	9.682825	9.942656	9.740169	10.259831	9.696334	9.938402	9.757931	10.242069	12
49	9.683055	9.942587	9.740468	10.259532	9.696554	9.938330	9.758224	10.241776	11
50	9.683284	9.942517	9.740767	10.259233	9.696775	9.938258	9.758517	10.241483	10
51	9.683514	9.942448	9.741066	10.258934	9.696995	9.938185	9.758810	10.241190	9
52	9.683743	9.942378	9.741365	10.258635	9.697215	9.938113	9.759102	10.240898	8
53	9.683972	9.942308	9.741664	10.258336	9.697435	9.938040	9.759395	10.240605	7
54	9.684201	9.942239	9.741962	10.258038	9.697654	9.937967	9.759687	10.240313	6
55	9.684430	9.942169	9.742261	10.257739	9.697874	9.937895	9.759979	10.240021	5
56	9.684658	9.942099	9.742559	10.257441	9.698094	9.937822	9.760272	10.239728	4
57	9.684887	9.942029	9.742858	10.257142	9.698313	9.937749	9.760564	10.239436	3
58	9.685115	9.941959	9.743156	10.256844	9.698532	9.937676	9.760856	10.239144	2
59	9.685343	9.941889	9.743454	10.256546	9.698751	9.937604	9.761148	10.238852	1
60	9.685571	9.941819	9.743752	10.256248	9.698970	9.937531	9.761439	10.238561	0
	Cosine	Sine	Cotang	Tang	Cosine	Sine	Cotang	Tang	'

′	30 Deg. Sine.	Cosine.	Tang.	Cotang.	31 Deg. Sine.	Cosine.	Tang.	Cotang.	
0	9·698970	9·937531	9·761439	10·238561	9·711839	9·933066	9·778774	10·221226	60
1	9·699189	9·937458	9·761731	10·238269	9·712050	9·932990	9·779060	10·220940	59
2	9·699407	9·937385	9·762023	10·237977	9·712260	9·932914	9·779346	10·220654	58
3	9·699626	9·937312	9·762314	10·237686	9·712469	9·932838	9·779632	10·220368	57
4	9·699844	9·937238	9·762606	10·237394	9·712679	9·932762	9·779918	10·220082	59
5	9·700062	9·937165	9·762897	10·237103	9·712889	9·932685	9·780203	10·219797	55
6	9·790280	9·937092	9·763188	10·236812	9·713098	9·932609	9·780489	10·219511	54
7	9·700498	9·937019	9·763479	10·236521	9·713308	9·932533	9·780775	10·219225	53
8	9·700716	9·936946	9·763770	10·236230	9·713517	9·932457	9·781060	10·218940	52
9	9·700933	9·936872	9·764061	10·235939	9·713726	9·932380	9·781346	10·218654	51
10	9·701151	9·936799	9·764352	10·235648	9·713935	9·932304	9·781631	10·218369	50
11	9·701368	9·936725	9·764643	10·235357	9·714144	9·932228	9·781916	10·218084	49
12	9·701585	9·936652	9·764933	10·235067	9·714352	9·932151	9·782201	10·217799	48
13	9·701802	9·936578	9·765224	10·234776	9·714561	9·932075	9·782486	10·217514	47
14	9·702019	9·936505	9·765514	10·234486	9·714769	9·931998	9·782771	10·217229	46
15	9·702236	9·936431	9·765805	10·234195	9·714978	9·931921	9·783056	10·216944	45
16	9·702452	9·936357	9·766095	10·233905	9·715186	9·931845	9·783341	10·216659	44
17	9·702669	9·936284	9·766385	10·233615	9·715394	9·931768	9·783626	10·216374	43
18	9·702885	9·936210	9·766675	10·233325	9·715602	9·931691	9·783910	10·216090	42
19	9·703101	9·936136	9·766965	10·233035	9·715809	9·931614	9·784195	10·215805	41
20	9·703317	9·936062	9·767255	10·232745	9·716017	9·931537	9·784479	10·215521	40
21	9·703533	9·935988	9·767545	10·232455	9·716224	9·931460	9·784764	10·215236	39
22	9·703749	9·935914	9·767834	10·232166	9·716432	9·931383	9·785048	10·214952	38
23	9·703964	9·935840	9·768124	10·231876	9·716639	9·931306	9·785332	10·214668	37
24	9·704179	9·935766	9·768414	10·221586	9·716846	9·931229	9·785616	10·214384	36
25	9·704395	9·935692	9·768703	10·231297	9·717053	9·931152	9·785900	10·214100	35
26	9·704610	9·935618	9·768992	10·231008	9·717259	9·931075	9·786184	10·213816	34
27	9·704825	9·935543	9·769281	10·230719	9·717466	9·930998	9·786468	10·213532	33
28	9·705040	9·935469	9·769571	10·230429	9·717673	9·930921	9·786752	10·213248	32
29	9·705254	9·935395	9·769860	10·230140	9·717879	9·930843	9·787036	10·212964	31
30	9·705469	9·935320	9·770148	10·239852	9·718085	9·939766	9·787319	10·212681	30
31	9·705683	9·935246	9·770437	10·229563	9·718291	9·930688	9·787603	10·212397	29
32	9·705898	9·935171	9·770726	10·229274	9·718497	9·930611	9·787886	10·212114	28
33	9·706112	9·935097	9·771015	10·228985	9·718703	9·930533	9·788170	10·211830	27
34	9·706326	9·935022	9·771303	10·228697	9·718909	9·930456	9·788453	10·211547	26
35	9·706539	9·934948	9·771592	10·228408	9·719114	9·930378	9·788736	10·211264	25
36	9·706753	9·934873	9·771880	10·228120	9·719320	9·930300	9·788019	10·210981	24
37	9·706967	9·934798	9·772168	10·227832	9·711525	9·930223	9·789302	10·210698	23
38	9·707180	9·934723	9·772457	10·227543	9·711730	9·930145	9·789585	10·210415	22
39	9·707393	9·934649	9·772745	10·227255	9·711935	9·930067	9·789868	10·210132	21
40	9·707606	9·934574	9·773033	10·226967	9·720140	9·929989	9·790151	10·209849	20
41	9·707819	9·934499	9·773321	10·226679	9·720345	9·929911	9·790434	10·209566	19
42	9·708032	9·934424	9·773608	10·226392	9·720549	9·929833	9·790716	10·209284	18
43	9·708245	9·934349	9·773896	10·226104	9·720754	9·929755	9·790999	10·209001	17
44	9·708458	9·934274	9·774184	10·225816	9·720958	9·929677	9·791281	10·208719	16
45	9·708670	9·934199	9·774471	10·225529	9·721162	9·929599	9·791563	10·208437	15
46	9·708882	9·934123	9·774759	10·225241	9·721366	9·929521	9·791846	10·208154	14
47	9·709094	9·934048	9·775046	10·224954	9·721570	9·929442	9·792128	10·207872	13
48	9·709306	9·933973	9·775333	10·224667	9·721774	9·929364	9·792410	10·207590	12
49	9·709518	9·933898	9·775621	10·224379	9·721978	9·929286	9·792692	10·207308	11
50	9·709730	9·933822	9·775908	10·224092	9·722181	9·929207	9·792974	10·207026	10
51	9·709941	9·933747	9·776195	10·223805	9·722385	9·929129	9·793256	10·206744	9
52	9·710153	9·933671	9·776482	10·223518	9·722588	9·929050	9·793538	10·206462	8
53	9·710364	9·933596	9·776769	10·223232	9·722791	9·928972	9·793819	10·206181	7
54	9·710575	9·933520	9·777055	10·222945	9·722994	9·928853	9·794101	10·205899	6
55	9·710786	9·933445	9·777342	10·222658	9·723197	9·928815	9·794383	10·205617	5
56	9·710997	9·933369	9·777628	10·222372	9·723400	9·928736	9·794664	10·205336	4
57	9·711208	9·933294	9·777915	10·222085	9·723603	9·928657	9·794946	10·205054	3
58	9·711419	9·933217	9·778201	10·221799	9·723805	9·928578	9·795227	10·204773	2
59	9·711629	9·933141	9·778488	10·221512	9·724007	9·928499	9·795508	10·204492	1
60	9·711839	9·933066	9·778774	10·221226	9·724210	9·928420	9·795789	10·204211	0
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.	′

	32 Deg.				33 Deg.				
′	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
0	9.724210	9.928420	9.795789	10.204211	9.736109	9.923591	9.812517	10.187483	60
1	9.724412	9.928342	9.796070	10.203930	9.736303	9.923509	9.812794	10.187206	59
2	9.724614	9.928263	9.796351	10.203649	9.736498	9.923427	9.813070	10.186930	58
3	9.724816	9.928183	9.796632	10.203368	9.736692	9.923345	9.813347	10.186653	57
4	9.725017	9.928104	9.796913	10.203087	9.736886	9.923263	9.813623	10.186377	56
5	9.725219	9.928025	9.797194	10.202806	9.737080	9.923181	9.813899	10.186101	55
6	9.725420	9.927946	9.797474	10.202526	9.737274	9.923098	9.814176	10.185824	54
7	9.725622	9.927867	9.797755	10.202245	9.737467	9.923016	9.814452	10.185548	53
8	9.72[illegible]	9.927787	9.798036	10.201964	9.737661	9.922933	9.814728	10.185272	52
9	9.726024	9.927708	9.798316	10.201684	9.737855	9.922851	9.815004	10.184996	51
10	9.726225	9.927629	9.798596	10.201404	9.738048	9.922768	9.815280	10.184720	50
11	9.726426	9.927549	9.798877	10.201123	9.738241	9.922686	9.815555	10.184445	49
12	9.726626	9.927470	9.799157	10.200843	9.738434	9.922603	9.815831	10.184169	48
13	9.726827	9.927390	9.799437	10.200563	9.738627	9.922520	9.816107	10.183893	47
14	9.727027	9.927310	9.799717	10.200283	9.738820	9.922438	9.816382	10.183618	46
15	9.727228	9.927231	9.799997	10.200003	9.739013	9.922355	9.816658	10.183342	45
16	9.727428	9.927151	9.800277	10.199723	9.739206	9.922272	9.816933	10.183067	44
17	9.727628	9.927071	9.800557	10.199445	9.739398	9.922189	9.817209	10.182791	43
18	9.727828	9.926991	9.800836	10.199164	9.739590	9.922106	9.817484	10.182516	42
19	9.728027	9.926911	9.801116	10.198884	9.739783	9.922023	9.817759	10.182241	41
20	9.728227	9.926831	9.801396	10.198604	9.739975	9.921940	9.818035	10.181965	40
21	9.728427	9.926751	9.801675	10.198325	9.740167	9.921857	9.818310	10.181690	39
22	9.728626	9.926671	9.801955	10.198045	9.740359	9.921774	9.818585	10.181415	38
23	9.728825	9.926591	9.802234	10.197766	9.740550	9.921691	9.818860	10.181140	37
24	9.729024	9.926511	9.802513	10.197487	9.740742	9.921607	9.819135	10.180865	36
25	9.729223	9.926431	9.802792	10.197208	9.740934	9.921524	9.819410	10.180590	35
26	9.729422	9.926351	9.803072	10.196928	9.741125	9.921441	9.819684	10.180316	34
27	9.729621	9.926270	9.803351	10.196649	9.741316	9.921357	9.819959	10.180041	33
28	9.729820	9.926190	9.803630	10.196370	9.741508	9.921274	9.820234	10.179766	32
29	9.730018	9.926110	9.803909	10.196091	9.741699	9.921190	9.820508	10.179492	31
30	9.730217	9.926029	9.804187	10.195813	9.741889	9.921107	9.820783	10.179217	30
31	9.730415	9.925949	9.804466	10.195534	9.742080	9.921023	9.821057	10.178943	29
32	9.730613	9.925868	9.804745	10.195255	9.742271	9.920939	9.821332	10.178668	28
33	9.730811	9.925788	9.805023	10.194977	9.742462	9.920856	9.821606	10.178394	27
34	9.731009	9.925707	9.805302	10.194698	9.742652	9.920772	9.821880	10.178120	26
35	9.731206	9.925626	9.805580	10.194420	9.742842	9.920688	9.822154	10.177846	25
36	9.731404	9.925545	9.805859	10.194141	9.743033	9.920604	9.822429	10.177571	24
37	9.731602	9.925465	9.806137	10.193863	9.743223	9.920520	9.822703	10.177297	23
38	9.731799	9.925384	9.806415	10.193585	9.743413	9.920436	9.822977	10.177023	22
39	9.731996	9.925303	9.806693	10.193307	9.743602	9.920352	9.823251	10.176749	21
40	9.732193	9.925222	9.806971	10.193029	9.743792	9.920268	9.823524	10.176476	20
41	9.732390	9.925141	9.807249	10.192751	9.743982	9.920184	9.823798	10.1[illegible]202	19
42	9.732587	9.925060	9.807521	10.192473	9.744171	9.920099	9.824072	10.175928	18
43	9.732784	9.924979	9.807805	10.192195	9.744361	9.920015	9.824345	10.175655	17
44	9.732980	9.924897	9.808083	10.191917	9.744550	9.919931	9.824619	10.175381	16
45	9.733177	9.924816	9.808361	10.191639	9.744739	9.919846	9.824893	10.175107	15
46	9.733373	9.924735	9.808638	10.191362	9.744928	9.919762	9.825166	10.174834	14
47	9.733569	9.924654	9.808916	10.191084	9.745117	9.919677	9.825439	10.174561	13
48	9.733765	9.924572	9.809193	10.190807	9.745306	9.919593	9.825713	10.174287	12
49	9.733961	9.924491	9.809471	10.190529	9.745494	9.919508	9.825986	10.174014	11
50	9.734157	9.924409	9.809748	10.190252	9.745683	9.919424	9.826259	10.173741	10
51	9.734353	9.924328	9.810025	10.189975	9.745871	9.919339	9.826532	10.173468	9
52	9.734549	9.924246	9.810302	10.189698	9.746060	9.919254	9.826805	10.173195	8
53	9.734744	9.924164	9.810580	10.189420	9.746248	9.919169	9.827078	10.172922	7
54	9.734939	9.924083	9.810857	10.189143	9.746436	9.919085	9.827351	10.172649	6
55	9.735135	9.924001	9.811134	10.188866	9.746624	9.919000	9.827624	10.172376	5
56	9.735330	9.923919	9.811410	10.188590	9.746812	9.918915	9.827897	10.172103	4
57	9.735525	9.923837	9.811687	10.188313	9.746999	9.918830	9.828170	10.171830	3
58	9.735719	9.923755	9.811964	10.188036	9.747187	9.918745	9.828442	10.171558	2
59	9.735914	9.923673	9.812241	10.187759	9.747374	9.918659	9.828715	10.171285	1
60	9.736109	9.923591	9.812517	10.187483	9.747562	9.918574	9.828987	10.171013	0
	Cosine.	Sine.	Cotan.	Tang.					

	34 Deg.				35 Deg.				
′	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
0	9·747562	9·918574	9·828987	10·171013	9·758591	9·913365	9·845227	10·154773	6
1	9·747749	9·918489	9·829260	10·170740	9·758772	9·913279	9·845496	10·154504	5
2	9·747936	9·918404	9·829532	10·170468	9·758952	9·913187	9·845764	10·154236	5
3	9·748123	9·918318	9·829805	10·170195	9·759132	9·913099	9·846033	10·153967	5
4	9·748310	9·918233	9·830077	10·169923	9·759312	9·913010	9·846302	10·153698	5
5	9·748497	9·918147	9·830349	10·169651	9·759492	9·912922	9·846570	10·153430	5
6	9·748683	9·918062	9·830621	10·169379	9·759672	9·912833	9·846839	10·153161	5
7	9·748870	9·917976	9·830893	10·169107	9·759852	9·912744	9·847108	10·152892	5
8	9·749056	9·917891	9·831165	10·168835	9·760031	9·912655	9·847376	10·152624	5
9	9·749243	9·917805	9·831437	10·168563	9·760211	9·912566	9·847644	10·152356	5
10	9·749429	9·917719	9·831709	10·168291	9·760390	9·912477	9·847913	10·152087	5
11	9·749615	9·917634	9·831981	10·168019	9·760569	9·912388	9·848181	10·151819	4
12	9·749801	9·917548	9·832253	10·167747	9·760748	9·912299	9·848449	10·151551	4
13	9·749987	9·917462	9·832525	10·167475	9·760927	9·912210	9·848717	10·151283	4
14	9·750172	9·917376	9·832796	10·167204	9·761106	9·912121	9·848986	10·151014	4
15	9·750358	9·917290	9·833068	10·166932	9·761285	9·912031	9·849254	10·150746	4
16	9·750543	9·917204	9·833339	10·166661	9·761464	9·911942	9·849522	10·150478	4
17	9·750729	9·917118	9·833611	10·166389	9·761642	9·911853	9·849790	10·150210	4
18	9·750914	9·917032	9·833882	10·166118	9·761821	9·911763	9·850057	10·149943	4
19	9·751099	9·916946	9·834154	10·165846	9·761999	9·911674	9·850325	10·149675	4
20	9·751284	9·916859	9·834425	10·165575	9·762177	9·911584	9·850593	10·149407	4
21	9·751469	9·916773	9·834696	10·165304	9·762356	9·911495	9·850861	10·149139	3
22	9·751654	9·916687	9·834967	10·165033	9·762534	9·911405	9·851129	10·148871	3
23	9·751839	9·916600	9·835238	10·164762	9·762712	9·911315	9·851396	10·148604	3
24	9·752023	9·916514	9·835509	10·164491	9·762889	9·911226	9·851664	10·148336	3
25	9·752208	9·916427	9·835780	10·164220	9·763067	9·911136	9·851931	10·148069	3
26	9·752392	9·916341	9·836051	10·163949	9·763245	9·911046	9·852199	10·147801	3
27	9·752576	9·916254	9·836322	10·163678	9·763422	9·910956	9·852466	10·147534	3
28	9·752760	9·916167	9·836593	10·163407	9·763600	9·910866	9·852733	10·147267	3
29	9·752944	9·916081	9·836864	10·163136	9·763777	9·910776	9·853001	10·146999	3
30	9·753128	9·915994	9·837134	10·162866	9·763954	9·910686	9·853268	10·146732	3
31	9·753312	9·915907	9·837405	10·162595	9·764131	9·910596	9·853535	10·146465	2
32	9·753495	9·915820	9·837675	10·162325	9·764308	9·910506	9·853802	10·146198	2
33	9·753679	9·915733	9·837946	10·162054	9·764485	9·910415	9·854069	10·145931	2
34	9·753862	9·915646	9·838210	10·161784	9·764662	9·910325	9·854336	10·145664	2
35	9·754046	9·915559	9·838487	10·161513	9·764838	9·910235	9·854603	10·145397	2
36	9·754229	9·915472	9·838757	10·161243	9·765015	9·910144	9·854870	10·145130	2
37	9·754412	9·915385	9·839027	10·160973	9·765191	9·910054	9·855137	10·144863	2
38	9·754595	9·915297	9·839297	10·160703	9·765367	9·909963	9·855404	10·144596	2
39	9·754778	9·915210	9·839568	10·160432	9·765544	9·909873	9·855671	10·144329	2
40	9·754960	9·915123	9·839838	10·160162	9·765720	9·909782	9·855938	10·144062	2
41	9·755143	9·915035	9·840108	10·159892	9·765896	9·909691	9·856204	10·143796	1
42	9·755326	9·914948	9·840378	10·159622	9·766072	9·909601	9·856471	10·143529	1
43	9·755508	9·914860	9·840648	10·159352	9·766247	9·909510	9·856737	10·143263	1
44	9·755690	9·914773	9·840917	10·159083	9·766423	9·909419	9·857004	10·142996	1
45	9·755872	9·914685	9·841187	10·158813	9·766598	9·909328	9·857270	10·142730	1
46	9·756054	9·914598	9·841457	10·158543	9·766774	9·909237	9·857537	10·142463	1
47	9·756236	9·914510	9·841727	10·158273	9·766949	9·909146	9·857803	10·142197	1
48	9·756418	9·914422	9·841996	10·158004	9·767124	9·909055	9·858069	10·141931	1
49	9·756600	9·914334	9·842266	10·157734	9·767300	9·908964	9·858336	10·141664	1
50	9·756782	9·914246	9·842535	10·157465	9·767475	9·908873	9·858602	10·141398	1
51	9·756963	9·914158	9·842805	10·157195	9·767649	9·908781	9·858868	10·141132	
52	9·757144	9·914070	9·843074	10·156926	9·767824	9·908690	9·859134	10·140866	
53	9·757326	9·913982	9·843343	10·156657	9·767999	9·908599	9·859400	10·140600	
54	9·757507	9·913894	9·843612	10·156388	9·768173	6·908507	9·859666	10·143340	
55	9·757688	9·913806	9·843882	10·156118	9·768348	9·908416	9·859932	10·140068	
56	9·757869	9·913718	9·844151	10·155849	9·768522	9·908324	9·860198	10·139802	
57	9·758050	9·913630	9·844420	10·155580	9·768697	9·908233	9·860464	10·139536	
58	9·758230	9·913541	9·844689	10·155311	9·768871	9·908141	9·860730	10·139270	
59	9·758411	9·913453	9·844958	10·155042	9·769045	9·908049	9·860995	10·139005	
60	9·758591	9·913365	9·845227	10·154773	9·769219	9·907958	9·861261	10·138739	
	Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotan.	Tang.	

	36 Deg.				37 Deg.				
′	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
0	9.769219	9.907958	9.861261	10.138739	9.779463	9.902349	9.877114	10.122886	60
1	9.769393	9.907866	9.861527	10.138473	9.779631	9.902253	9.877377	10.122623	59
2	9.769566	9.907774	9.861792	10.138208	9.779798	9.902158	9.877640	10.122360	58
3	9.769740	9.907682	9.862058	10.137942	9.779966	9.902063	9.877903	10.122097	57
4	9.769913	9.907590	9.862323	10.137677	9.780133	9.901967	9.878165	10.121835	56
5	9.770087	9.907498	9.862589	10.137411	9.780300	9.901872	9.878428	10.121572	55
6	9.770260	9.907406	9.862854	10.137146	9.780467	9.901776	9.878691	10.121309	54
7	9.770433	9.907314	9.863119	10.136881	9.780634	9.901681	9.878953	10.121047	53
8	9.770606	9.907222	9.863385	10.136615	9.780801	9.901585	9.879216	10.120784	52
9	9.770779	9.907129	9.863650	10.136350	9.780968	9.901490	9.879478	10.120522	51
10	9.770952	9.907037	9.863915	10.136085	9.781134	9.901394	9.879741	10.120259	50
11	9.771125	9.906945	9.864180	10.135820	9.781301	9.901298	9.880003	10.119997	49
12	9.771298	9.906852	9.864445	10.135555	9.781468	9.901202	9.880265	10.119735	48
13	9.771470	9.906760	9.864710	10.135290	9.781634	9.901106	9.880528	10.119472	47
14	9.771643	9.906667	9.864975	10.135025	9.781800	9.901010	9.880790	10.119210	46
15	9.771815	9.906575	9.865240	10.134760	9.781966	9.900914	9.881052	10.118948	45
16	9.771987	9.906482	9.865505	10.134495	9.782132	9.900818	9.881314	10.118686	44
17	9.772159	9.906389	9.865770	10.134230	9.782298	9.900722	9.881577	10.118423	43
18	9.772331	9.906296	9.866035	10.133965	9.782464	9.900626	9.881839	10.118161	42
19	9.772503	9.906204	9.866300	10.133700	9.782630	9.900529	9.882101	10.117899	41
20	9.772675	9.906111	9.866564	10.133436	9.782796	9.900433	9.882363	10.117637	40
21	9.772847	9.906018	9.866829	10.133171	9.782961	9.900337	9.882625	10.117375	39
22	9.773018	9.905925	9.867094	10.132906	9.783127	9.900240	9.882887	10.117113	38
23	9.773190	9.905832	9.867358	10.132642	9.783292	9.900144	9.883148	10.116852	37
24	9.773361	9.905739	9.867623	10.132377	9.783458	9.900047	9.883410	10.116590	36
25	9.773533	9.905645	9.867887	10.132113	9.783623	9.899951	9.883672	10.116328	35
26	9.773704	9.905552	9.868152	10.131848	9.783788	9.899854	9.883934	10.116066	34
27	9.773875	9.905459	9.868416	10.131584	9.783953	9.899757	9.884196	10.115804	33
28	9.774046	9.905366	9.868680	10.131320	9.784118	9.899660	9.884457	10.115543	32
29	9.774217	9.905272	9.868945	10.131055	9.784282	9.899564	9.884719	10.115281	31
30	9.774388	9.905179	9.869209	10.130791	9.784447	9.899467	9.884980	10.115020	30
31	9.774558	9.905085	9.869473	10.130527	9.784612	9.899370	9.885242	10.114758	29
32	9.774729	9.904992	9.869737	10.130263	9.784777	9.899273	9.885504	10.114496	28
33	9.774899	9.904898	9.870001	10.129999	9.784941	9.899176	9.885765	10.114235	27
34	9.775070	9.904804	9.870265	10.129735	9.785105	9.899078	9.886026	10.113974	26
35	9.775240	9.904711	9.870529	10.129471	9.785269	9.898981	9.886288	10.113712	25
36	9.775410	9.904617	9.870793	10.129207	9.785433	9.898884	9.886549	10.113451	24
37	9.775580	9.904523	9.871057	10.128943	9.785597	9.898787	9.886811	10.113189	23
38	9.775759	9.904429	9.871321	10.128679	9.785761	9.898689	9.887072	10.112928	22
39	9.775920	9.904335	9.871585	10.128415	9.785925	9.898592	9.887333	10.112067	21
40	9.776090	9.904241	9.871849	10.128151	9.786089	9.898494	9.887594	10.112406	20
41	9.776259	9.904147	9.872112	10.127888	9.786252	9.898397	9.887855	10.112145	19
42	9.776429	9.904053	9.872376	10.127624	9.786416	9.898299	9.888116	10.111884	18
43	9.776598	9.903959	9.872640	10.127360	9.786579	9.898202	9.888378	10.111622	17
44	9.776768	9.903864	9.872903	10.127097	9.786742	9.898104	9.888639	10.111361	16
45	9.776937	9.903770	9.873167	10.126833	9.786906	9.898006	9.888900	10.111100	15
46	9.777106	9.903676	9.873430	10.126570	9.787069	9.897908	9.889161	10.110839	14
47	9.777275	9.903581	9.873694	10.126306	9.787232	9.897810	9.889421	10.110579	13
48	9.777444	9.903487	9.873957	10.126043	9.787395	9.897712	9.889682	10.110318	12
49	9.777613	9.903392	9.874220	10.125780	9.787557	9.897614	9.889943	10.110057	11
50	9.777781	9.903398	9.874484	10.125516	9.787720	9.897516	9.890204	10.109796	10
51	9.777950	9.903203	9.874747	10.125253	9.787883	9.897418	9.890465	10.109535	9
52	9.778119	9.903108	9.875010	10.124990	9.788045	9.897320	9.890725	10.109275	8
53	9.778287	9.903014	9.875273	10.124727	9.788208	9.897222	9.890986	10.109014	7
54	9.778455	9.902919	9.875537	10.124463	9.788370	9.897123	9.891247	10.108753	6
55	9.778624	9.902824	9.875800	10.124200	9.788532	9.897025	9.891507	10.108493	5
56	9.778792	9.902729	9.876063	10.123937	9.788694	9.896926	9.891768	10.108232	4
57	9.778960	9.902634	9.876326	10.123674	9.788856	9.896828	9.892028	10.107972	3
58	9.779128	9.902539	9.876589	10.123411	9.789018	9.896729	9.892289	10.107711	2
59	9.779295	9.902444	9.876852	10.123148	9.789180	9.896631	9.892549	10.107451	1
60	9.779463	9.902349	9.877114	10.122886	9.789342	9.896532	9.892810	10.107190	0
	Cosine.	Sine.	Cotan.	Tang.					

	38 Deg.				39 Deg.				
′	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
0	9.789342	9.896532	9.892810	10.107190	9.798872	9.890503	9.908369	10.091631	60
1	9.789504	9.896433	9.893070	10.106930	9.799028	9.890400	9.908628	10.091372	59
2	9.789665	9.896335	9.893331	10.106669	9.799184	9.890298	9.908886	10.091114	58
3	9.789827	9.896236	9.893591	10.106409	9.799339	9.890195	9.909144	10.090856	57
4	9.789988	9.896137	9.893851	10.106149	9.799495	9.890093	9.909402	10.090598	56
5	9.790149	9.896038	9.894111	10.105889	9.799651	9.889990	9.909660	10.090340	55
6	9.790310	9.895939	9.894372	10.105628	9.799806	9.889888	9.909918	10.090082	54
7	9.790471	9.895840	9.894632	10.105368	9.799962	9.889785	9.910177	10.089823	53
8	9.790632	9.895741	9.894891	10.105108	9.800117	9.889682	9.910435	10.089565	52
9	9.790793	9.895641	9.895152	10.104848	9.800222	9.889579	9.910693	10.089307	51
10	9.790954	9.895542	9.895412	10.104588	9.800477	9.889477	9.910951	10.089049	50
11	9.791115	9.895443	9.895672	10.104328	9.800582	9.889374	9.911209	10.088791	49
12	9.791275	9.895343	9.895932	10.104068	9.800737	9.889271	9.911467	10.088533	48
13	9.791436	9.895244	9.896192	10.103808	9.800892	9.889168	9.911725	10.088275	47
14	9.791596	9.895145	9.896452	10.103548	9.801047	9.889064	9.911982	10.088018	46
15	9.791757	9.895045	9.896712	10.103288	9.801201	9.888961	9.912240	10.087760	45
16	9.791917	9.894945	9.896971	10.103029	9.801356	9.888858	9.912498	10.087502	44
17	9.792077	9.894846	9.897231	10.102769	9.801511	9.888755	9.912756	10.087244	43
18	9.792237	9.894746	9.897491	10.102509	9.801665	9.888651	9.913014	10.086986	42
19	9.792397	9.894646	9.897751	10.102249	9.801819	9.888548	9.913271	10.086729	41
20	9.792557	9.894546	9.898010	10.101990	9.801973	9.888444	9.913529	10.086471	40
21	9.792716	9.894446	9.898270	10.101730	9.802128	9.888341	9.913787	10.086213	39
22	9.792876	9.894346	9.898530	10.101470	9.802282	9.888237	9.914044	10.085956	38
23	9.793035	9.894246	9.898789	10.101211	9.802436	9.888134	9.914302	10.085698	37
24	9.793195	9.894146	9.899049	10.100951	9.802589	9.888030	9.914560	10.085440	36
25	9.793354	9.894046	9.899308	10.100692	9.802743	9.887926	9.914817	10.085183	35
26	9.793514	9.893946	9.899568	10.100432	9.802897	9.887822	9.915075	10.084925	34
27	9.793673	9.893846	9.899827	10.100173	9.803050	9.887718	9.915332	10.084668	33
28	9.793832	9.893745	9.900087	10.099913	9.803204	9.887614	9.915590	10.084410	32
29	9.793991	9.893645	9.900346	10.099654	9.803357	9.887510	9.915847	10.084153	31
30	9.794150	9.893544	9.900605	10.099395	9.803511	9.887406	9.916104	10.083896	30
31	9.794308	9.893444	9.900864	10.099136	9.803664	9.887302	9.916362	10.083638	29
32	9.794467	9.893343	9.901124	10.098876	9.803817	9.887198	9.916619	10.083381	28
33	9.794626	9.893243	9.901383	10.098617	9.803970	9.887093	9.916877	10.083123	27
34	9.794784	9.893142	9.901642	10.098358	9.804123	9.886989	9.917134	10.082866	26
35	9.794942	9.893041	9.901901	10.098099	9.804276	9.886885	9.917391	10.082609	25
36	9.795101	9.892940	9.902160	10.097840	9.804428	9.886780	9.917648	10.082352	24
37	9.795259	9.892839	9.902420	10.097580	9.804581	9.886676	9.917906	10.082094	23
38	9.795417	9.892739	9.902679	10.097321	9.804734	9.886571	9.918163	10.081837	22
39	9.795575	9.892638	9.902938	10.097062	9.804886	9.886466	9.918420	10.081580	21
40	9.795733	9.892536	9.903197	10.096803	9.805039	9.886362	9.918677	10.081323	20
41	9.795891	9.892432	9.903456	10.096544	9.805191	9.886257	9.918934	10.081066	19
42	9.796049	9.892334	9.903714	10.096286	9.805343	9.886152	9.919191	10.080809	18
43	9.796206	9.892233	9.903973	10.096027	9.805495	9.886047	9.919448	10.080552	17
44	9.796364	9.892132	9.904232	10.095768	9.805647	9.885942	9.919705	10.080295	16
45	9.796521	9.892030	9.904491	10.095509	9.805799	9.885837	9.919962	10.080038	15
46	9.796679	9.891929	9.904750	10.095250	9.805951	9.885732	9.920219	10.079781	14
47	9.796836	9.891827	9.905008	10.094992	9.806103	9.885627	9.920476	10.079524	13
48	9.796993	9.891726	9.905267	10.094733	9.806253	9.885522	9.920733	10.079267	12
49	9.797150	9.891624	9.905526	10.094474	9.806406	9.885416	9.920990	10.079010	11
50	9.797307	9.891523	9.905785	10.094215	9.806557	9.885311	9.921247	10.078753	10
51	9.797464	9.891421	9.906043	10.093957	9.806709	9.885205	9.921503	10.078497	9
52	9.797621	9.891319	9.906302	10.093698	9.806860	9.885100	9.921760	10.078240	8
53	9.797777	9.891217	9.906560	10.093440	9.807111	9.884994	9.922017	10.077983	7
54	9.797934	9.891115	9.906819	10.093181	9.807163	9.884889	9.922274	10.077726	6
55	9.798091	9.891013	9.907077	10.092923	9.807314	9.884783	9.922530	10.077470	5
56	9.798247	9.890911	9.907336	10.092664	9.807465	9.884677	9.922784	10.077213	4
57	9.798403	9.890809	9.907594	10.092406	9.807615	9.884572	9.923034	10.076956	3
58	9.798560	9.890707	9.907853	10.092147	9.807766	9.884466	9.923300	10.076700	2
59	9.798716	9.890605	9.908111	10.091889	9.807917	9.884360	9.923557	10.076443	1
60	9.798872	9.890503	9.908369	10.091631	9.808067	9.884254	9.923814	10.076186	
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.	

	40 Deg.				41 Deg.				
′	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	′
0	9.808067	9.884254	9.923814	10.076186	9.816943	9.877780	9.939163	10.060837	60
1	9.808218	9.884148	9.924070	10.075930	9.817088	9.877670	9.939418	10.060582	59
2	9.808368	9.884042	9.924327	10.075673	9.817233	9.877560	9.939673	10.060327	58
3	9.808519	9.883936	9.924583	10.075417	9.817379	9.877450	9.939928	10.060072	57
4	9.808669	9.883829	9.924840	10.075160	9.817524	9.877340	9.940183	10.059817	56
5	9.808819	9.883723	9.925096	10.074904	9.817668	9.877230	9.940439	10.059561	55
6	9.808969	9.883617	9.925352	10.074648	9.817813	9.877120	9.940694	10.059306	54
7	9.809119	9.883510	9.925609	10.074391	9.817958	9.877010	9.940949	10.059051	53
8	9.809269	9.883404	9.925865	10.074135	9.818103	9.876899	9.941204	10.058796	52
9	9.809419	9.883297	9.926122	10.073878	9.818247	9.876789	9.941459	10.058541	51
10	9.809569	9.883191	9.926378	10.073622	9.818392	9.876678	9.941713	10.058287	50
11	9.809718	9.883084	9.926634	10.073366	9.818536	9.876568	9.941968	10.058032	49
12	9.809868	9.882977	9.926890	10.073110	9.818681	9.876457	9.942223	10.057777	48
13	9.810017	9.882871	9.927147	10.072853	9.818825	9.876347	9.942478	10.057522	47
14	9.810167	9.881764	9.927403	10.072597	9.818969	9.876236	9.942733	10.057267	46
15	9.810316	9.882657	9.927659	10.072341	9.819113	9.876125	9.942988	10.057012	45
16	9.810465	9.882550	9.927915	10.072085	9.819257	9.876014	9.943243	10.056757	44
17	9.810614	9.882443	9.928171	10.071829	9.819401	9.875904	9.943498	10.056502	43
18	9.810763	9.882336	9.928427	10.071573	9.819545	9.875793	9.943752	10.056248	42
19	9.810912	9.882229	9.928684	10.071316	9.819689	9.875682	9.944007	10.055993	41
20	9.811061	9.882121	9.928940	10.071060	9.819832	9.875571	9.944262	10.055738	40
21	9.811210	9.882014	9.929196	10.070804	9.819976	9.875459	9.944517	10.055483	39
22	9.811358	9.881907	9.929452	10.070548	9.820120	9.875348	9.944771	10.055229	38
23	9.811507	9.881799	9.929708	10.070292	9.820263	9.875237	9.945026	10.054974	37
24	9.811655	9.881692	9.929964	10.070036	9.820406	9.875126	9.945281	10.054719	36
25	9.811804	9.881554	9.930220	10.069780	9.820550	9.875014	9.945535	10.054465	35
26	9.811952	9.881477	9.930475	10.069525	9.820693	9.874903	9.945790	10.054210	34
27	9.812100	9.881369	9.930731	10.069269	9.820836	9.874791	9.946044	10.053955	33
28	9.812248	9.881261	9.930987	10.069013	9.820979	9.874680	9.946299	10.053701	32
29	9.812396	9.881153	9.931243	10.068757	9.821122	9.874568	9.946554	10.053446	31
30	9.812544	9.881046	9.931499	10.068501	9.821265	9.874456	9.946808	10.053192	30
31	9.812692	9.880938	9.931755	10.068245	9.821407	9.874344	9.947063	10.052937	29
32	9.812840	9.880830	9.932010	10.067990	9.821550	9.874232	9.947318	10.052682	28
33	9.812988	9.880722	9.932266	10.067734	9.821693	9.874121	9.947572	10.052428	27
34	9.813135	9.880613	9.932522	10.067478	9.821835	9.874009	9.947827	10.052173	26
35	9.813283	9.880505	9.932778	10.067222	9.821977	9.873896	9.948081	10.051919	25
36	9.813430	9.880397	9.933033	10.066967	9.822120	9.873784	9.948335	10.051665	24
37	9.813578	9.880289	9.933289	10.066711	9.822262	9.873672	9.948590	10.051410	23
38	9.813725	9.880180	9.933545	10.066455	9.822404	9.873560	9.948844	10.051156	22
39	9.813872	9.880072	9.933800	10.066200	9.822546	9.873448	9.949099	10.050901	21
40	9.814019	9.879963	9.934056	10.065944	9.822688	9.873335	9.949353	10.050647	20
41	9.814166	9.879855	9.934311	10.065689	9.822830	9.873223	9.949608	10.050392	19
42	9.814313	9.879746	9.934567	10.065433	9.822972	9.873110	9.949862	10.050138	18
43	9.814460	9.879637	9.934822	10.065178	9.823114	9.872998	9.950116	10.049884	17
44	9.814607	9.879529	9.935078	10.064922	9.823255	9.872885	9.950371	10.049629	16
45	9.814753	9.879420	9.935333	10.064667	9.823397	9.872772	9.950625	10.049375	15
46	9.814900	9.879311	9.935589	10.064411	9.823539	9.872659	9.950879	10.049121	14
47	9.815046	9.879202	9.935844	10.064156	9.823680	9.872547	9.951133	10.048867	13
48	9.815193	9.879093	9.936100	10.063900	9.823821	9.872434	9.951388	10.048612	12
49	9.815339	9.878984	9.936355	10.063645	9.823963	9.872321	9.951642	10.048358	11
50	9.815485	9.878875	9.936611	10.063389	9.824104	9.872208	9.951896	10.048104	10
51	9.815632	9.878766	9.936866	10.063134	9.824245	9.872095	9.952150	10.047850	9
52	9.815778	9.878656	9.937121	10.062879	9.824386	9.871981	9.952405	10.047595	8
53	9.815924	9.878547	9.937377	70.062623	9.824527	9.871868	9.952659	10.047341	7
54	9.816069	9.878433	9.937632	10.062368	9.824668	9.871755	9.952913	10.047087	6
55	9.816215	9.878328	9.937887	10.062113	9.824808	9.871641	9.953167	10.046833	5
56	9.816361	9.878219	9.938142	10.061858	9.824949	9.871528	9.953421	10.046579	4
57	9.816507	9.878109	9.938398	10.061602	9.825090	9.871414	9.953675	10.046325	3
58	9.816652	9.877999	9.938653	10.061347	9.825230	9.871301	9.953929	10.046071	2
59	9.816798	9.877890	9.938908	10.061092	9.825371	9.871187	9.954183	10.045817	1
60	9.816943	9.877780	9.939163	10.060837	9.425511	9.871073	9.954437	10.045563	0

	42 Deg.				43 Deg.				
'	Sine.	Cosine.	Tang.	Cotang.	Sine	Cosine.	Tang.	Cotang.	
0	9.825511	9.871073	9.954437	10.045563	9.833783	9.864127	9.969656	10.030344	60
1	9.825651	9.870960	9.954691	10.045309	9.833919	9.864010	9.969909	10.030091	59
2	9.825791	9.870846	9.954946	10.045054	9.834054	9.863892	9.970162	10.029838	58
3	9.825931	9.870732	9.955200	10.044800	9.834189	9.863774	9.970416	10.029584	57
4	9.826071	9.870618	9.955454	10.044546	9.834325	9.863656	9.970669	10.029331	56
5	9.826211	9.870504	9.955708	10.044292	9.834460	9.863538	9.970922	10.029078	55
6	9.826351	9.870390	9.955961	10.044039	9.834595	9.863419	9.971175	10.028825	54
7	9.826491	9.870276	9.956215	10.043785	9.834730	9.863301	9.971429	10.028571	53
8	9.826631	9.870161	9.956469	10.043531	9.834865	9.863183	9.971682	10.028318	52
9	9.826770	9.870047	9.956723	10.043277	9.834999	9.863064	9.971935	10.028065	51
10	9.826910	9.869933	9.956977	10.043023	9.835134	9.862946	9.972188	10.027812	50
11	9.827049	9.869818	9.957231	10.042769	9.835269	9.862827	9.972441	10.027559	49
12	9.827189	9.869704	9.957485	10.042515	9.835403	9.862709	9.972695	10.027305	48
13	9.827329	9.869589	9.957739	10.042261	9.835538	9.862590	9.972948	10.027052	47
14	9.827467	9.869474	9.957993	10.042007	9.835672	9.862471	9.973201	10.026799	46
15	9.827606	9.869360	9.958247	10.041753	9.835807	9.862353	9.973454	10.026546	45
16	9.827745	9.869245	9.958500	10.041500	9.835941	9.862234	9.973707	10.026293	44
17	9.827884	9.869130	9.958754	10.041246	9.836075	9.862115	9.973960	10.026040	43
18	9.828023	9.869015	9.959008	10.040992	9.836209	9.861996	9.974213	10.025787	42
19	9.828162	9.868900	9.959262	10.040738	9.836343	9.861877	9.974466	10.025534	41
20	9.828301	9.868785	9.959516	10.040484	9.836477	9.861758	9.974720	10.025280	40
21	9.828439	9.868670	9.959769	10.040231	9.836611	9.861638	9.974973	10.025027	39
22	9.828578	9.868555	9.960023	10.039977	9.836745	9.861519	9.975226	10.024774	38
23	9.828716	9.868440	9.960277	10.039723	9.836878	9.861400	9.975479	10.024521	37
24	9.828855	9.868324	9.960530	10.039470	9.837012	9.861280	9.975732	10.024268	36
25	9.828993	9.868209	9.960784	10.039216	9.837146	9.861161	9.975985	10.024015	35
26	9.829131	9.868093	9.961038	10.038962	9.837279	9.861041	9.976238	10.023762	34
27	9.829269	9.867978	9.961292	10.038708	9.837412	9.860922	9.976491	10.023509	33
28	9.829407	9.867862	9.961545	10.038455	9.837546	9.860802	9.976744	10.023256	32
29	9.829545	9.867747	9.961799	10.038201	9.837679	9.860682	9.976997	10.023003	31
30	9.829683	9.867631	9.962052	10.037948	9.837812	9.860562	9.977250	10.022750	30
31	9.829821	9.867515	9.962306	10.037694	9.837945	9.860442	9.977503	10.022497	29
32	9.829959	9.867399	9.962560	10.037440	9.838078	9.860322	9.977756	10.022244	28
33	9.830097	9.867283	9.962813	10.037187	9.838211	9.860202	9.978009	10.021991	27
34	9.830234	9.867167	9.963067	10.036933	9.838344	9.860082	9.978262	10.021738	26
35	9.830372	9.867051	9.963320	10.036680	9.838477	9.859962	9.978515	10.021485	25
36	9.830509	9.866935	9.963574	10.036426	9.838610	9.859842	9.978768	10.021232	24
37	9.830646	9.866819	9.963828	10.036172	9.838742	9.859721	9.979021	10.020979	23
38	9.830784	9.866703	9.964081	10.035919	9.838875	9.859601	9.979274	10.020726	22
39	9.830921	9.866586	9.964335	10.035665	9.839007	9.859480	9.979527	10.020473	21
40	9.831058	9.866470	9.964588	10.035412	9.839140	9.859360	9.979780	10.020220	20
41	9.831195	9.866353	9.964842	10.035158	9.839272	9.859239	9.980033	10.019967	19
42	9.831332	9.866237	9.965095	10.034905	9.839404	9.859119	9.980286	10.089714	18
43	9.831469	9.866120	9.965349	10.034651	9.839536	9.858998	9.930538	10.019462	17
44	9.831606	9.866004	9.965602	10.034398	9.839668	9.858877	9.980791	10.019209	16
45	9.831742	9.865887	9.965855	10.034145	9.839800	9.858756	9.981044	10.018956	15
46	9.831879	9.865770	9.966109	10.033891	9.839932	9.858635	9.981297	10.018703	14
47	9.832015	9.865653	9.966362	10.033638	9.840064	9.858514	9.981550	10.018450	13
48	9.832152	9.865536	9.966616	10.033384	9.840196	9.858393	9.981803	10.018197	12
49	9.832288	9.865419	9.966869	10.033131	9.840328	9.858272	9.982056	10.017944	11
50	9.832425	9.865302	9.967123	10.032877	9.840459	9.858151	9.982309	10.017691	10
51	9.832561	9.865185	9.967376	10.032624	9.840591	9.858029	9.982562	10.017438	9
52	9.832697	9.865068	9.967629	10.032371	9.840722	9.857908	9.982814	10.017186	8
53	9.832833	9.864950	9.967883	10.032117	9.840854	9.857786	9.983067	10.016933	7
54	9.832969	9.864833	9.968136	10.031864	9.840985	9.857665	9.983320	10.016680	6
55	9.833105	9.864716	9.968389	10.031611	9.841116	9.857543	9.983573	10.016427	5
56	9.833241	9.864598	9.968643	10.031357	9.842347	9.857422	9.983826	10.016174	4
57	9.833377	9.864481	9.968896	10.031104	9.841378	9.857300	9.984079	10.015921	3
58	9.833512	9.864363	9.969149	10.030851	9.841509	9.857178	9.984332	10.015668	2
59	9.833648	9.864245	9.969403	10.030597	9.841640	9.857056	9.984584	10.015416	1
60	9.833783	9.864127	9.969656	10.030344	9.841771	9.856934	9.984837	10.015163	0
	Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	

44 Deg.

′	Sine.	Cosine.	Tang.	Cotang.	′
0	9·841771	9·856934	9·984837	10·015163	60
1	9·841902	9·856812	9·985090	10·014910	59
2	9·842033	9·856690	9·985343	10·014657	58
3	9·842163	9·856568	9·985596	10·014404	57
4	9·842294	9·856446	9·985848	10·014152	56
5	9·842424	9·856323	9·986101	10·013899	55
6	9·842555	9·856201	9·986354	10·013646	54
7	9·842685	9·856078	9·986607	10·013393	53
8	9·842815	9·855956	9·986860	10·013140	52
9	9·842946	9·855833	9·987112	10·012888	51
10	9·843076	9·855711	9·987365	10·012635	50
11	9·843206	9·855588	9·987618	10·012382	49
12	9·843336	9·855465	9·987871	10·012129	48
13	9·843466	9·855342	9·988123	10·011877	47
14	9·843595	9·855219	9·988376	10·011624	46
15	9·843725	9·855096	9·988629	10·011371	45
16	9·843855	9·854973	9·988882	10·011118	44
17	9·843984	9·854850	9·989134	10·010866	43
18	9·844114	9·854727	9·989387	10·010613	42
19	9·844243	9·854603	9·989640	10·010360	41
20	9·844372	9·854480	9·989893	10·010107	40
21	9·844502	9·854356	9·990145	10·009855	39
22	9·844631	9·854233	9·990398	10·009602	38
23	9·844760	9·854109	9·990651	10·009349	37
24	9·844889	9·853986	9·990903	10·009097	36
25	9·845018	9·853862	9·991156	10·008844	35
26	9·845147	9·853738	9·991409	10·008591	34
27	9·845276	9·853614	9·991662	10·008338	33
28	9·845405	9·853490	9·991914	10·008086	32
29	9·845533	9·853366	9·992167	10·007833	31
30	9·845662	9·853242	9·992420	10·007580	30
31	9·845790	9·853118	9·992672	10·007328	29
32	9·845919	9·852994	9·992925	10·007075	28
33	9·846047	9·852869	9·993178	10·006822	27
34	9·846175	9·852745	9·993431	10·006569	26
35	9·846304	9·852620	9·993683	10·006317	25
36	9·846432	9·852496	9·993936	10·006064	24
37	9·846560	9·852371	9·994189	10·005811	23
38	9·846688	9·852247	9·994441	10·005559	22
39	9·846816	9·852122	9·994694	10·005306	21
40	9·846944	9·851997	9·994947	10·005053	20
41	9·847071	9·851872	9·995199	10·004801	19
42	9·847199	9·851747	9·995452	10·004548	18
43	9·847327	9·851622	9·995705	10·004295	17
44	9·847454	9·851497	9·995957	10·004043	16
45	9·847582	9·851372	9·996210	10·003790	15
46	9·847709	9·851246	9·996463	10·003537	14
47	9·847836	9·851121	9·996715	10·003285	13
48	9·847964	9·850996	9·996968	10·003032	12
49	9·848091	9·850870	9·997221	10·002779	11
50	9·848218	9·850745	9·997473	10·002527	10
51	9·848345	9·850619	9·997726	10·002274	9
52	9·848472	9·850493	9·997979	10·002021	8
53	9·848599	9·850368	9·998231	10·001769	7
54	9·848726	9·850242	9·998484	10·001516	6
55	9·848852	9·850116	9·998737	10·001263	5
56	9·848979	9·849990	9·998989	10·001011	4
57	9·849106	9·849864	9·999242	10·000758	3
58	9·849232	9·849738	9·999495	10·000505	2
59	9·849359	9·849611	9·999747	10·000253	1
60	9·849485	9·849485	10.000000	10·000000	0